Evaluation of the non-Gaussianity of two-mode entangled states over a bosonic memory channel via cumulant theory and quadrature detection

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(Received 2 August 2017; revised manuscript received 29 October 2017; published 4 April 2018)

We study the properties of the cumulants of multimode boson operators and introduce the phase-averaged quadrature cumulants as the measure of the non-Gaussianity of multimode quantum states. Using this measure, we investigate the non-Gaussianity of two classes of two-mode non-Gaussian states: photon-number entangled states and entangled coherent states traveling in a bosonic memory quantum channel. We show that such a channel can skew the distribution of two-mode quadrature variables, giving rise to a strongly non-Gaussian correlation. In addition, we provide a criterion to determine whether the distributions of these states are super- or sub-Gaussian.

DOI: 10.1103/PhysRevA.97.042303

I. INTRODUCTION

The non-Gaussianity has been shown to play a prominent role in statistical problems in various fields of astrophysics and general physics over the last decades. For example, the non-Gaussianity has become an important probe of the fundamental origin and the late time evolution of universe structures in cosmography [1] and a powerful tool for measuring the independence of signals in signal processing [2] and the correlations among many-body quantum systems [3]. In particular, a quantum version of the non-Gaussianity (QNG) has already been well explored in the quantum optical context [4] and quantum information processing such as quantum teleportation [5,6] and quantum error correction [7].

Recently, the various measures have been suggested for quantifying or detecting the quantum non-Gaussianity of quantum states. Genoni *et al.* [8] first used the Hilbert-Schmidt distance to quantify the non-Gaussian character of a bosonic quantum state and evaluated the non-Gaussianity of some relevant states. Subsequently, they developed the entropic measure of the non-Gaussianity based on the quantum relative entropy [9], by which they investigated the performance of conditional Gaussification toward twin-beam and de-Gaussification processes driven by Kerr interaction. On the other hand, the non-Gaussianity was experimentally measured via relative entropy for single-photon added coherent states [10] and phase-averaged coherent states [11], respectively. It should be noted that these measures are based on the distinguishability of a given quantum state itself and its reference Gaussian state with the same first and second moments. However, it has recently been proven that they could not discriminate between quantum non-Gaussian states and mixtures of Gaussian states. Thus, to accomplish this goal, a quantum non-Gaussianity witness in phase space was proposed, whose main idea is to seek the violation of a lower bound for the values that the phase-space quasiprobability distributions can take in a particular point of phase space [12,13]. It has been shown that the Husimi Q-function-based witnesses are more than effective than other criteria in detecting the quantum non-Gaussianity of various kinds of non-Gaussian states evolving in lossy channel [13]. We shall emphasize that the computation of the above-mentioned measures is intractable for multimode continuous-variable (CV) quantum states. Thus, a natural question arises: Are there other simple ways to characterize the non-Gaussianity for quantum states of CV systems with an arbitrary number of modes?

The purpose of this research is to address this issue. As is well known, in quantum optics the quantum state of a bosonic system can be identified with its density matrix or phase-space functions such as the Glauber-Sudarshan P function [14], the Husimi Q function [15], and the Wigner function [16]; the density matrix could be related to these distribution functions by a suitable Fourier transform [16,17]. So they contain the same information. On the other hand, it has been shown in Ref. [18] that the phase-space distributions can be obtained from the measured appropriate probability distributions of the rotated quadrature operator. That is to say, the full information on the quantum state can be reconstructed from the statistics of the quadrature components of a given light field, and then allowing evaluation of the properties of quantum-state-like nonclassicality [19] and statistic properties [20]. Therefore, at this point, the knowledge of the quadrature variable is equivalent to knowledge of quantum states in describing the quantum non-Gaussianity, and vice versa. Consequently, we can safely say that the non-Gaussianity of the quadrature distribution is meant to that of quantum states of the multimode CV system. Or, more precisely, a quantum state is non-Gaussian if the quantum phase-space distribution functions of quadrature operators have a non-Gaussian form. In fact, the research in this area has been reported in recent years. Olsen and Corney [21] investigated the non-Gaussian statistic of the Kerr-squeezed state by calculating higher-order cumulants of quadrature variables. It was found that the nonlinear interaction can skew the distribution of the quadrature variables, giving rise to large third- and fourth-order cumulants for sufficiently long interaction times. We characterized the non-Gaussianity of CV quantum states by means of the cumulant theory and

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studied the dynamics of various orderings of the bosonic operators in two-mode single-photon squeezed Bell states in two different decoherence modes with the fourth-order cumulant method [22]. It was shown that in a two-reservoir model, all the fourth-order cumulants are very fragile, while in the single-reservoir model, the fourth-order cumulants for the two-mode squeezed singlet Bell state are insensitive to thermal noise, showing the time-invariant non-Gaussianity. The main disadvantage of this method in Ref. [22] lies in the fact that to exactly detect the non-Gaussianity, it requires one to analyze all the cumulants of various combinations of the operators acting on quantum states. As a result, this limits the proposals to analyze quantum systems with a large number of degrees of freedom, and thus we aspire to further improve it.

Among quantum channels, a lossy bosonic memory channel provides an exactly solvable theoretical model and can be realized experimentally [23]. Recently, a lot of efforts have been devoted to the study of the capacities of such a quantum channel. Ruggeri *et al.* [24] studied the information transmission through a lossy bosonic channel with memory effects. It was shown that the entangled inputs can improve the rate of transmission of such a channel. Pilyavets *et al.* [25] demonstrated that entangled inputs can enhance the classical capacity with respect to the memoryless case. In their research, the environment modes are assumed to be initially in a multimode squeezed vacuum in such a way that such a quantum channel can be viewed as a bosonic Gaussian channel, which transforms Gaussian random variables (Gaussian states) into Gaussian random variables (Gaussian states). Unlike the treatment in pervious works, in this paper we will regard such a channel as a phase-sensitive reservoir being in a multimode squeezed number state [26] and investigate how such a channel affects the non-Gaussianities of two-mode non-Gaussian entangled states using the phase-averaged cumulant technique of quadrature operators.

The paper is organized as follows. In Sec. II we briefly recall the definition of cumulants and propose some lemmas for the computation of the cumulants of multimode quadrature operators under the specific conditions. In particular, we put forward an explicit expression of the fourth-order cumulant of two noncommutative quantum operators when a quantum system is entangled. In Sec. III we review the quantum memory channel and find its solution with respect to quadrature operators. Without loss of generality, we introduce a measure of the phase-average cumulant to detect the non-Gaussianity of N signals and provide a scheme of multimode homodyne tomography with a single-mode local oscillator. By virtue of these results we investigate the effects of the quantum memory channel on the quantum non-Gaussianities of two classes of two-mode non-Gaussian entangled states: photon number entangled states and coherent entangled states, in Sec. IV. Section V summarizes our results.

II. MOMENTS AND CUMULANTS OF OPERATORS

According to the standard interpretation of quantum mechanics, the density matrix ρ contains the complete information about a given quantum system, whose full statistics can, however, be obtained from the probability distribution of observable quantities. Let $\hat{X}$ be a Hermitian operator with a one-dimensional continuous spectrum of real eigenvalues x to right-hand eigenstates $|x\rangle$, then its eigenvalue equation is written as $\hat{X}|x\rangle = x|x\rangle$ and the corresponding characteristic function is expressed as

$$\tilde{f}(\xi) \equiv \langle e^{\xi \hat{X}} \rangle, \tag{1}$$

where the bracket $\langle \hat{O} \rangle$ denotes the expectation of the operator $\hat{O}$. Using the Maclaurin series expansion, we can further write the above equation as

$$\tilde{f}(\xi) = 1 + \sum_{j=1}^{n} \frac{1}{n!} \mu_n(\hat{X}) \xi^n, \tag{2}$$

where we have referred to $\mu_n(\hat{X})$ as the nth order moment of operator $\hat{X}$, i.e., $\mu_n(\hat{X}) = \langle \hat{X}^n \rangle$.

Similar to the probability theory and statistics, we can define the cumulant-generating function $G(\xi)$ via the natural logarithm of the characteristic function of the operator, that is,

$$G(\xi) = \ln\langle e^{\xi \hat{X}} \rangle. \tag{3}$$

It follows that the nth cumulant of operator $\hat{X}$ can be obtained by differentiating the above expansion n times and evaluating the result at zero:

$$\kappa_n(\hat{X}) = \frac{\partial^n}{\partial \xi^n} G(\xi) \bigg|_{\xi=0}. \tag{4}$$

Thus, with these cumulants $\kappa_n(\hat{X})$, we can write the characteristic function of operator $\hat{X}$ in another form:

$$\langle e^{\xi \hat{X}} \rangle = \exp\left(\sum_{n=1}^{\infty} \frac{\xi^n}{n!} \kappa_n(\hat{X}) \right). \tag{5}$$

Applying the identity [27],

$$(a_1 + a_2 + \cdots + a_m)^n = \sum_{\{l_m\}} n! \left(\prod_{s=1}^{m} \frac{a_s^{l_s}}{l_s!} \right), \tag{6}$$

where $\sum_{\{l_m\}}$ is the sum of all possible combination of $\{l_1, l_2, \ldots, l_m\}$ with $l_1 + l_2 + \cdots + l_m = n$, and comparing Eqs. (2) and (5), we can obtain the relationship between the moments and cumulants as

$$\kappa_n(\hat{X}) = \mu_n(\hat{X}) - \sum_{m=1}^{n-1} \binom{n-1}{m-1} \mu_{n-m}(\hat{X}) \kappa_m(\hat{X}), \tag{7}$$

where $\binom{n}{m} = \frac{n!}{m!(n-m)!}$. It can clearly be seen that a cumulant can be explicitly represented only by the lower moments and vice versa. For example, the third- and fourth-order cumulants can be written as

$$\kappa_3(\hat{X}) = \mu_3(\hat{X}) - 3\mu_2(\hat{X})\mu_1(\hat{X}) + 2\mu_1^3(\hat{X}), \tag{8a}$$

$$\kappa_4(\hat{X}) = \mu_4(\hat{X}) - 3\mu_2^2(\hat{X}) - 4\mu_1(\hat{X})\mu_3(\hat{X}) + 12\mu_1^2(\hat{X})\mu_2(\hat{X}) - 6\mu_1^4(\hat{X}). \tag{8b}$$

These cumulants have certain geometric meanings and characterize the asymmetry and the sharpness of the characteristic function of operator $\hat{X}$, respectively. As is well known, the third-order cumulant will vanish for symmetric distributions.

Therefore, the information contained in the nonzero fourth-order cumulant is enough to determine whether a given distribution is Gaussian or not. Namely, with the positive κ_4 the distribution of operator $\hat{X}$ is called super-Gaussian or platykurtotic, and that with negative κ_4 it is called sub-Gaussian or leptokurtotic.

With these notations and definitions at hand, a generalization to a multimode collective operator is straightforward. However, in calculating the joint cumulants of such operators, we should consider two facts. One is the commutation relations among these operators. Another important fact is that one should know whether or not a quantum system is entangled. Thus we introduce the following lemmas in terms of quantum-mechanical conventional wisdom, respectively.

Lemma 1. (Additivity). Suppose that $\hat{A}_1,\hat{A}_2,\dots,\hat{A}_N$ are N mutually commutative Hermitian operators with $\hat{A}_j$ acting on the Hilbert space of the subsystem j and the whole system is prepared in a produce state, i.e., $\rho=\bigotimes_{j=1}^N \rho_j$, with ρ_j being the density matrix of the subsystem j, then the nth-order cumulant of their sum is equal to the sum of their nth-order cumulants.

Proof. Let $\hat{A}=\sum_{j=1}^N \hat{A}_j$ be a collective operator. We can write the joint characteristic function as

$$\tilde{f}(\xi)=\langle \exp(\xi\hat{A})\rangle_\rho=\left\langle \exp\left(\sum_{j=1}^N \xi\hat{A}_j\right)\right\rangle_\rho. \tag{9}$$

Since these operators satisfy the commutation relations $[\hat{A}_l,\hat{A}_m]=0$, we have $[e^{\hat{A}_l},e^{\hat{A}_m}]=0$ and $\langle e^{\sum_{l=1}^N \hat{A}_l}\rangle=\langle\prod_{l=1}^N e^{\hat{A}_l}\rangle$. Considering the case of $\rho=\bigotimes_{j=1}^N\rho_j$, Eq. (9) can be further expressed as the sum of the corresponding marginal characteristic functions as

$$\tilde{f}(\xi)=\prod_{j=1}^N\langle \exp(\xi\hat{A}_j)\rangle_{\rho_j}=\tilde{f}_1(\xi)\otimes\tilde{f}_2(\xi)\otimes\cdots\otimes\tilde{f}_N(\xi). \tag{10}$$

Furthermore, the cumulant generating function is given by

$$K(\xi)=\sum_{j=1}^N \ln\tilde{f}_j(\xi)=\sum_{j=1}^N K_j(\xi). \tag{11}$$

According to the definition of the cumulant, we can easily obtain

$$\kappa_n(\hat{A})=\sum_{j=1}^N \kappa_n(\hat{A}_j). \tag{12}$$

This ends the proof. ■

Lemma 2. (Homogeneity). Suppose $\hat{X}$ is an operator with an nth-order cumulant, then for any scalar $\lambda\in\mathbb{R}$, the operator $\lambda\hat{X}$ has the nth-order cumulant, which is given by $\lambda^n\kappa_n(\hat{X})$.

Proof. According to the definition (4), we can write the characteristic function of operator $\lambda\hat{X}$ in terms of the cumulants as

$$\langle e^{\xi\lambda\hat{X}}\rangle=\exp\left(\sum_{n=1}^\infty\frac{\xi^n}{n!}\kappa_n(\lambda\hat{X})\right)=\exp\left(\sum_{n=1}^\infty\frac{(\lambda\xi)^n}{n!}\kappa_n(\hat{X})\right). \tag{13}$$

Then, we have

$$\kappa_n(\lambda\hat{X})=\lambda^n\kappa_n(\hat{X}), \tag{14}$$

which completes the proof. ■

Lemma 3. (Semi-invariance). Suppose $\hat{X}$ is an operator with an nth cumulant and let $\mathbb{G}:\hat{X}\mapsto\hat{Y}=a\hat{X}+b$ be a quantum affine transformation with a and b being any real constants. Under such a transformation, the cumulants of $\hat{Y}$ are given by

$$\kappa_n(\hat{Y})=\begin{cases} a\kappa_n(\hat{X})+b, & \text{if } n=1,\\ a^n\kappa_n(\hat{X}), & \text{if } n\geqslant 2.\end{cases} \tag{15}$$

Proof. The characteristic function of $\hat{Y}$ with a real parameter ξ can be written as

$$\begin{aligned}\tilde{f}(\xi)&=\langle\exp(\xi\hat{Y})\rangle=\langle\exp[\xi(a\hat{X}+b)]\rangle\\ &=e^{\xi b}\langle\exp(\xi a\hat{X})\rangle,\end{aligned} \tag{16}$$

so that the cumulant generating function is given by

$$K(\xi)=\xi b+\ln\langle\exp(\xi a\hat{X})\rangle. \tag{17}$$

As is well known, the constant operator has a nonzero first-order cumulant with other cumulants vanishing. Thus according to Lemma 2, we can obtain the higher-order cumulants of operator $\hat{Y}$ as

$$\kappa_n(\hat{Y})=a^n\kappa_n(\hat{X}). \tag{18}$$

That is, the lemma is proven. ■

Lemma 4. (Gaussian cumulant). For a set of Gaussian states in an infinite-dimensional Hilbert space $\mathcal{H}$, then the cumulants are equal to the corresponding mean values of quadratures and elements of the covariance matrix, respectively.

Proof. For any N-mode Gaussian state, its characteristic function is fully determined in $\mathcal{H}$ by the first and second moments of the quadrature operators, namely

$$\begin{aligned}\chi[\hat{\rho}_G](\boldsymbol{\xi})&=\mathrm{Tr}\left[\hat{\rho}_G\bigotimes_{j=1}^N\hat{D}(\xi_j)\right]\\ &=\exp[-\boldsymbol{\mu}^T\boldsymbol{\nu}\boldsymbol{\mu}-i\langle\mathbf{R}^T\rangle_{\hat{\rho}_G}\boldsymbol{\mu}],\end{aligned} \tag{19}$$

where $\hat{D}(\xi_j)=\exp(\xi_j\hat{a}_j^+-\xi_j^*\hat{a}_j)$, $\boldsymbol{\nu}\in M_{2N}(\mathbb{R})$ is the real, symmetric, and positive $2N\times 2N$ covariance matrix of the state $\hat{\rho}_G$. $\mathbf{R}:=(\hat{R}_1,\hat{R}_2,\dots,\hat{R}_{2N-1},\hat{R}_{2N})^T=(\hat{x}_1,\hat{p}_1,\dots,\hat{x}_N,\hat{p}_N)^T$ is a $1\times 2N$ row matrix whose entries are the $2N$ quadrature operators of the modes, $\hat{x}_j=(\hat{a}_j+\hat{a}_j^+)/\sqrt{2},\hat{p}_j=(\hat{a}_j-\hat{a}_j^+)/(i\sqrt{2})$. The vector $\boldsymbol{\mu}\in\mathbb{R}^{2N}$ is related to the expectation values $\bar{\mathbf{R}}:=\sqrt{2}\left\{\mathrm{Re}(\xi_1),\mathrm{Im}(\xi_1),\dots,\mathrm{Re}(\xi_N),\mathrm{Im}(\xi_N)\right\}$ of the quadrature operators in an N-mode coherent state $|\xi_j\rangle^{\otimes N}$ by the linear transformation $\boldsymbol{\mu}=-\Omega_N\bar{\mathbf{R}}$ with $\Omega_N=\bigoplus_{k=1}^N i\sigma_y$ being a $2N\times 2N$ orthogonal symplectic matrix.

Thus by the definition of the cumulant,

$$\kappa_{\mathbf{m}}(\mathbf{R})=\frac{1}{i^{\mathbf{m}}}\frac{\partial^{\mathbf{m}}}{\partial z_1^n\partial w_1^m\cdots\partial z_N^p\partial w_N^q}\ln\{\chi[\hat{\rho}_G](\boldsymbol{\xi})\}\bigg|_{\boldsymbol{\xi}=0}, \tag{20}$$

where $\mathbf{m} = n + m + \cdots + p + q$ and $\xi_j = w_j + i z_j$, one can easily show that the first-order cumulants are given by

$$\kappa_1(\hat{R}_j) = \langle \hat{R}_j^T \rangle_{\hat{\rho}_G}, \tag{21}$$

and the second-order cumulants are the vectors of the covariance matrix, i.e.,

$$\kappa_2(\hat{R}_l \hat{R}_m) = \mathrm{vec}(\boldsymbol{v}_{l,m}), \quad (l,m = 1,2,\ldots,2N), \tag{22}$$

which completes the proof. ■

Now we briefly review the cumulants of some Gaussian states. It follows that an arbitrary Gaussian state $\hat{\rho}_G$ can be generated from a thermal state $\hat{\rho}_G$ with a Gaussian operator $\hat{O}_G$ [28], i.e., $\hat{\rho}_G = \hat{O}_G \hat{\rho}_G \hat{O}_G^+$. Additionally, since the first moments of the Gaussian state can always be made to vanish via local operations, we focus on calculating the second-order cumulants for some subsets of Gaussian states. A single-mode Gaussian state is a simple class of Gaussian states and can be parametrized in an optimal form as $\rho_{sG} = \hat{D}(\alpha)\hat{S}(r,\phi)\rho_{th}(\bar{n})\hat{S}^+(r,\phi)\hat{D}^+(\alpha)$, in which $\hat{S}^+(r,\phi) = \exp[-\frac{r}{2}(e^{i\phi}\hat{a}^{+2} - e^{-i\phi}\hat{a}^2)]$ is the squeezing operator with r being the squeezing strength and ϕ being the axis, and $\rho_{th}(\bar{n}) = \sum_{n=0}^{\infty}[\frac{\bar{n}^n}{(1+\bar{n})^{1+n}}]|n\rangle\langle n|$ is a thermal state with $\bar{n}$ being the mean photon number. We see that a pure squeezed state is obtained when $\bar{n} = 0$ and $\alpha = 0$. The state is a thermal squeezed one when $\alpha = 0$. When $r = 0$ there is no squeezing and the corresponding states are pure coherent ones or thermal coherent ones. The implicit expressions of the second-order cumulants are given by $\kappa_2(\hat{x}^2) = \frac{1}{2}(\bar{n} + 1/2)[\cosh(2r) - \sinh(2r)\cos(\phi)]$, $\kappa_2(\hat{p}^2) = \frac{1}{2}(\bar{n} + 1/2)[\cosh(2r) + \sinh(2r)\cos(\phi)]$, and $\kappa_2(\hat{x}\hat{p}) = \kappa_2(\hat{p}\hat{x}) = \frac{1}{2}(\bar{n} + 1/2)\sinh(2r)\sin(\phi)$, respectively. Another typical example of Gaussian states is the N-mode squeezed state, a SU(1,1) coherent state, given by [29] $|\Psi\rangle_{NMSS} = \hat{S}(N,r)|0\rangle^{\otimes N}$, where the squeezing operator $\hat{S}(N,r) = \exp[r(W_1 - W_2)]$ with $W_1 = \frac{2-N}{2N}\sum_{l=1}^{N}\hat{a}_l^{+2} + \frac{2}{N}\sum_{l<m=1}^{N}\hat{a}_l^+\hat{a}_m^+$ and $W_2 = \frac{2-N}{2N}\sum_{l=1}^{N}\hat{a}_l^2 + \frac{2}{N}\sum_{l<m=1}^{N}\hat{a}_l\hat{a}_m$. Thus, the state $|\Psi\rangle_{NMSS}$ becomes the original, normalized EPR state in the infinite squeezing limit for $N = 2$. After some lengthy calculation, we can obtain the second-order cumulants of quadratures as: $\kappa_2(\hat{x}_j^2) = \frac{N\cosh(2r)+(N-2)\sinh(2r)}{4N}$, $\kappa_2(\hat{p}_j^2) = \frac{N\cosh(2r)-(N-2)\sinh(2r)}{4N}$, and $\kappa_2(\hat{x}_l\hat{x}_m) = \kappa_2(\hat{p}_l\hat{p}_m) = -\frac{\sinh(2r)}{2N}, (l \neq m)$.

Lemma 5. (Normal ordering). Let $\hat{X}$ and $\hat{Y}$ be two noncommuting quantum mechanical operators in a quantum system, each having an nth cumulant. If such a quantum system is prepared in an entangled state, then for any $(g,h) \in \mathbb{R}$, the normally ordered nth-order cumulant of the combined operator $\hat{F} = g\hat{X} + h\hat{Y}$ is formally expressed as

$$\kappa_n(\hat{F}) = \sum_{j=0}^{n} g^j h^{n-j} \kappa(:\hat{X}^j\hat{Y}^{n-j}:), \tag{23}$$

where the symbol :: denotes the normal ordering (all $\hat{X}$ stand on the left of all $\hat{Y}$).

Proof. Since these two operators $\hat{X}$ and $\hat{Y}$ have noncommutativity and the system is in a nonseparable state, we have $\langle e^{g\hat{X}+h\hat{Y}}\rangle \neq \langle e^{g\hat{X}}\rangle\langle e^{h\hat{Y}}\rangle$. Consequently, taking the exponential polynomial expansion, we can write the characteristic function $\tilde{f}_{\hat{F}}$ of operator $\hat{F}$ as

$$\begin{aligned}\tilde{f}_{\hat{F}}(\xi) &= \langle \exp[\xi(g\hat{X} + h\hat{Y})]\rangle \\ &= 1 + \left\langle \sum_{n=1}^{\infty}\frac{\xi^n}{n!}(g\hat{X} + h\hat{Y})^n \right\rangle \\ &= 1 + \sum_{n=1}^{\infty}\sum_{j=0}^{n}\frac{\xi^n}{n!}g^j h^{n-j}\langle :\hat{X}^j\hat{Y}^{n-j}:\rangle,\end{aligned} \tag{24}$$

where we have used the normal order of the quantum operator in the last line of the above Eq. (24). Furthermore, in light of the definition of the cumulant, we have

$$\begin{aligned}\ln \tilde{f}_{\hat{F}}(\xi) &\equiv \sum_{n=1}^{\infty}\frac{\xi^n}{n!}\kappa_n(\hat{F}) \\ &= \sum_{m=1}^{\infty}\frac{(-)^{m+1}}{m!}\left[\sum_{n=1}^{\infty}\sum_{j=0}^{n}\frac{\xi^n}{n!}g^j h^{n-j}\langle :\hat{X}^j\hat{Y}^{n-j}:\rangle\right]^m.\end{aligned} \tag{25}$$

Thus, we can obtain the formula (23) by comparing the coefficient ξ on both sides of Eq. (25) and by a recursion relation (7). But doing so it is quite cumbersome. To better understand the formula (23), we shall give a concrete example for normally ordered operators. For example, the normally ordered operator $\hat{X}\hat{Y}^3$ can be written as $:\hat{X}\hat{Y}^3: = \hat{X}\hat{Y}^3 + \hat{Y}\hat{X}\hat{Y}^2 + \hat{Y}^2\hat{X}\hat{Y} + \hat{Y}^3\hat{X}$. Thus, its fourth-order cumulant can be defined as $\kappa(:\hat{X}\hat{Y}^3:) \equiv \kappa(\hat{X}\hat{Y}^3) + \kappa(\hat{Y}\hat{X}\hat{Y}^2) + \kappa(\hat{Y}^2\hat{X}\hat{Y}) + \kappa(\hat{Y}^3\hat{X})$, where $\kappa(\hat{X}\hat{Y}\hat{Z}\hat{W}) = \langle\hat{X}\hat{Y}\hat{Z}\hat{W}\rangle - \langle\hat{X}\hat{Y}\rangle\langle\hat{Z}\hat{W}\rangle - \langle\hat{X}\hat{Z}\rangle\langle\hat{Y}\hat{W}\rangle - \langle\hat{X}\hat{W}\rangle\langle\hat{Y}\hat{Z}\rangle$ if the operators $(\hat{X},\hat{Y},\hat{Z},\hat{W})$ have zero means. Additionally, we can write the formula (23) in another two forms: antinormal ordering and Weyl ordering. The mutual transformation formulas for these ordering bosonic-operator functions have been explicitly given in some references [16,30–33]. For example, the mutual transformations between the antinormal and normal orderings for the bosonic creation and annihilation operators are given by [32]

$$\hat{a}^n(\hat{a}^+)^m = \sum_{l=0}^{\min(m,n)} l!\binom{n}{l}\binom{m}{l}(\hat{a}^+)^{m-l}\hat{a}^{n-l}, \tag{26a}$$

$$(\hat{a}^+)^n\hat{a}^m = \sum_{l=0}^{\min(m,n)} (-1)^l l!\binom{n}{l}\binom{m}{l}\hat{a}^{m-l}(\hat{a}^+)^{n-l}. \tag{26b}$$

For convenience later, we consider a special case in which two bosonic operators $(\hat{X},\hat{Y})$ have zero means and obey the c-number commutation relation, i.e., $[\hat{X},\hat{Y}] = c$ with c being a constant. According to the formula (23), the normally ordered fourth-order cumulant of the combined operator $\hat{F} = a\hat{X} + b\hat{Y}$ with a and b being two constants is calculated as

$$\begin{aligned}\kappa_4(\hat{F}) &= a^4\kappa(\hat{X}^4) + b^4\kappa(\hat{Y}^4) + 4a^3b\kappa(\hat{X}^3\hat{Y}) + 4ab^3\kappa(\hat{X}\hat{Y}^3) \\ &\quad + 6a^2b^2\kappa(\hat{X}^2\hat{Y}^2) - 6a^3bc\kappa(\hat{X}^2) - 6ab^3c\kappa(\hat{Y}^2) \\ &\quad - 12a^2b^2c\kappa(\hat{X}\hat{Y}) + 3a^2b^2c^2,\end{aligned} \tag{27}$$

where we show again that if and only if the quantum system is in a separable state and these quantum operators commute with each other, the statement of Lemma 5 is consistent with its classical version, which denotes that the observables are statistically mutually independent [27]. Namely, the formula (27) may simplify to $\kappa_n(\hat{F}) = \sum_{j=0}^{n} \binom{n}{j} g^j h^{n-j} \kappa(\hat{X}^j \hat{Y}^{n-j})$. Additionally, we also see from Lemma 3 that the Gaussian maps such as the single-mode displacement and squeezing operations do not change the Gaussian characters of the distribution of a quantum operator, but can alter its non-Gaussianity. For example, taking into account a single-mode squeezing operator $\hat{S}(r,\phi)$ with $\phi = 0$, we have $\hat{S}^{-1}(\zeta)(\hat{x},\hat{p})^T \hat{S}(\zeta) = \mathrm{diag}(e^{-r},e^{r})(\hat{x},\hat{p})^T$. According to Lemma 3, the n-order cumulants of quadrature operators $\hat{x}$ and $\hat{p}$ after squeezing can be easily calculated as $\kappa_n(\hat{x}^{'}) = e^{-nr}\kappa_n(\hat{x})$ and $\kappa_n(\hat{p}^{'}) = e^{nr}\kappa_n(\hat{p})$, respectively. Thus we see that the Gaussian characters of the distribution of these quadrature operators can keep unchanged if and only if they are initially Gaussian, i.e., $\kappa_{n\geqslant 3}(\hat{p}) = 0, \kappa_{n\geqslant 3}(\hat{x}) = 0$. However, when they are initially non-Gaussian, it is clear to see that the higher-order cumulants for $\hat{x}^{'}$ approach to zero in the case of r being enough larger, implying that its non-Gaussianity vanishes. Therefore, in what follows we investigate the quantum non-Gaussianities of two-mode non-Gaussian entangled states traveling in a bosonic non-Gaussian memory channel with the help of the proposed lemmas.

III. LOSSY BOSONIC MEMORY CHANNEL

In quantum information science, a quantum channel is a communication channel which can transmit classical and quantum information. This information, however, can be easily destroyed due to the unavoidable interaction with their noisy environments. According to whether the noise affecting communication is correlated or uncorrelated, quantum channels can be classified into two categories. One is a memoryless quantum channel where the noise acts identically and independently on each channel use. In other words, the output of a channel at a given time depends only upon the corresponding input and not any previous ones. Memoryless processes are often recognized as Markovian. The other is a quantum memory channel, which is characterized by a correlated source of noise where the future states of a channel are directly or indirectly influenced by its past input states. In this paper we consider a model of a bosonic memory channel, which is particularly interesting because the memory effects can be controlled by a multimode squeezer [25,34,35]. Just as stated in Ref. [25], such a noise correlation is introduced by contiguous mode interactions which results in an exponential decay of the correlations over channel uses (modes), thus making the noise in the channel non-Markovian and the channel not forgetful. This quantum channel is thus named the non-Markovian memory quantum channel. We shall stress that there is no inextricable link between the non-Markovianity and non-Gaussianity of the quantum channel. It is generally believed that a quantum channel is Gaussian if and only if it maps Gaussian input states (variables) into Gaussian output ones (variables). Therefore, some of the non-Markovian quantum channels are Gaussian quantum channels [36,37] and others can be non-Gaussian quantum channels [38].

Figure 1 shows the schematic model of our quantum memory channel. Unlike the quantum channel models used in Ref. [24,25,34], we assume that the environment is initially in a multimode Fock state. We see from Fig. 1 that the quantum channel only consists of n beam splitters and a multimode squeezer so that it has two input ports. One is to input a collection of n bosonic modes with the ladder operators $\{\hat{a}_j,\hat{a}_j^{+}\}_{j=1,2,\ldots,n}$ and the other connects all the channel environment modes with $\{\hat{b}_j,\hat{b}_j^{+}\}_{j=1,2,\ldots,n}$. Thus, the transformation of such a channel is described as $\hat{U}(\eta,s) = \hat{U}(\eta)\otimes\hat{S}(s)$, in which the n parallel beam splitters can be characterized by n local unitary operators: $\hat{U}(\eta) = \bigotimes_{j=1}^{n} \hat{U}_j(\eta)$ with

$$\hat{U}_j(\eta) = \exp\left[(\hat{a}_j^{+}\hat{c}_j - \hat{a}_j\hat{c}_j^{+})\arctan\left(\sqrt{\frac{1-\eta}{\eta}}\right)\right], \quad (28)$$

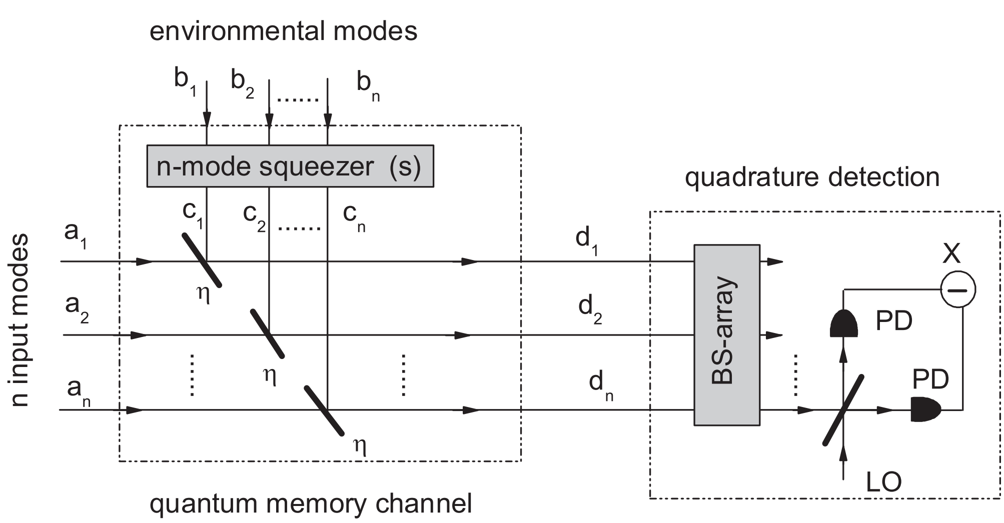

FIG. 1. Scheme of quantum memory channel and multimode quadrature measurement. Each input mode $\hat{a}_j$ interacts with the corresponding environment mode $\hat{b}_k$ through a beam splitter with transmittivity $\eta \in [0,1]$, which is the modeled lossy channel. The memory effect in the quantum channel relies on the squeezing parameter s of the multimode squeezed operation. Thus, we can use the parameters η and s to characterize the channel dynamics. After a BS array, a probability distribution of a linear superposition of n single-mode quadratures is measured by means of standard single-mode homodyne detection. BS, beam splitter; LO, local oscillator; PD, photodetectors.

where $\eta \in [0,1]$ represents the channel dissipation or losses. It leads to the following transformation:

$$\hat{U}_j(\eta)\begin{pmatrix}\hat{a}_j\\ \hat{c}_j\end{pmatrix}\hat{U}_j^{+}(\eta)=\begin{pmatrix}\sqrt{\eta} & -\sqrt{1-\eta}\\ \sqrt{1-\eta} & \sqrt{\eta}\end{pmatrix}\begin{pmatrix}\hat{a}_j\\ \hat{c}_j\end{pmatrix}. \quad (29)$$

The squeezing operator $\hat{S}(s)$ of the n-mode squeezer is described by [34]

$$\hat{S}(s)=\exp\left[s\sum_{k,k'=1,k\neq k'}^{n}(\hat{b}_k^{+}\hat{b}_{k'}^{+}-\hat{b}_k\hat{b}_{k'})\right], \quad (30)$$

where the parameter $s\in\mathbb{R}$ is the memory strength. The quantum channel is a memoryless one if $s=0$; otherwise it represents the memory channel.

By introducing the quadrature operators,

$$\hat{X}_k=\frac{\hat{b}_k+\hat{b}_k^{+}}{\sqrt{2}}, \quad (31a)$$

$$\hat{P}_k=\frac{\hat{b}_k-\hat{b}_k^{+}}{i\sqrt{2}}, \quad (31b)$$

we can write the multimode squeezing operator $\hat{S}(s)$ of Eq. (30) in the compact form,

$$\hat{S}(s)=\exp[i\mathbf{X}\mathbf{\Lambda}(s)\mathbf{P}^T], \quad (32)$$

where $\mathbf{X}=(\hat{X}_1,\hat{X}_2,\ldots,\hat{X}_n)$, $\mathbf{P}=(\hat{P}_1,\hat{P}_2,\ldots,\hat{P}_n)$, and a $n\times n$ matrix $\mathbf{\Lambda}(s)$ is given by

$$\mathbf{\Lambda}(s)=\begin{pmatrix}0 & s & s & \cdots & s\\ s & 0 & s & \cdots & s\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ s & s & s & \cdots & 0\end{pmatrix}. \quad (33)$$

Moreover we can obtain

$$e^{\mathbf{\Lambda}(s)}=\frac{1}{n}\begin{pmatrix}G(s) & F(s) & F(s) & \cdots & F(s)\\ F(s) & G(s) & F(s) & \cdots & F(s)\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ F(s) & F(s) & F(s) & \cdots & G(s)\end{pmatrix}, \quad (34)$$

where $G(s)=(n-1)e^{-s}+e^{(n-1)s}$ and $F(s)=e^{(n-1)s}-e^{-s}$.

From the well-known Baker-Hausdorff formula,

$$e^{\hat{A}}\hat{B}e^{-\hat{A}}=\hat{B}+[\hat{A},\hat{B}]+\frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]]+\cdots, \quad (35)$$

and by the symmetry of $e^{\mathbf{\Lambda}(s)}$, the multimode squeezing operator acts on the annihilation operator of the environmental modes as

$$\hat{S}(s)\hat{b}_k\hat{S}^{+}(s)=\sum_{j=1}^{n}[\Gamma_{kj}^{+}(-s)\hat{b}_j+\Gamma_{kj}^{-}(-s)\hat{b}_j^{+}], \quad (36)$$

where $\Gamma_{kj}^{\pm}(s)=\frac{1}{2}\left[(e^{\mathbf{\Lambda}(s)})_{kj}\pm(e^{-\mathbf{\Lambda}(s)})_{kj}\right]$ with $(e^{\mathbf{\Lambda}(s)})_{kj}$ being the k,j entry of matrix $e^{\mathbf{\Lambda}(s)}$.

After such a squeezing transformation, the annihilation operator for the jth environmental mode can be written as

$$\begin{aligned}\hat{c}_j=&\frac{1}{n}\{(n-1)\cosh(s)+\cosh[(n-1)s]\}\hat{b}_j\\ &+\frac{1}{n}\{(n-1)\sinh(s)-\sinh[(n-1)s]\}\hat{b}_j^{+}\\ &+\frac{1}{n}\{\cosh[(n-1)s]-\cosh(s)\}\sum_{k=1}^{n}{}'\hat{b}_k\\ &-\frac{1}{n}\{\sinh[(n-1)s]+\sinh(s)\}\sum_{k=1}^{n}{}'\hat{b}_k^{+},\end{aligned} \quad (37)$$

where the summation $\sum'$ is taken over all the indexes excluding $j=k$. In this way, for $n=2$, we recover the usual two-mode squeezing case [39].

Furthermore, the jth signal mode interferes with the resulting environmental mode at the jth beam splitter. We can obtain the operator of the jth signal mode leaving the beam splitter as

$$\begin{aligned}\hat{d}_j=\sqrt{\eta}\hat{a}_j-\frac{\sqrt{1-\eta}}{n}\Bigg\{&[(n-1)\cosh(s)+\cosh[(n-1)s]]\hat{b}_j\\ &+[(n-1)\sinh(s)-\sinh[(n-1)s]]\hat{b}_j^{+}\\ &+[\cosh[(n-1)s]-\cosh(s)]\sum_{l=1}^{n}{}'\hat{b}_l\\ &-[\sinh[(n-1)s]+\sinh(s)]\sum_{l=1}^{n}{}'\hat{b}_l^{+}\Bigg\},\end{aligned} \quad (38)$$

where the summation $\sum'$ is taken over all the indexes excluding $j=l$.

As is well known, the homodyne detection is one of the most important continuous-variable quantum measurement schemes and corresponds to the following phase-shifted quadrature amplitude:

$$\hat{X}(\theta)=\hat{X}\cos\theta+\hat{P}\sin\theta, \quad (39)$$

with θ being the phase of the local oscillator associated with the homodyne detection device such that $0\leqslant\theta\leqslant 2\pi$. For a signal field state described by the density operator ρ, the measured quadrature probability distribution is given by $P(x,\theta)=\langle\hat{X}(\theta)|\rho|\hat{X}(\theta)\rangle$. Therefore, we can obtain the geometrical morphology of such a distribution based on the cumulant theory, especially using the fourth-order cumulant technique.

At the channel output, we perform the homodyne detection measurement scheme for the n output signal modes introduced in Ref. [40], in which the total quadrature operators have the following form (see Appendix A for more information):

$$\hat{X}^{\text{out}}(\theta)=\sqrt{\eta}\hat{X}^{\text{in}}(\theta)+\sqrt{1-\eta}\hat{X}_{\text{env}}(\theta), \quad (40)$$

where

$$\hat{X}^{\text{in}}(\theta)=\sum_{j=1}^{n}\hat{X}_j(\theta)=\sum_{j=1}^{n}(\hat{X}_j\cos\theta+\hat{P}_j\sin\theta), \quad (41a)$$

$$\hat{X}_{\text{env}}(\theta) = -\sum_{j=1}^{n}\left[e^{-(n-1)s}\hat{X}_{\hat{b}_j}\cos\theta + e^{(n-1)s}\hat{P}_{\hat{b}_j}\sin\theta\right]. \tag{41b}$$

IV. NON-GAUSSIAN STATISTICS OF INPUT SIGNALS THROUGH LOSSY BOSONIC MEMORY CHANNEL

Generally, the signal modes and their environment are initially in an uncorrelated state:

$$\rho = \rho_s \otimes \rho_{\text{env}}. \tag{42}$$

According to Lemma 1 and Eq. (40), we can easily obtain the input-output relationship of the mth-order cumulant of quadrature operators of the n input signals via the lossy bosonic memory channel as

$$\kappa_m(\hat{X}^{\text{out}}(\theta)) = \eta^{\frac{m}{2}}\kappa_m(\hat{X}^{\text{in}}(\theta)) + (1-\eta)^{\frac{m}{2}}\kappa_m(\hat{X}_{\text{env}}(\theta)), \tag{43}$$

which shows the additivity properties of the cumulant for quantum random variables under such a memory quantum channel. In order to obtain complete information on the measured optical operators, we apply the phase-averaged processing technique in Ref. [41] and thus the phase-averaged cumulant can be written as

$$\begin{aligned}\kappa_m(\hat{X}^{\text{out}}) &\equiv \frac{1}{2\pi}\int_0^{2\pi}\kappa_m(\hat{X}^{\text{out}}(\theta))d\theta \\ &= \eta^{\frac{m}{2}}\kappa_m(\hat{X}^{\text{in}}) + (1-\eta)^{\frac{m}{2}}\kappa_m(\hat{X}_{\text{env}}).\end{aligned} \tag{44}$$

Lemma 6. (Multimode non-Gaussian qualifier). Consider an N-mode quantum system; each mode is described by a pair of conjugate dimensionless quadratures $\hat{X}_k$ and $\hat{P}_k$, $(k = 1,2,\ldots,N)$. Let $\hat{X}(\theta) = \sum_{k=1}^{N}\hat{X}_k(\theta) = \sum_{k=1}^{N}\left(\hat{X}_k\cos\theta + \hat{P}_k\sin\theta\right)$ be an N-mode rotated quadrature operator with the tunable phase parameter θ. Then the phase-averaged cumulants of such a quadrature operator can be viewed as a good statistics for quantifying the genuine quantum non-Gaussianity for any quantum state.

Proof. Quantum homodyne tomography is a powerful tool allowing us to reconstruct the density matrix ρ or equivalently the Wigner function. One can obtain the full statistics of the rotated quadrature observable in terms of the quadrature probability distribution, which contains complete information on the corresponding quantum state when varying the tunable phase angle θ at the interval $[0,2\pi]$. Let x_j and $|x_j(\theta)\rangle$ be the eigenvalue and eigenstate of the jth rotated quadrature operator $\hat{X}_j(\theta)$, i.e., $\hat{X}_j(\theta)|x_j(\theta)\rangle = x_j|x_j(\theta)\rangle$. The characteristic function of multimode quadrature distribution is a Fourier transform of the characteristic function described by the density operator ρ [18], that is,

$$\begin{aligned}P(\{x_j\},\theta) = \frac{1}{\pi^N}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\chi(\{\xi_j = i\eta_j e^{i\theta}\}) \\ \times e^{-i\sqrt{2}\boldsymbol{\eta}\cdot\boldsymbol{x}}d^N\{\eta_j\},\end{aligned} \tag{45}$$

where $\boldsymbol{\eta} = (\eta_1,\eta_2,\ldots,\eta_N)$, $\boldsymbol{x} = (x_1,x_2,\ldots,x_N)$, the characteristic function of the state ρ is given by $\chi(\{\xi_j\}) = \text{Tr}[\rho\Pi_{j=1}^{N}\exp(\xi_j\hat{a}_j^{+} - \xi_j^{*}\hat{a}_j)]$, and $d^N\{\eta_j\} = d\eta_1\cdots d\eta_N$. So the quadrature distribution contains complete information on the statistics of a given quantum state because the Fourier transform is fully invertible.

By using the cumulant expansion (5), the probability distribution of the phase-averaged quadrature operator can be formally expanded as

$$\bar{P}(\{x_j\}) = \frac{1}{2\pi}\int_0^{2\pi}P(\{x_j\},\theta)d\theta = \exp\left[\sum_{j=1}^{N}\frac{\xi^j}{j!}\bar{\kappa}_j(\hat{X})\right], \tag{46}$$

where $\bar{\kappa}_j(\hat{X})$ are the phase-averaged cumulants, some of which are given by

$$\bar{\kappa}_1(\hat{X}) = \left\langle\sum_{j=1}^{N}\hat{X}_j(\theta)\right\rangle_{pa}, \tag{47a}$$

$$\bar{\kappa}_2(\hat{X}) = \left\langle\left(\sum_{j=1}^{N}\hat{X}_j(\theta)\right)^2\right\rangle_{pa} - \bar{\kappa}_1^2(\hat{X}), \tag{47b}$$

$$\bar{\kappa}_3(\hat{X}) = \left\langle\left(\sum_{j=1}^{N}\hat{X}_j(\theta)\right)^3\right\rangle_{pa} - \bar{\kappa}_1^3(\hat{X}) - 3\bar{\kappa}_1(\hat{X})\bar{\kappa}_2(\hat{X}), \tag{47c}$$

where the subscript "pa" denotes the phase-average value. Hence the proposed phase-averaged cumulant of the multimode rotated quadrature observable is a good measure of the quantum non-Gaussianity of multimode continuous-variable quantum states. Just as in the above discussion, we can say safely that a given quantum state is Gaussian if all the higher-order phase-average cumulants of the multimode rotated quadrature operator vanish identically; otherwise, it is non-Gaussian. The cumulants below (beyond) the second are said to be Gaussian (non-Gaussian) cumulants. ■

In our scenario, when leaving out the initial state of the environment, the quantum channel is equivalent to a canonical linear transformation and thus can be described by a Gaussian map: $\hat{\mathcal{L}}(s,\eta) = \hat{S}(s)\hat{U}(\eta)$. According to the formula (43), we see that when the environment is in a Gaussian state, we have $\kappa_{>2}(\hat{X}_{\text{env}}) = 0$ for such a Gaussian quantum channel, then the cumulant of the output signal modes $\kappa_m(\hat{X}_j^{\text{out}}) = \eta^{\frac{m}{2}}\kappa_m(\hat{X}_j^{\text{in}})$, implying that whether a given quantum variable is Gaussian or not, its cumulant-based quantum non-Gaussianity remains invariant for the Gaussian quantum channel. But for a case of the environment to be in a non-Gaussian state, i.e., the non-Gaussian quantum channel, it drives all the input signals into non-Gaussian ones. Our result shows that the initial states of the environment play a key role in characterizing the quantum memory channel and then affecting the statistical properties of the output signals. Therefore, in the following we assume the bosonic memory channel to be initially in a multimode squeezed Fock state, i.e., $\hat{S}(s)|p\rangle^{\otimes n}$.

A. Fourth-order cumulant of environmental modes

To begin with, we compute the fourth-order cumulant of the environmental canonical operators in Eq. (41b). The environmental modes are assumed to be uncorrelated to each other and each environmental mode to have p photons. According to Lemmas 1 and 2, the fourth-order cumulant of the combined operators for the environment is a sum of that of its individual mode:

$$\kappa_4(\hat{X}_{\rm env}(\theta)) = \sum_{l=1}^{n} \kappa_4(\hat{X}_{l-{\rm env}}(\theta)). \tag{48}$$

On the other hand, the matrix elements of the quadrature operators in the Fock-state representation $\{|m\rangle,|p\rangle\}$ can be written as

$$\langle m|\hat{X}^q|p\rangle = \sum_{j=\max(0,\frac{m'}{2})}^{\min(m,p)} \frac{q!\sqrt{p!m!}}{\sqrt{2^q}\, j!(p-j)!(m-j)!v!!}, \tag{49a}$$

$$\langle m|\hat{P}^q|p\rangle = \sum_{j=\max(0,\frac{m'}{2})}^{\min(m,p)} \frac{i^{m-p}q!\sqrt{p!m!}}{\sqrt{2^q}\, j!(p-j)!(m-j)!v!!}, \tag{49b}$$

where $m' = \frac{m+p-q}{2}$, $v = 2j+q-m-p$, and $(2p)!! = 2^p p!$. It is easy to see that $\langle m|\hat{P}^q|p\rangle = i^{m-p}\langle m|\hat{X}^q|p\rangle$, so the matrix elements of their combination can be calculated as $\langle m|\hat{X}^\alpha\hat{P}^\beta|p\rangle = \sum_k \langle m|\hat{X}^\alpha|k\rangle\langle k|\hat{P}^\beta|p\rangle = \sum_k i^{(k-p)}\langle m|\hat{X}^\alpha|k\rangle\langle k|\hat{X}^\beta|p\rangle$. At the same time, we see from these relations that when q or $\alpha+\beta$ are odd, all the matrix elements of the canonical operators and their combinations vanish.

It follows from the formula (49) and Lemma 5 that the fourth-order cumulant of the environmental quadrature operator is given by

$$\kappa_4(\hat{X}_{\rm env}(\theta)) = -\tfrac{3}{2}np(p+1)w^4(\theta), \tag{50}$$

where

$$w^2(\theta) = e^{-2(n-1)s}\cos^2\theta + e^{2(n-1)s}\sin^2\theta. \tag{51}$$

Furthermore, we can obtain the phase-averaged fourth-order cumulant of the n-mode environmental field as

$$\kappa_4(\hat{X}_{\rm env}) = -\tfrac{3}{8}np(p+1)\{3\cosh[4(n-1)s]+1\}. \tag{52}$$

It is clear from Eq. (52) that the photon numbers of the environment and the degree of memory of the quantum squeezer can contribute to the quantum non-Gaussianity. As they increase, the non-Gaussian behavior of the quadrature observable is more pronounced. We show again that a quantum memory channel proposed here is a Gaussian quantum channel when its environment is in a vacuum state, i.e., $p = 0$.

B. Two-mode photon-number entangled states

Multiphoton entangled states are important for developing studies of photonic quantum networking and quantum computation. Of particular interest are multiphoton Greenberger-Horne-Zeilinger (GHZ) states since they are basic ingredients for testing quantum nonlocality against local hidden theories [42] and for constructing ballistic universal quantum computation [43] and quantum communication [44]. Various ingenious schemes for generating such states have been put forward experimentally in the last years. For example, experimental generations of three- [45], eight- [46], and 10-photon [47] GHZ states, respectively, have been already demonstrated by

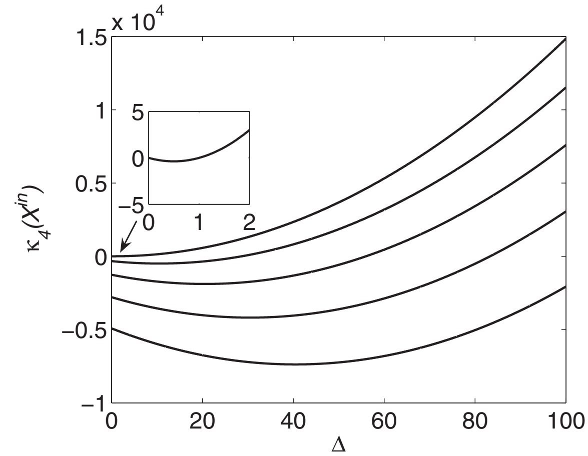

FIG. 2. Phase-averaged $\kappa_4(\hat{X}^{\rm in})$ of two-mode quadrature operators in $NNMM$ states as a function of Δ for different values of N. In each plot the curves from top to bottom are for $N = 0,10,20,30,40$. The inset shows $\kappa_4(\hat{X}^{\rm in})$ for Δ ranging between zero and two and for $N = 0$ in more detail.

using spontaneous down-version (SPDC) processes and linear optical circuits. Additionally, the stability [48], robustness [49], and scalability [50] properties of GHZ entanglement in decoherence have been recently discussed, showing that the fragility of these entanglement increases with the size of the system and the effective number of entangled subsystems decreases with time. Quantum Fisher information of GHZ states has also been investigated in three decoherence channels: the amplitude-damping channel (ADC), phase-damping channel (PDC), and depolarizing channel (DPC) [51]. It has been shown that the decay and sudden change of the quantum Fisher information in all three channels are observed. However, so far, few efforts have been devoted to the study of the cumulant-based quantum non-Gaussianity of GHZ states in the presence of quantum memory channel. Therefore, we consider this point and use a more general two-mode photon-number entangled state (PNES), originally introduced in Ref. [52],

$$|\Phi\rangle_{\rm in} = \frac{1}{\sqrt{2}}(|N\rangle^{\otimes 2} + |M\rangle^{\otimes 2}), \tag{53}$$

as the input signal for the quantum channel. Here $|M\rangle^{\otimes n}$ denotes the exact M photons in each mode. This state is called the $NNMM$ state for later convenience.

According to the formula (27) and Eq. (49), we can compute the phase-averaged fourth-order cumulant of the two-mode quadrature operator $\hat{X}^{\rm in}$ in Eq. (41a) (see Appendix B for more information) as

$$\kappa_4(\hat{X}^{\rm in}) = -\tfrac{3}{2}[2N^2 + 2N + (1+2N)\Delta - \Delta^2], \tag{54}$$

where $\Delta = |M-N|$. In Fig. 2 we plot the behavior of the fourth-order cumulant of the two-mode quadrature operators as a function of Δ for different values of N. One can observe that there exists a critical value that determines the behavior of the quantum non-Gaussianity of $\kappa_4(\hat{X}^{\rm in})$. This critical threshold can be calculated from $\frac{\partial}{\partial\Delta}\kappa_4(\hat{X}^{\rm in}) = 0$. Thus, for a given N, we have $\Delta = \frac{1+2N}{2}$. We can obtain the result that the negative values of $\kappa_4(\hat{X}^{\rm in})$ increase with the increasing of Δ when $\Delta \leqslant$

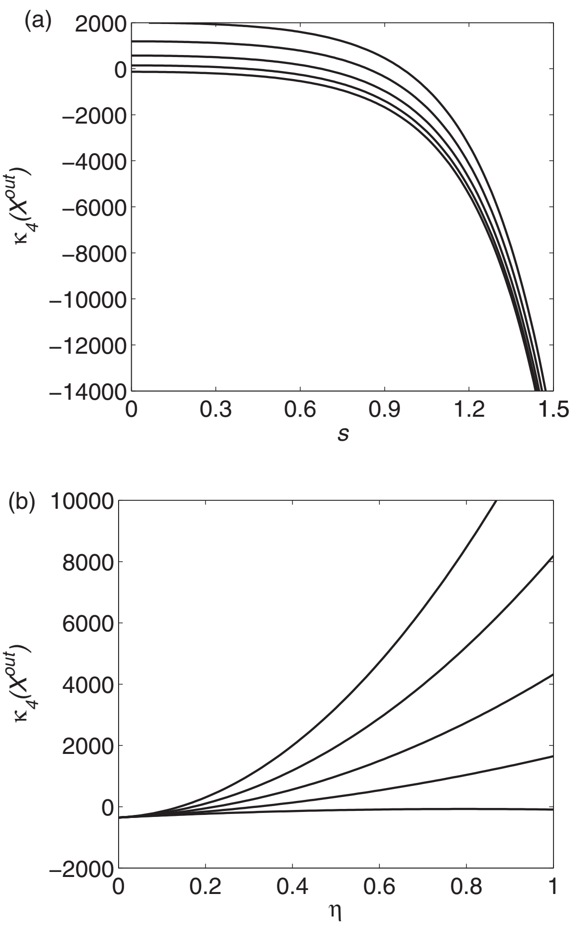

FIG. 3. (a) The quantity $\kappa_4(\hat{X}^{\text{out}})$ is plotted versus s with $\eta = 0.4$; (b) the quantity $\kappa_4(\hat{X}^{\text{out}})$ is plotted versus η with $s = 0.1$. The curves from bottom to top correspond to $\Delta = 0,40,60,80,100$, respectively. The values of other parameters are $p = 10$ and $N = 5$.

$\frac{1+2N}{2}$; otherwise its negative values decrease and even the sign flip of the fourth-order cumulants is possible. In any case, they indicate a stronger quantum non-Gaussianity.

By substituting Eqs. (52) and (54) into Eq. (44), we can explore the effect of the parameters (s,η) of the quantum channel on the behavior of the phase-averaged fourth-order cumulant of two-mode quadrature operators. In Fig. 3(a) we show the quantity $\kappa_4(\hat{X}^{\text{out}})$ as a function of the memory parameter s for various values of Δ and for $\eta = 0.4, p = 2$, and $n_{\text{tot}} = 100$, while in Fig. 3(b) the $\kappa_4(\hat{X}^{\text{out}})$ are shown as a function of the transmittivity η of the beam splitter for various values of Δ and for $s = 0.1, p = 2$, and $n_{\text{tot}} = 100$, where the total photon number of signal modes $n_{\text{tot}} = \text{Tr}[|\phi_{\text{in}}\rangle\langle\phi_{\text{in}}|(\hat{a}_1^+\hat{a}_1 + \hat{a}_2^+\hat{a}_2)] = 2N + \Delta$. From Fig. 3(a), it can be seen that the non-Gaussianities of collective quadrature operators of all the output states show a similar behavior. In the case of weak memory effects, the $\kappa_4(\hat{X}^{\text{out}})$ change gradually with the memory degree s. But for the increasing memory, the negative values of the $\kappa_4(\hat{X}^{\text{out}})$ are dramatically increased and tend to an infinite value, implying that the N-mode squeezer can result in a significant departure of the distribution of the two-mode quadrature operator from the Gaussian one and then exhibiting a strong non-Gaussian behavior. This phenomenon can be explained as follows: In the limit of weak memory effects, we have $\cosh[4(n-1)s] \sim 1 + 8(n+1)^2s^2$. Thus the quantum non-Gaussianity of the two-mode output quadrature variable slowly changes when $s \ll 1$. But in the case of strong memory effects, we have $\cosh[4(n-1)s] \sim e^{4(n+1)s}$, an exponential function of s, so that for higher levels of squeezing, the κ_{env} of the environment modes are the main component of the $\kappa_4(\hat{X}^{\text{out}})$ of output modes in Eq. (43) and moreover dominates the behavior of $\kappa_4(\hat{X}^{\text{out}})$. In addition, one may find that under the action of the quantum channel, the cumulant-based quantum non-Gaussianity of $NNMM$ states exhibits a quite different behavior with the transmissivity of the quantum channel. If originally, the quantum non-Gaussianity of input states is much lager than that of the environment modes, then the $\kappa_4(\hat{X}^{\text{out}})$ exhibit a monotonically increasing behavior with η, whereas for the case of $\kappa_4(\hat{X}^{\text{in}}) < \kappa_4(\hat{X}_{\text{env}})$, it has an opposite behavior. It is interesting to see that the freezing behavior of $\kappa_4(\hat{X}^{\text{out}})$ can appear for the case of $\kappa_4(\hat{X}^{\text{in}}) = \kappa_4(\hat{X}_{\text{env}})$.

C. Two-mode entangled coherent states

As a second non-Gaussian example we consider two-mode entangled coherent state (ECS),

$$|\Phi\rangle_{\text{in}} = \frac{1}{\mathcal{N}_\pm}(|\alpha\rangle^{\otimes 2} \pm |-\alpha\rangle^{\otimes 2}), \tag{55}$$

as the input signal, where $|\alpha\rangle$ is the coherent state with α being real for simplicity and the normalization factor $\mathcal{N}_\pm^2 = 2(1 \pm p^2)$ with $p = e^{-2|\alpha|^2}$. The sign $(\pm)$ of the superposition refers to the two-mode even entangled coherent states and odd entangled coherent state, respectively. Their nonclassical properties such as entanglement [53], nonclassical photon statistics, and squeezing [54] were discussed in the past decades. Of course, such states have played a key role in some potential applications such as quantum teleportation [55] and quantum cryptography [56]. It has also been found that the entanglement of such states are insensitive to decoherence due to photon absorption [6].

We can readily obtain the reduced density matrix of the singled-out jth field mode by tracing out the degrees of freedom of another mode:

$$\rho_{j\pm} = \frac{1}{\mathcal{N}_\pm^2}[|\alpha\rangle_{jj}\langle\alpha| + |-\alpha\rangle_{jj}\langle-\alpha| \pm e^{-2\alpha^2}(|\alpha\rangle_{jj}\langle-\alpha| + |-\alpha\rangle_{jj}\langle\alpha|)], j = 1,2. \tag{56}$$

Taken together with the expressions of quadrature operators in the coherent-state representation,

$$\langle\alpha|\hat{X}^l|\beta\rangle = \sum_{j=0}^{[\frac{l}{2}]} \frac{l!(2j-1)!!}{2^l(2j)!(l-2j)!}\left(\frac{\alpha^* + \beta}{\sqrt{2}}\right)^{l-2j} \times \exp\left(-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \alpha^*\beta\right), \tag{57a}$$

$$\langle\alpha|\hat{P}^l|\beta\rangle = \sum_{j=0}^{[\frac{l}{2}]} \frac{i^{l-2j}l!(2j-1)!!}{2^l(2j)!(l-2j)!}\left(\frac{\alpha^* - \beta}{\sqrt{2}}\right)^{l-2j} \times \exp\left(-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \alpha^*\beta\right), \tag{57b}$$

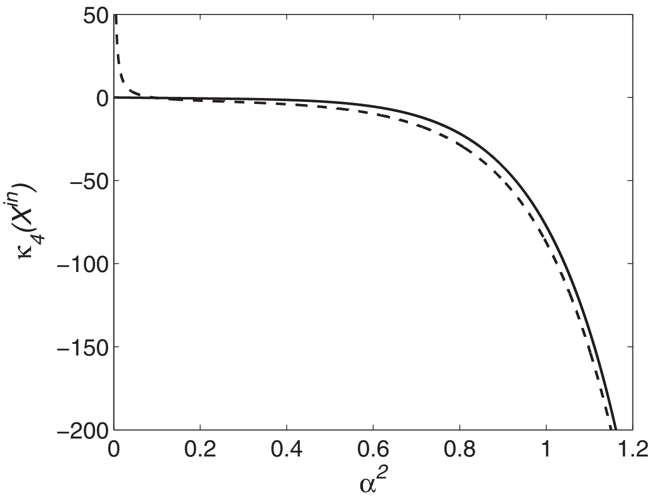

FIG. 4. The quantity $\kappa_4(\hat{X}^{\rm in})$ is plotted versus α^2. The dashed line stands for odd ECS and the solid line for even ECS.

and by analogy with the derivation of PNES case, we have the phase-averaged fourth-order cumulant of the two-mode input signals in the form,

$$\begin{aligned}
&\kappa_{4\pm}(\hat{X}^{\rm in})\\
&=\frac{3}{8\mathcal{N}_\pm^2}\big[(7\alpha^4-12\alpha^2+2)e^{2\alpha^2}+(\alpha^4-4\alpha^2+2)e^{\alpha^2}\\
&\quad\pm(7\alpha^4+12\alpha^2+2)e^{-2\alpha^2}\pm(\alpha^4+4\alpha^2+2)e^{-3\alpha^2}\big]\\
&\quad-\frac{3}{2\mathcal{N}_\pm^4}\{2\alpha^2(3\alpha^2-2)e^{3\alpha^2}+(3\alpha^4-4\alpha^2+2)e^{2\alpha^2}\\
&\quad\pm\alpha^2e^{-\alpha^2}\pm2(2-\alpha^4)e^{-2\alpha^2}+2\alpha^4[3\cosh(4\alpha^2)\\
&\quad\mp1]-\alpha^2e^{-5\alpha^2}+(3\alpha^4+4\alpha^2+2)e^{-6\alpha^2}\}. \qquad (58)
\end{aligned}$$

In deriving the above formula we have used the properties of the Hermite polynomials and presented in Appendix C the explicit expression of each term appearing in Eqs. (B2)–(B5) for the case of two-mode entangled coherent states (51). Figure 4 shows the phase-average fourth-order cumulants of the two-mode quadrature operator for even ECS and odd ECS as a function of α^2, respectively. It can be seen that their phase-average fourth-order cumulants have a slight difference for smaller values of α^2. In particular, in the limit of $\alpha\rightarrow0$ such a difference is more clear. That is due to the fact that a two-mode odd ECS $|\alpha,\alpha\rangle-|-\alpha,-\alpha\rangle$ becomes a single-photon entangled state $|0,1\rangle+|1,0\rangle$, showing the stronger quantum non-Gaussianity, while the even ECS $|\alpha,\alpha\rangle+|-\alpha,-\alpha\rangle$ becomes a vacuum product state $|0,0\rangle$ without the quantum non-Gaussian character. Additionally, it is interesting to note that for the large coherent amplitudes, the phase-average fourth-order cumulants of two-mode even entangled coherent states are in agreement with those of odd entangled coherent states. According to experimental reports in Ref. [57], the single-mode coherent state superposition of light with large amplitudes and high-fidelities, e.g., $\alpha=1.76$ and $F=0.59$, have been achieved with current technology, with which entangled coherent states can be realized using linear optical elements. Therefore, we restrict our attention to input even ECS with large amplitudes below.

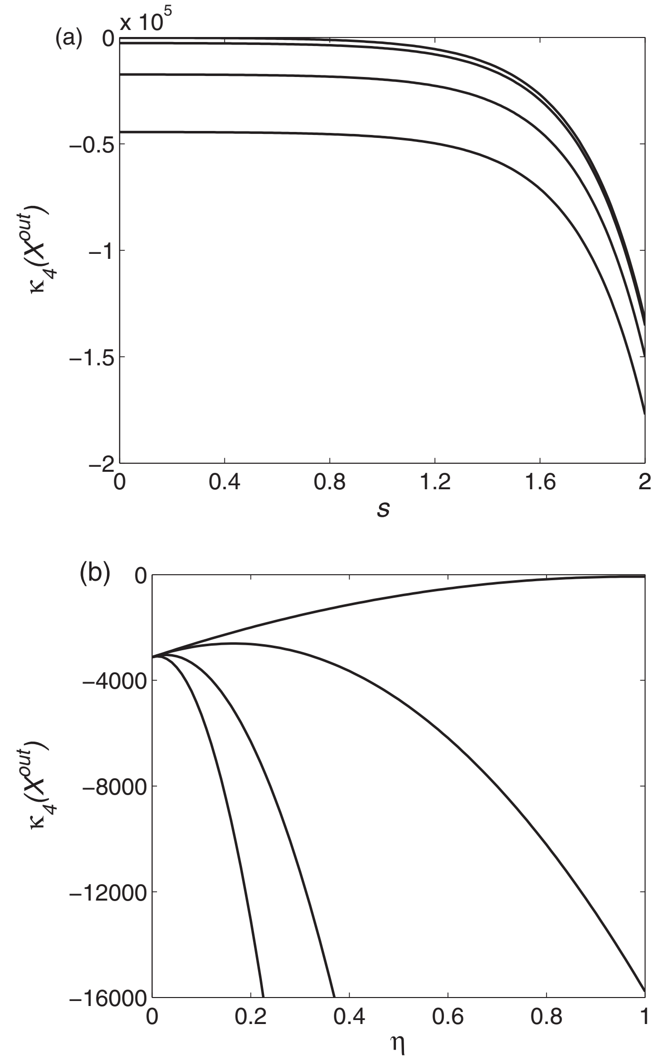

FIG. 5. (a) The quantity $\kappa_4(\hat{X}^{\rm out})$ is plotted versus s with $\eta=0.4$; (b) the quantity $\kappa_4(\hat{X}^{\rm out})$ is plotted versus η with $s=0.8$. In all subfigures, the quantum system is initially in the even ECS. The curves from top to bottom correspond to $\alpha^2=(1,2,2.4,2.6)$. The values of other parameters are $p=10$.

Figure 5 shows the dependence of the $\kappa_{4+}(\hat{X}^{\rm out})$ corresponding to two-mode even entangled coherent states on the parameters (s,η) of the quantum memory channel for different values of α^2 at the photon numbers of the environment $p=10$. As is seen from Fig. 5(a), in the regime of small initial squeezing s, the $\kappa_{4+}(\hat{X}^{\rm out})$ remain unchanged. As soon as the degree of memory increases, the negative values of the $\kappa_{4+}(\hat{X}^{\rm out})$ increase for all these states and exhibit the same behavior for strong memory effects. Increasing the coherent amplitudes gives rise to increasing negative value in $\kappa_{4+}(\hat{X}^{\rm out})$. On the other hand, we observe in Fig. 5(b) that different even entangled coherent states have different behaviors of $\kappa_{4+}(\hat{X}^{\rm out})$ when varying the loss η of the quantum channel. For even ECS with small amplitude, the negative values of $\kappa_{4+}(\hat{X}^{\rm out})$ decrease and tend to zero, implying that a vanishing quantum non-Gaussianity, while its negative values increase for the case of the large amplitudes. This behavior can be well understood from Eq. (44). In our channel protocol, the input signals are mixed with the environmental noises via beam splitters. Obviously, one sees that the input signals are gradually decoupled from their environments when increasing η. In the limit $\eta \to 1$, this interaction almost completely vanishes and then we have $\kappa_{4+}(\hat{X}^{\rm out}) = \kappa_{4+}(\hat{X}^{\rm in})$, as expected. While in the limit $\eta \to 0$, we only perform a measurement on the environment and as a result, we have $\kappa_{4+}(\hat{X}^{\rm out}) = \kappa_4(\hat{X}_{\rm env})$. In other cases, the amount of the environmental non-Gaussianity can be partially transferred into that of input signals via the beam splitters, that is, $\kappa_4(\hat{X}_{\rm env}) \to \kappa_{4+}(\hat{X}^{\rm in})$. At the same time, it is not difficult to see that a condition that reveals the relationship between the quantum non-Gaussianity behavior and channel dissipation is exactly the same as that of the PNES case.

We end this section with four remarks. First, the memory effects in our channel come from the long-range correlations (non-Markovian) among environments acting on different channel uses and lead to monotonically increasing the negativity of the phase-average fourth cumulant of the input signal modes over the quantum channel, without the oscillation phenomenon in other non-Markovian quantum channels [36]. Meanwhile, the proposed method is very general and can be applied to other contexts, e.g., Bose-Einstein condensate [58] and optomechanical systems [59]. Second, just as stated in Ref. [21], the fourth-order cumulant can be regarded as an indicator of testing non-Gaussian behavior for the probability distribution of the quadrature amplitudes. The larger the absolute values of the cumulants, the more significant the distribution departs from Gaussianity, meaning that it is much more useful for quantum information processing and quantum computation. Thus our analysis is important for correctly inferring on the non-Gaussian natures of some non-Gaussian states in propagating light fields by using the phase-averaged cumulant method. Third, the quantum homodyne tomographies are an important topic in the field of quantum optics and quantum information science, by which we can reconstruct the quantum state from the statistics of the quadrature components of the signal mode. Among these tomographies, single-mode optical homodyne tomography is now a well-established quantitative method and involves a balanced lossless beam splitter, two photodetectors, and a strong coherent local oscillator (LO). However, for a multimode homodyne tomography, such a detection technique, i.e., the requirement of the detection for each mode, complicates the experiment. So the multimode homodyne tomography with a single-mode LO [40,60] is highly desirable in the experiment. Finally, the theoretical description of the multimode squeezer has been proposed via nonlinear optical processes and four-wave mixing [61] and the higher-level multimode squeezing, e.g., −4.9 dB, has been experimentally achieved by using one type-II optical parametric amplifier [62]. Remarkably, recently the fourth-order cumulant of electrical noise has been experimentally implemented via measuring the current noise generated in a mesoscopic conductor by macroscopic quantum tunneling (MQT) in a current biased Josephson junction placed parallel to the conductor [63]. Therefore, we are looking forward so that our cumulant-based non-Gaussianity can be evaluated within current experimental technology.

V. SUMMARY AND CONCLUSIONS

In this paper, we have presented a measure of the non-Gaussianity of multimode quadrature operators based on the cumulant theory and exploited the proposed measure to investigate the behavior of the non-Gaussianity of two families of two-mode non-Gaussian entangled states traveling through a bosonic memory channel, which is initially in multimode squeezed number states. It has been shown that the presence of memory effects always increases the phase-averaged fourth-order cumulant of the two-mode quadrature operator, and thus gives rise to enhancing the cumulant-based non-Gaussianity. Additionally, we have shown that such a non-Gaussianity is very sensitive to the noise parameter of the quantum channel. We emphasize that our cumulant-based non-Gaussianity can be directly calculated from the sampling function of multimode quadrature homodyne tomography with one homodyne detector and thus more easily be measured in a different experimental setup using the current technology.

Finally, we should point out that by using the non-Gaussianity measure proposed in Ref. [22], one needs to calculate the whole higher-order joint cumulants of quadrature operators and their combinations in continuous variable quantum states. This is a tedious and time-consuming task. For instance, it contains $16N^4$ fourth-order cumulants for an N-mode quantum state. Nevertheless, in our approach one only deals with a single quadrature of quantum system, which is a linear superposition of single-mode quadrature operators of all involved modes. This mitigates the complexity involved in calculating non-Gaussianity and provides a potentially useful tool for studying non-Gaussianity of quantum states in other decoherence models.

ACKNOWLEDGMENTS

This work was supported by the National Natural Science Foundation of China, under Grants No. 11174100, No. 11474120, and No. 11374264; the Natural Science Foundation of Hunan Province, China under Grant No. 2017JJ2214, and the Key Project Foundation of Hunan Provincial Education Department, China under Grant No. 14A114.

APPENDIX A: DERIVATION OF EQ. (40)

Using Eq. (39) we can write the quadrature formulas of the jth outgoing signal mode at the output port of the jth beam splitter as

$$\hat{X}_j^{\rm out}(\theta) = \sqrt{\eta}\hat{X}_j^{\rm in} + \sqrt{1-\eta}\hat{X}_{\rm env}, \tag{A1}$$

where

$$\hat{X}_j^{\rm in} = \hat{X}_j^{\rm in}\cos\theta + \hat{P}_j^{\rm in}\sin\theta, \tag{A2}$$

$$\begin{aligned}\hat{X}_{\rm env} = &-\frac{1}{n}\big\{[(n-1)e^{s} + e^{-(n-1)s}]\hat{X}_{\hat{b}_j}\cos\theta \\ &+ [(n-1)e^{-s} + e^{(n-1)s}]\hat{P}_{\hat{b}_j}\sin\theta \\ &+ \sum_{l=1}^{n}{}'\big[(e^{-(n-1)s} - e^{s})\hat{X}_{\hat{b}_l}\cos\theta \\ &- (e^{(n-1)s} - e^{-s})\hat{P}_{\hat{b}_l}\sin\theta\big]\big\}.\end{aligned} \tag{A3}$$

Then, using multimode optical homodyne tomography with a single local oscillator [40], one measures a probability distribution of the quadrature $\hat{X}^{\rm out}$ of N signal modes, in which the collective quadrature $\hat{X}^{\rm out}$ is given by

$$\hat{X}^{\rm out}(\theta) = \sum_{j=1}^{N} \hat{X}_j^{\rm out}(\theta) = \frac{1}{\sqrt{2}} \sum_{j=1}^{N} [\hat{d}_j(\theta) + \hat{d}_j^{+}(\theta)], \quad \text{(A4)}$$

where $\hat{d}_j(\theta) = \hat{d}_j e^{-i\theta}$. Moreover, substituting Eq. (A1) into (A4) yields the result in Eq. (40).

APPENDIX B: EXPLICIT EXPRESSIONS OF FOURTH-ORDER CUMULANT FOR NNMM STATES

Let $\hat{G}_1 = \hat{X}_1 \cos\theta + \hat{P}_1 \sin\theta$ and $\hat{G}_2 = \hat{X}_2 \cos\theta + \hat{P}_2 \sin\theta$. Then, they are two commutative operators with zero means. Assume that the whole signal modes are in a nonseparable state (53). In this case, the fourth-order cumulant of sum operator $\hat{G} = \hat{G}_1 + \hat{G}_2$ is thus given, according to Lemma 5, by

$$\kappa_4(\hat{G}) = \kappa_4(\hat{G}_1) + \kappa_4(\hat{G}_2) + 4\kappa\big(\hat{G}_1^3\hat{G}_2\big) + 4\kappa\big(\hat{G}_1\hat{G}_2^3\big) + 6\kappa\big(\hat{G}_1^2\hat{G}_2^2\big), \quad \text{(B1)}$$

where

$$\kappa_4(\hat{G}_j) = \langle \hat{G}_j^4\rangle - 3\langle \hat{G}_j^2\rangle^2, \quad j = 1,2, \quad \text{(B2)}$$

$$\kappa\big(\hat{G}_1^3 G_2\big) = \langle \hat{G}_1^3\hat{G}_2\rangle - 3\langle \hat{G}_1^2\rangle\langle \hat{G}_1\hat{G}_2\rangle, \quad \text{(B3)}$$

$$\kappa\big(\hat{G}_1\hat{G}_2^3\big) = \langle \hat{G}_1\hat{G}_2^3\rangle - 3\langle \hat{G}_1\hat{G}_2\rangle\langle \hat{G}_2^2\rangle, \quad \text{(B4)}$$

$$\kappa\big(\hat{G}_1^2\hat{G}_2^2\big) = \langle \hat{G}_1^2\hat{G}_2^2\rangle - \langle \hat{G}_1^2\rangle\langle \hat{G}_2^2\rangle - 2\langle \hat{G}_1\hat{G}_2\rangle^2. \quad \text{(B5)}$$

The reduced density matrix of each input signal is given by

$$\rho_j = \tfrac{1}{2}(|N\rangle\langle N| + |M\rangle\langle M|), j = 1,2. \quad \text{(B6)}$$

By means of the formula (49), we can obtain the phase-average cumulant $\kappa_4(\hat{G}_j)$ as

$$\kappa_4(\hat{G}_j) = -\tfrac{3}{4}(N + M + 2NM), j = 1,2. \quad \text{(B7)}$$

Similarity, we can arrive at the phase-average fourth-order joint cumulants,

$$\kappa\big(\hat{G}_1^2\hat{G}_2^2\big) = \frac{1}{4}(N - M)^2, \quad \text{(B8)}$$

$$\kappa\big(\hat{G}_1^3\hat{G}_2\big) = \kappa\big(\hat{G}_1\hat{G}_2^3\big) = 0, \quad \text{(B9)}$$

where we have used the following relation:

$$\int_0^{2\pi} e^{i(n-m)\theta} d\theta = \begin{cases} 2\pi & \text{if } n = m, \\ 0 & \text{otherwise.} \end{cases} \quad \text{(B10)}$$

Combining Eqs. (B7)–(B9) we can obtain the average fourth-order cumulant of the two-mode input signals.

APPENDIX C: MATRIX ELEMENTS OF QUADRATURE OPERATORS FOR ENTANGLED COHERENT STATES

In this Appendix, we give the phase-average expectation values of quadrature operators in evaluating the fourth-order cumulant of the input signals, which are initially in an entangled coherent state (55). By using Eq. (57) and after some algebra, we have

$$\langle \hat{G}_j^4\rangle_{pa} = \frac{3}{16\mathcal{N}_\pm^2}\big[e^{\alpha^2} f_-(\alpha) \pm e^{-3\alpha^2} f_+(\alpha)\big], \quad \text{(C1)}$$

$$\langle \hat{G}_1\hat{G}_2\rangle_{pa}^2 = \frac{\alpha^4}{4\mathcal{N}_\pm^4}[3\cosh[4\alpha^2] \mp 1], \quad \text{(C2)}$$

$$\langle \hat{G}_1^3\hat{G}_2\rangle_{pa} = \frac{3\alpha^2}{16\mathcal{N}_\pm^2}\big[e^{2\alpha^2}(\alpha^2 - 2) \pm e^{-2\alpha^2}(\alpha^2 + 2)\big], \quad \text{(C3)}$$

$$\langle \hat{G}_1^2\hat{G}_2^2\rangle_{pa}^2 = \frac{1}{16\mathcal{N}_\pm^2}\big[e^{2\alpha^2} f_-(\alpha) \pm e^{-2\alpha^2} f_+(\alpha)\big], \quad \text{(C4)}$$

$$\langle \hat{G}_j^2\rangle_{pa}^2 = \frac{1}{8\mathcal{N}_\pm^4}\big[e^{2\alpha^2} f_-(\alpha) \pm 2e^{-2\alpha^2}(2 - \alpha^4) + e^{-6\alpha^2} f_+(\alpha)\big], \quad \text{(C5)}$$

$$\big(\langle \hat{G}_1^2\rangle\langle \hat{G}_1\hat{G}_2\rangle\big)_{pa} = \frac{\alpha^2}{8\mathcal{N}_\pm^4}\big[e^{3\alpha^2}(3\alpha^2 - 2) \pm e^{-\alpha^2} - e^{-5\alpha^2}(3\alpha^2 + 2)\big], \quad \text{(C6)}$$

where $f_\pm(\alpha) = \alpha^4 \pm 4\alpha^2 + 2$, $\hat{G}_j$ $(j = 1,2)$ have the same meaning as Appendix B, and the subscript "pa" denotes the phase-average value in the interval $[0,2\pi]$.

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