Einstein-Podolsky-Rosen steering and coherence in the family of entangled three-qubit states

J. K. Kalaga and W. Leoński*

Quantum Optics and Engineering Division, Faculty of Physics and Astronomy, University of Zielona Góra, Prof. Z. Szafrana 4a, 65-516 Zielona Góra, Poland

J. Peřina, Jr.

Joint Laboratory of Optics of Palacký University and Institute of Physics of the Czech Academy of Sciences, Faculty of Science, Palacký University, 17. listopadu 12, 771 46 Olomouc, Czech Republic

(Received 19 December 2017; published 13 April 2018)

Considering the system of three interacting qubits, we analyze four families of states from the point of view of bipartite correlations appearing in two-qubit subsystems of a three-qubit model, such as Einstein-Podolsky-Rosen steering, entanglement, and coherence. We reveal mutual relations among the steering parameter, concurrence, and three measures of coherence (degree of coherence, first-, and second-order correlation functions). Analyzing in parallel the steerable and unsteerable states, we derive analytical formulas giving the maximal and minimal values of coherence measures as concurrence varies.

DOI: 10.1103/PhysRevA.97.042110

I. INTRODUCTION

Einstein-Podolsky-Rosen (EPR) steering, or just quantum steering, plays an essential role in the behavior of quantum physics systems. The steering as one type of nonlocal behavior gives the ability of one system to steer the states of others via a local measurement. The concept of steering exploiting the EPR correlations was described first by Schrödinger [1] in 1935. Only many years later, the steering was understood sufficiently deeply to allow Ried to introduce the necessary criteria for demonstrating EPR steering [2]. The Ried criteria, that are based upon the uncertainty relation, were used by Ou *et al.* [3] to experimentally realize steering in the system of spatially separated and correlated light modes. Subsequently, other steering criteria were formulated. For instance, Walborn *et al.* [4,5] introduced the entropic steering inequality. Or, in 2014, Chowdhury *et al.* [6] showed that the Ried criteria are not successfully applicable to the states exhibiting higher-order correlations. For such states, we need to use the entropic steering inequality.

The EPR steering represents only one, though one of the most interesting, possible form of correlations observed in quantum systems. Various types of quantum correlations were extensively studied in various physical systems, including two neighboring atoms [7], multimode twin beams [8], two ionized electrons [9], and two strongly coupled bosonic modes [10], to name the few. On the other hand, internal coherence of physical systems was addressed for the first time by Zernike [11] in 1938 in the area of classical field propagation where he introduced the concept of the degree of coherence. Later in the 1950s, Hanbury Brown and Twiss investigated the higher-order coherence (intensity correlations) [12–14]. Finally in 1963, Glauber [15,16] and Sudarshan [17] and later Metha and Sudarshan [18] formulated the quantum coherence theory. Comprehensive presentation of the classical and quantum coherence theory can be found in [19] and [20,21], respectively. These well-established theories have recently been successfully applied in other areas of physics, including condensed-matter physics, field of quantum algorithms, etc. [22–24]. Only in the past years, mutual relation between the coherence and entanglement has come into attention and has been investigated in various quantum systems [25]. For example, atomic ensembles in high-Q cavities [26], optomechanical systems [27], and three coupled nonlinear oscillators [28] were addressed recently from the point of view of this relation.

EPR steerability as one type of nonlocality has been addressed, together with Bell-type nonlocality, by Wiseman *et al.* in 2007 [29,30]. They showed that the steerable states form a subset in the set of nonseparable states. On the other hand, the Bell nonlocal states form a subset in the set of steerable states. In other words, the steering means stronger correlations than those known as the entanglement, but weaker correlations than those found in the Bell nonlocal states. All these types of quantum correlations and their mutual relations were widely investigated for various systems, to understand these relations in detail. Small Bose-Hubbard chains [31], nondegenerate optical parametric oscillators [32], or a system containing an atomic ensemble inside a cavity comprising an oscillating mirror [33] serve just as typical examples.

Recently, the steering has been intensively studied not only in two-partite systems but also in more general, N-partite systems by He and Reid [34]. They derived inequalities which allow one to demonstrate a multipartite EPR steering for the Greenberger-Horne-Zeilinger and Gaussian continuous variable states. Special attention was paid to the significance of tripartite steering for secure quantum communications. In 2014, Teh and Reid proposed the criteria to quantify a genuine three-partite EPR paradox. The experimental observations of the multipartite EPR steering in the three intense optical beams system have been performed by Armstrong *et al.* [35].

*Corresponding author: W.Leonski@if.uz.zgora.pl

2469-9926/2018/97(4)/042110(12)

What is important is that the steering is such type of the nonlocality which is less sensitive to the noise and decoherence effects than the Bell nonlocality. In this sense, the steerable states are more useful to quantum communication protocols. Additionally, the confirmation of steering between subsystems certifies entanglement between them [29,30,36]. This is a very important feature in a quantum cryptography scenario, when not all parties are trusted [37]. In 2017, Máttar *et al.* [38] showed that W state could be implemented in asymmetric cryptographic protocols. They have shown that all kinds of entanglement of W state can be verified by each party in the tripartite steering scenario, even if other parties are not trusted. Recently, nonclassical correlations of a single object at different times were introduced and they are referred to as *temporal steering* [39–41]. It was proposed that such kind of correlations could be applied to test the security of quantum key distribution protocols [42].

The reason for these numerous investigations of quantum correlations lies in the fact that they play the crucial role in future development of quantum computation and quantum information processing. That is why revealing and understanding the relations among various types of quantum correlations represents a very important task with many implications. Whereas a lot of work has been done in characterizing these correlations in bipartite systems that can be treated relatively easily, quantum correlations in more complex systems consisting of more than two subsystems have not been understood well up to now and so they require future extensive studies. The purpose of this article is to contribute to these studies by addressing the relationship among EPR steering, entanglement, and coherence in several families of three-qubit states.

The quantum coherence plays a crucial role in quantum optics research [15,16]. As the EPR steering can be applied to secure quantum communication [43,44], the studies of the relations among the steering, entanglement, and quantum coherence are relevant in the context of their applications in the quantum cryptography and quantum communication protocols. Such correlations certify the entanglement between two subsystems when one of the measurements is untrusted [29]. The relationships discussed here can also be useful in the secret sharing procedure [45,46], when in the three-partite system we send the quantum encryption key separately to the two parties. We assume there that Alice, Bob, and Charlie participate in the distribution quantum key and that Alice's measurement device is trusted but that belonging to Bob is not. When Charlie confirms EPR steering between Alice and Bob, Charlie can deduce the level of security of the correlations between Alice and Bob [47]. It should be emphasized that quantum protocols based on the steering are less secure than those based on the Bell nonlocality. Obviously, they are more secure than the standard protocols. The implementations of the steering-based protocols are also playing a significant role. One-device-independent protocols are easier to implement than the device-independent scenarios [48]. Application of the steering and coherence at the same time seems to be also promising, due to the recent research. Mondal *et al.* [49] derived complementarity relations for the measures of coherence and then they obtained conditions that allow one to generate steerable states.

The paper is organized as follows. In Sec. II, we describe two families of states in a three-qubit system. In particular, we derive formulas giving the steering parameters for the single and double excited system. For both cases, we analyze the degree of coherence in Sec. III. In this section, we also study the relationship between steering and coherence in the case of a qubit-qubit subsystem and we find limitations to the strength of coherence both for steerable and unsteerable states. First-order and second-order correlation functions are analyzed in Secs. IV and V for the double excited system, respectively. Their relation to steerable and unsteerable mixed states is also revealed.

II. DIFFERENT FAMILIES OF THREE-QUBIT STATES

In our study, we concentrate on the steering properties of states belonging to a general three-qubit system. In particular, we are interested in a mutual relation between quantum steering and coherence characterizing two-qubit subsystems. The general the three-qubit system that is analyzed here can represent, e.g., three interacting two-level atoms [50] or three interacting two-state spin systems [51]. In our investigations we restrict our attention to the states whose evolution is closed within two basis states—the ground $|0\rangle$ and excited $|1\rangle$ states.

Four different groups of states can be identified in the analyzed three-qubit system. Two of them contain the states with one and two excitations. The remaining two cases are reserved for the limiting trivial situations: when we have no excitation in the system and when all three qubits are excited. For all four groups of states, we discuss the relation between the degree of coherence, expressed by the first- and second-order correlation functions, and the concurrence considering separately steerable and unsteerable mixed states of two "glued" qubits.

In this paper, we consider the bipartite steering effect appearing in two-qubit subsystems of the system containing three qubits. In such a three-qubit system the bipartite steering is shared among two subsystems—we are not dealing with genuine three-partite steering which is shared among all three qubits simultaneously. One should mention that, when the bipartite steering is generated, we do not observe genuine tripartite steering [34].

To distinguish steerable states from those which do not exhibit steering effect, we apply here the inequality introduced by Cavalcanti *et al.* [52]. For the cases when two subsystems are taken into account, it takes the following form:

$$|\langle \hat{a}_i \hat{a}_j^\dagger \rangle|^2 \leqslant \langle \hat{a}_i^\dagger \hat{a}_i (\hat{a}_j^\dagger \hat{a}_j + 1/2)\rangle, \quad (1)$$

where $\hat{a}^\dagger$ and $\hat{a}$ are the boson creation and annihilation operators, respectively, and the indices i and j label the qubits. As the steering appears in the system, the inequality (1) is violated. For such a case, the qubit j steers qubit i. In practice, we use the steering parameter S_{ij} which bases on the inequality (1)

$$S_{ij} = \langle \hat{a}_i \hat{a}_j^\dagger \rangle \langle \hat{a}_i^\dagger \hat{a}_j \rangle - \langle \hat{a}_i^\dagger \hat{a}_i (\hat{a}_j^\dagger \hat{a}_j + 1/2)\rangle. \quad (2)$$

When the parameter S_{ij} takes positive values, the qubit j steers that labeled by i. As the indices i and j exchange their position ($i \leftrightarrow j$), the parameter S_{ji} quantifies the steering in the opposite direction.

However, if we are dealing with the opposite situation, i.e., $S_{ij} \leqslant 0$, we cannot answer whether the steering effect appears or not. It should be emphasized that the steering parameter used here is only a witness of the steering, not its measure. Therefore, in the further parts of the paper, when we use the term "unsteerable states," it means that the discussed states are nonsteerable with respect to the criterion (1).

Labeling the qubits with 1, 2, and 3, the condition $\langle\hat{n}\rangle \equiv \langle\hat{n}_1\rangle + \langle\hat{n}_2\rangle + \langle\hat{n}_3\rangle = 1$ identifies the states with one excitation whose wave function can be written in the form

$$|\psi^{(I)}\rangle = C_{001}|001\rangle + C_{010}|010\rangle + C_{100}|100\rangle \tag{3}$$

using the complex probability amplitudes C_{ijk}. The corresponding density matrix is then expressed as

$$\rho^{(I)} = |\psi^{(I)}\rangle\langle\psi^{(I)}| = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & P_{001} & C^*_{001}C_{010} & 0 & C^*_{001}C_{100} & 0 & 0 & 0 \\ 0 & C^*_{010}C_{001} & P_{010} & 0 & C^*_{010}C_{100} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C^*_{100}C_{001} & C^*_{100}C_{010} & 0 & P_{100} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{4}$$

and the symbols $P_{ijk} \equiv |C_{ijk}|^2$ stand for the corresponding probabilities. Superscript (I) in the density matrix $\rho^{(I)}$ indicates the presence of one excitation in the system. Considering different pairs of qubits described by the partially reduced density matrix $\rho^{(I)}$ in Eq. (4), the appropriate steering parameters are expressed in terms of probabilities P_{ijk} as follows:

$$\begin{aligned} S^{(I)}_{12} &= P_{010}P_{100} - P_{100}/2, \\ S^{(I)}_{21} &= P_{010}P_{100} - P_{010}/2, \\ S^{(I)}_{13} &= P_{001}P_{100} - P_{100}/2, \\ S^{(I)}_{31} &= P_{001}P_{100} - P_{001}/2, \\ S^{(I)}_{23} &= P_{001}P_{010} - P_{010}/2, \\ S^{(I)}_{32} &= P_{001}P_{010} - P_{001}/2. \end{aligned} \tag{5}$$

According to the relations (5), qubit j steers qubit i, i.e., $S^{(I)}_{ij} > 0$, if the corresponding probability is greater than 1/2. For example, if $P_{010} > 1/2$, S_{12} is positive and thus qubit 2 steers qubit 1. Different configurations of the steering relations among three qubits were discussed in detail in [53–55].

Now, let us analyze steering in the next group involving the states with two excitations, i.e., $\langle\hat{n}\rangle = 2$. In this case, the system is described by the wave function $|\psi^{(II)}\rangle$,

$$|\psi^{(II)}\rangle = C_{011}|011\rangle + C_{101}|101\rangle + C_{110}|110\rangle, \tag{6}$$

or the corresponding density matrix $\rho^{(II)}$,

$$\rho^{(II)} = |\psi^{(II)}\rangle\langle\psi^{(II)}| = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & P_{011} & 0 & C^*_{011}C_{101} & C^*_{011}C_{110} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C^*_{101}C_{011} & 0 & P_{101} & C^*_{101}C_{110} & 0 \\ 0 & 0 & 0 & C^*_{110}C_{011} & 0 & C^*_{110}C_{101} & P_{110} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{7}$$

The steering parameters S_{ij} appropriate for different pairs of qubits i and j attain the form

$$\begin{aligned} S^{(II)}_{12} &= P_{011}P_{101} + 3P_{011}/2 + P_{101} - 3/2, \\ S^{(II)}_{21} &= P_{011}P_{101} + 3P_{101}/2 + P_{011} - 3/2, \\ S^{(II)}_{13} &= P_{011}P_{110} + 3P_{011}/2 + P_{110} - 3/2, \\ S^{(II)}_{31} &= P_{011}P_{110} + 3P_{110}/2 + P_{011} - 3/2, \\ S^{(II)}_{23} &= P_{101}P_{110} + 3P_{101}/2 + P_{110} - 3/2, \\ S^{(II)}_{32} &= P_{101}P_{110} + 3P_{110}/2 + P_{101} - 3/2. \end{aligned} \tag{8}$$

The conditions determining the occurrence of steering effects can be deduced from Eqs. (8), but they take more complicated forms compared to those derived for the system with single excitation. For example, the inequality $P_{011}P_{101} + 3P_{011}/2 + P_{101} > 3/2$ guarantees steering of qubit 1 by qubit 2.

In the last two cases, no steering occurs. If all three qubits are in their ground states, i.e., $|\psi^{(0)}\rangle = |000\rangle$, all six steering parameters are equal to zero. Similarly, when all three qubits are observed in their excited states, so that $|\psi^{(III)}\rangle = |111\rangle$, all steering parameters are negative. But even more importantly, both states $|000\rangle$ and $|111\rangle$ are the product states. Independent of the steering criteria, the reduced states of such three-qubit states clearly allow for a "local hidden state model" and they can be confirmed as being unsteerable.

III. DEGREE OF COHERENCE AND STEERABILITY

We need to quantify coherence in a bipartite system to understand its relation to quantum entanglement and steering effects. The simplest way for quantifying this coherence is based on the determination of the degrees D_i and D_j of first-order coherence in qubits i and j along the formula [21]

$$D_k = \sqrt{2\,\mathrm{Tr}\left(\rho_k^2\right) - 1}, \quad k = i,j, \tag{9}$$

that relies on the reduced density matrix ρ_k of qubit k and subsequent application of the definition of the degree D_{ij}^2 of coherence in the whole bipartite system [21,25]:

$$D_{ij}^2 = \left(D_i^2 + D_j^2\right)/2. \tag{10}$$

The parameter D_{ij}^2 defined in Eq. (10) takes on values from zero (no coherence) to unity (maximal, full coherence).

For a bipartite system, concurrence C_{ij} defined in [56,57] is a suitable measure of bipartite entanglement:

$$C_{ij} = \max(\sqrt{\lambda_I} - \sqrt{\lambda_{II}} - \sqrt{\lambda_{III}} - \sqrt{\lambda_{IV}}, 0). \tag{11}$$

Parameters λ_l appearing in Eq. (11) stand for the eigenvalues of matrix $R = \rho_{ij}\tilde{\rho_{ij}}$ where $\tilde{\rho_{ij}}$ is defined as $\tilde{\rho_{ij}} = \sigma_y \otimes \sigma_y \rho_{ij}^* \sigma_y \otimes \sigma_y$ and σ_y is the usual 2×2 Pauli matrix.

In our analysis, we first pay attention to the relation between entanglement and coherence for two distinct groups of states: steerable and unsteerable states with single excitation described by the density matrix $\rho_{ijk}^{(I)}$ given in Eq. (4). The diagram depicting mutual relations between the degree of coherence and the concurrence is plotted in Fig. 1: steerable states are found in the green area, whereas unsteerable states form the blue area. The diagrams in Fig. 1 were obtained by analyzing an ensemble of $\sim 10^6$ randomly generated three-qubit states containing single excitation.

To reveal the boundary conditions for the steerable states as plotted by solid and dashed-dotted curves in Fig. 1(a), we have to start with expressing the degree D_{ij}^2 of coherence in terms of probabilities P:

$$\begin{aligned} D_{12}^2 &= 1 + 2\left(P_{100}^2 - P_{100} + P_{010}^2 - P_{010}\right),\\ D_{13}^2 &= 1 + 2\left(P_{100}^2 - P_{100} + P_{001}^2 - P_{001}\right),\\ D_{23}^2 &= 1 + 2\left(P_{010}^2 - P_{010} + P_{001}^2 - P_{001}\right). \end{aligned} \tag{12}$$

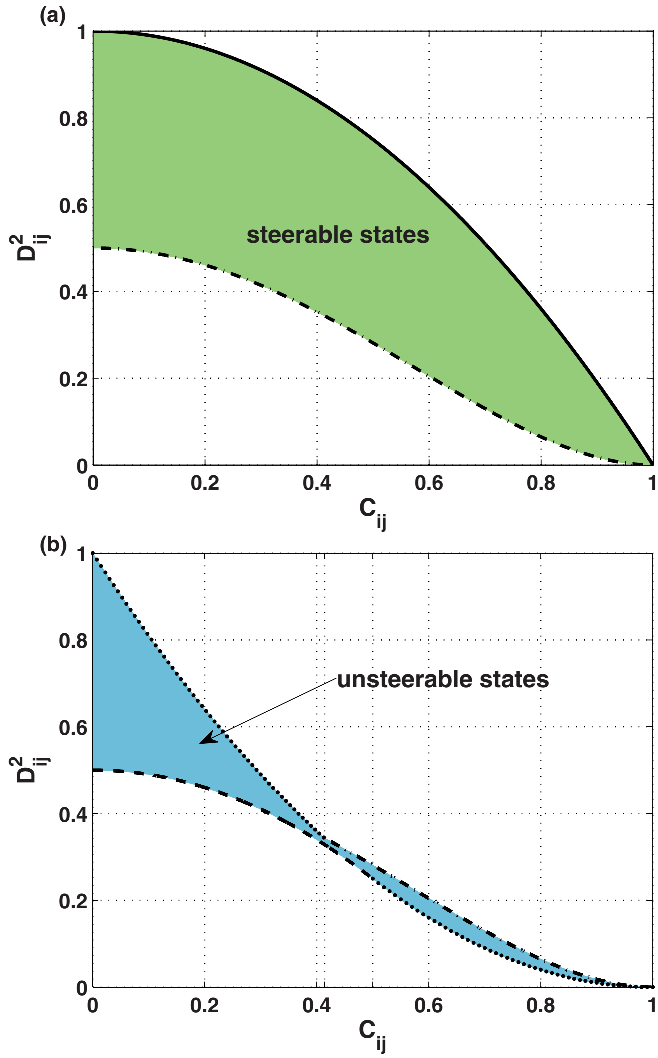

FIG. 1. Degree of coherence D_{ij}^2 versus concurrence C_{ij} for steerable (a) and unsteerable (b) two-qubit states derived from the density matrix $\rho_{ijk}^{(I)}$ given in Eq. (4). Steerable (unsteerable) states are found in green (blue) areas. Black border curves are drawn according to the corresponding analytical formulas.

Then, applying formula (11) we arrive at the expressions for concurrence for different pairs of qubits:

$$\begin{aligned} C_{12} &= \sqrt{4P_{100}P_{010}},\\ C_{13} &= \sqrt{4P_{100}P_{001}},\\ C_{23} &= \sqrt{4P_{010}P_{001}}. \end{aligned} \tag{13}$$

We first analyze the boundary formed by steerable states with the lowest possible coherence assuming the fixed concurrence C_{ij}. Numerical analysis reveals that the steering parameter S_{ij} equals zero for these states. Substitution of Eq. (12) for the degree of coherence and Eq. (13) for the concurrence into Eq. (5) for the steering parameter leaves us with the looked-for formula:

$$D_{ij}^2 = \left(C_{ij}^2 - 1\right)^2/2. \tag{14}$$

On the other hand, the boundary giving the maximal attainable coherence for the fixed concurrence C_{ij} of steerable states

TABLE I. Occurrence of steerable and unsteerable states in the plane (D_{ij}^2, C_{ij}) spanned by the degree of coherence and concurrence.

Concurrence	Degree of coherence	Generated states
$C_{ij} < \sqrt{2}-1$	$(C_{ij}-1)^2 < D_{ij}^2 \leqslant 1 - C_{ij}^2$	Only steerable states
	$(C_{ij}^2-1)^2/2 < D_{ij}^2 \leqslant (C_{ij}-1)^2$	Steerable or unsteerable states
	$1/2 - C_{ij}^2 \leqslant D_{ij}^2 \leqslant (C_{ij}^2-1)^2/2$	Only unsteerable states
$\sqrt{2}-1 \leqslant C_{ij} \leqslant 1/2$	$(C_{ij}-1)^2 < D_{ij}^2 \leqslant 1 - C_{ij}^2$	Only steerable states
	$1/2 - C_{ij}^2 \leqslant D_{ij}^2 \leqslant (C_{ij}^2-1)^2/2$	Only unsteerable states
$C_{ij} \geqslant 1/2$	$(C_{ij}-1)^2 < D_{ij}^2 \leqslant 1 - C_{ij}^2$	Only steerable states
	$(C_{ij}-1)^2 \leqslant D_{ij}^2 \leqslant (C_{ij}^2-1)^2/2$	Only unsteerable states

contains pure states, whose degree D_{ij}^2 of coherence takes the following form:

$$\begin{aligned} D_{12}^2 &= 2P_{100}^2 + 2P_{010}^2 - 1, \\ D_{13}^2 &= 2P_{100}^2 + 2P_{001}^2 - 1, \\ D_{23}^2 &= 2P_{010}^2 + 2P_{001}^2 - 1. \end{aligned} \tag{15}$$

Applying now Eq. (15) instead of Eq. (12) for the degree of coherence, we reveal, instead of Eq. (14), the following formula:

$$D_{ij}^2 = 1 - C_{ij}^2. \tag{16}$$

Combining Eqs. (14) and (16) together, we see that the degree C_{ij} of coherence of steerable states lies in the interval $D_{ij}^2 \in ([C_{ij}^2-1]^2/2, 1-C_{ij}^2)$. This means that, with the increasing value of concurrence C_{ij}, decrease of the degree D_{ij}^2 of coherence is expected on average. We even have for pure states $D_{ij}^2 + C_{ij}^2 = 1$. We note that this condition is analogous to that proposed by Svozilik *et al.* [25] for the parameter describing maximal violation of the Clauser-Horne-Shimony-Holt (CHSH) inequality and the coherence.

Now, we address the boundaries for unsteerable states drawn in Fig. 1(b). For $C_{ij} > \sqrt{2}-1$ the maximal attainable degree D_{ij}^2 of coherence of unsteerable states is given by formula $D_{ij}^2 = (C_{ij}^2 - 1)^2/2$ already derived for steerable states. On the other hand, simultaneous analysis of Eqs. (12) and (13) for the degree of coherence and concurrence, respectively, gives us the minimal achievable degree D_{ij}^2 of coherence in the region $C_{ij} > 1/2$:

$$D_{ij}^2 = (C_{ij}-1)^2. \tag{17}$$

The relation (17) corresponds to the family of mixed Werner states in a two-qubit system that are defined as mixtures of the Bell states and separable states. The corresponding density matrix is written as

$$\rho_W^{(I)} = \begin{bmatrix} 1-\alpha & 0 & 0 & 0 \\ 0 & \alpha/2 & \alpha/2 & 0 \\ 0 & \alpha/2 & \alpha/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{18}$$

and the introduced parameter α directly gives the concurrence:

$$C\left(\rho_W^{(I)}\right) = \alpha. \tag{19}$$

Detailed analysis reveals that Eq. (17) also determines the maximal degree D_{ij}^2 of coherence for $C_{ij} < \sqrt{2}-1$. The last boundary in Fig. 1(b) giving the minimal degree D_{ij}^2 of coherence for $C_{ij} \leqslant 1/2$ is parametrized as

$$D_{ij}^2 = 1/2 - C_{ij}^2, \tag{20}$$

and the underlying states have the following density matrix ρ_{ij}:

$$\rho_{ij} = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/2-\alpha & \sqrt{(1/2-\alpha)\alpha} & 0 \\ 0 & \sqrt{(1/2-\alpha)\alpha} & \alpha & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{21}$$

We note that for fixed qubits i and j fulfilling Eq. (21), the steering parameters S_{ik} and S_{jk} involving the remaining third qubit k are zero.

The analysis of attainable values of the degree D_{ij}^2 of coherence and concurrence C_{ij} as plotted in the diagrams of Fig. 1 allows us to identify different regions in the plane (D_{ij}^2, C_{ij}) from the point of view of steerability. Results of this analysis are summarized in Table I. According to this analysis, only steerable states occur in certain areas of the plane (D_{ij}^2, C_{ij}).

According to the diagram for steerable states shown in Fig. 1(a), the degree D_{ij}^2 of coherence "statistically" increases at the expense of concurrence C_{ij}. Moreover, steering is observed only for entangled states with sufficiently strong quantum correlations. As a consequence, increase of the degree D_{ij}^2 of coherence should be accompanied by the decrease of the steering parameter S_{ij}. This behavior has really been observed for steerable states being in the analyzed randomly generated ensemble of states, as documented in the diagram in Fig. 2 drawn in the plane (S_{ij}, D_{ij}^2). The maximal value of steering parameter S_{ij} for the fixed degree D_{ij}^2 of coherence is reached for pure states that give

$$S_{ij} = \left(D_{ij} - D_{ij}^4\right)/4. \tag{22}$$

The absolute maximal value of steering parameter S_{ij} is reached for $D_{ij}^2 = 1/4$ and it characterizes the states with the following density matrix:

$$\rho_{ij}^{S\max} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \alpha & \sqrt{\alpha(1-\alpha)} & 0 \\ 0 & \sqrt{\alpha(1-\alpha)} & 1-\alpha & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{23}$$

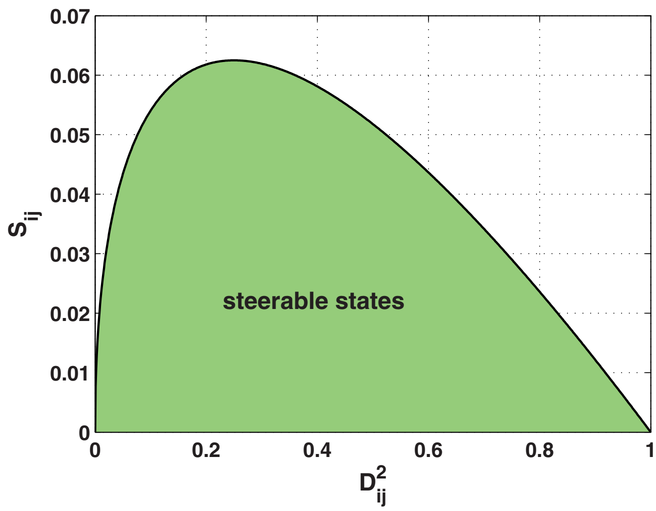

FIG. 2. Steering parameter S_{ij} versus the degree of coherence D^2_{ij} for steerable states.

and α equal to $1/4$ or $3/4$ (for more details, see the discussion in [53]). We note that the same relation between the degree D^2_{ij} of coherence and steering parameter S_{ij}, as plotted in Fig. 2, is obtained for the three-qubit states with double excitation.

Now, we move to the second group of states of the system of three interacting qubits that contain the double excitation. They are described by the density matrix $\rho^{(II)}$ written in Eq. (7) and some of them allow for steering. Similarly, as in the case of states with a single excitation, it is useful to analyze the relationship between the degree D^2_{ij} of coherence and concurrence C_{ij} independently for steerable and unsteerable states. Numerical analysis, performed for a large number of randomly generated states with doubled excitation, provided the corresponding diagrams shown in Fig. 3. Comparison of these diagrams reveals that steerable and unsteerable states occupy different areas in the plane (D^2_{ij}, C_{ij}), but these areas have a common border.

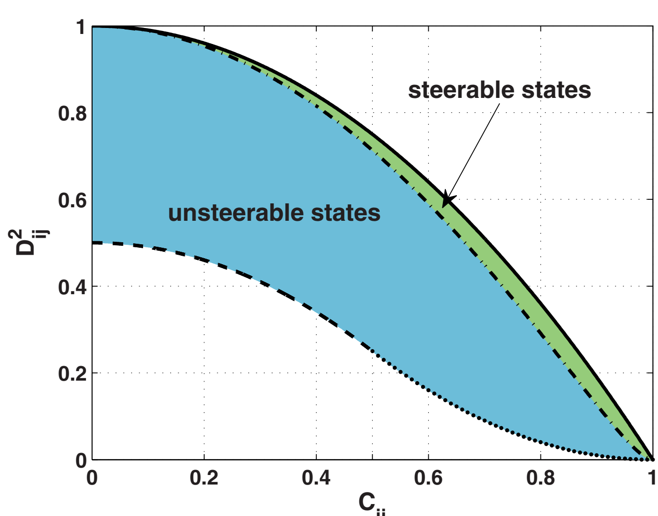

FIG. 3. Degree of coherence D^2_{ij} versus concurrence C_{ij} for steerable (green) and unsteerable (blue) two-qubit states described by the numerically calculated density matrix $\rho^{(II)}_{ij}$ and observed in three-qubit system with double excitation. Black border curves are drawn following the corresponding analytical formulas.

To determine the boundaries in the diagrams in Fig. 3, we first express the degree D^2_{ij} of coherence as a function of probabilities P:

$$\begin{aligned} D^2_{12} &= 1 + 2\big(P^2_{101} - P_{101} + P^2_{011} - P_{011}\big), \\ D^2_{13} &= 1 + 2\big(P^2_{011} - P_{011} + P^2_{110} - P_{110}\big), \\ D^2_{23} &= 1 + 2\big(P^2_{101} - P_{101} + P^2_{110} - P_{110}\big). \end{aligned} \tag{24}$$

Also the concurrence C_{ij} defined in Eq. (11) can be determined in terms of probabilities P:

$$\begin{aligned} C_{12} &= \sqrt{4P_{011}P_{101}}, \\ C_{13} &= \sqrt{4P_{011}P_{110}}, \\ C_{23} &= \sqrt{4P_{101}P_{110}}. \end{aligned} \tag{25}$$

The boundary between the steerable and unsteerable states is characterized by the condition $S_{ij} = 0$. Thus, combining Eq. (24) for the degree of coherence D^2_{ij} and Eq. (25) for the concurrence C_{ij}, we get the formula which defines the boundary:

$$\begin{aligned} D^2_{ij} = \frac{1}{144}\Big[&13C^4_{ij} - 18\Big(-14 + \sqrt{36 - 36C^2_{ij} + C^4_{ij}}\Big) \\ &+ C^2_{ij}\Big(-252 + 5\sqrt{36 - 36C^2_{ij} + C^4_{ij}}\Big)\Big]. \end{aligned} \tag{26}$$

For the region closed to the border defined by Eq. (26), steerable states with the minimal value of D^2_{ij} for the fixed concurrence C_{ij} occur. On the other hand, pure steerable states attain their maximal possible values of the degree D^2_{ij} of coherence when the concurrence C_{ij} is fixed. The following relations valid for pure states

$$\begin{aligned} D^2_{12} &= 2P^2_{011} + 2P^2_{101} - 1, \\ D^2_{13} &= 2P^2_{011} + 2P^2_{110} - 1, \\ D^2_{23} &= 2P^2_{101} + 2P^2_{110} - 1 \end{aligned} \tag{27}$$

allow us, together with the consideration of Eq. (25) for concurrence C_{ij}, to write down the formula for the maximal degree D^2_{ij} of coherence of steerable states:

$$D^2_{ij} = 1 - C^2_{ij}. \tag{28}$$

We note that the maximal attainable degree D^2_{ij} of coherence observed for the fixed concurrence C_{ij} is the same for the states with single and double excitation [compare Eqs. (28) and (16)]. Contrary to this fact, the minimal attainable values of the parameter D^2_{ij} describing the degree of coherence for the fixed concurrence C_{ij} are reached by unsteerable states, as apparent from the diagram in Fig. 3. The boundary in the diagram shown in Fig. 3, given by these values, is parametrized by two analytical curves. For $C_{ij} \geqslant 1/2$, at the boundary there occur the Werner states characterized by the two-qubit density matrices parametrized by α

$$\rho^{(II)}_W = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \alpha/2 & \alpha/2 & 0 \\ 0 & \alpha/2 & \alpha/2 & 0 \\ 0 & 0 & 0 & 1-\alpha \end{bmatrix} \tag{29}$$

and the corresponding concurrence C given as

$$C\left(\rho_W^{(II)}\right) = \alpha. \tag{30}$$

In this case, the formula determining the boundary becomes

$$D_{ij}^2 = (C_{ij} - 1)^2. \tag{31}$$

For the remaining values of concurrence C_{ij} fulfilling $C_{ij} < 1/2$, the density matrix ρ_{ij},

$$\rho_{ij} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2-\alpha & \sqrt{(1/2-\alpha)\alpha} & 0 \\ 0 & \sqrt{(1/2-\alpha)\alpha} & \alpha & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}, \tag{32}$$

describes the states at the boundary. The following relation between the degree D_{ij}^2 of coherence and concurrence C_{ij} is derived for the density matrix ρ_{ij} in Eq. (32):

$$D_{ij}^2 = 1/2 - C_{ij}^2. \tag{33}$$

Comparing in parallel Eqs. (17) and (20) and Eqs. (31) and (33) that give the minimal achievable value of the degree of coherence D_{ij}^2 for the fixed concurrence C_{ij} for unsteerable states for the single and double excitation cases, we observe equivalence (similar to the case of the maximal attainable value of the degree of coherence D_{ij}^2 reached for the fixed concurrence C_{ij} by the steerable states). Finally, we note that the "fluctuation" of D_{ij}^2 for the fixed concurrence C_{ij} is quite low for steerable states, as evidenced in the diagram in Fig. 3.

IV. FIRST-ORDER CROSS-CORRELATION FUNCTION AND STEERING

Coherence in a system is an important attribute of not only the constituting subsystems but also of the relationship among the subsystems. In this case, the coherence theory introduces cross-correlation functions of different orders to quantify coherence. The first- and second-order cross-correlation functions that quantify the coherence between fields' amplitudes and intensities, respectively, are the most important from the practical point of view. In this section, we study in detail the relation between steering and the first-order cross-correlation function $g_{ij}^{(1)}$ defined for subsystems i and j (mutual coherence) [58,59] as

$$g_{ij}^{(1)} = \frac{|\langle \hat{a}_i^\dagger \hat{a}_j\rangle|}{\sqrt{\langle \hat{a}_i^\dagger \hat{a}_i\rangle\langle \hat{a}_j^\dagger \hat{a}_j\rangle}}. \tag{34}$$

The maximal reachable value of function $g_{ij}^{(1)}$ equals 1 and it corresponds to the maximally coherent field. When no coherence between subsystems i and j exists $g_{ij}^{(1)} = 0$.

The application of definition (34) to the above-considered states with single excitation gives us $g_{ij}^{(1)} = 1$, i.e., these states are maximally coherent independently whether they are steerable or unsteerable. On the other hand, the values of function $g_{ij}^{(1)}$ reached for the states with double excitation range from 0 to 1 and so the relationship between first-order coherence and steering needs a detailed analysis.

Numerical investigation of the ensemble of randomly generated states with double excitation as described by the density matrix $\rho^{(II)}$ given in Eq. (7) reveals that the steerable and unsteerable states occupy neighboring areas in the diagram in the plane spanned by the first-order correlation function $g_{ij}^{(1)}$ and concurrence C_{ij}, as shown in Fig. 4. We can see in Fig. 4 that steerable states, contrary to the unsteerable ones, are endowed with high values of the first-order correlation function $g_{ij}^{(1)}$ more or less independently on the value of concurrence C_{ij}. Also, greater values of concurrence C_{ij} naturally imply greater values of the first-order correlation function $g_{ij}^{(1)}$, independently on steerability of the analyzed state.

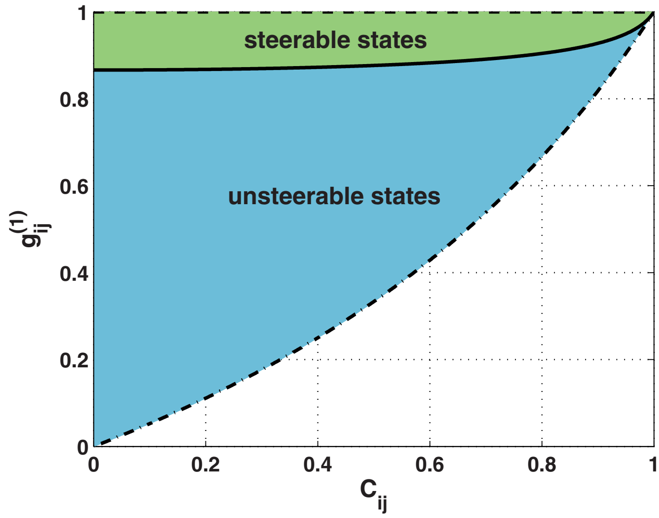

FIG. 4. First-order correlation function $g_{ij}^{(1)}$ versus concurrence C_{ij} for steerable (green area) and unsteerable states (blue area). Black border curves are drawn according to the corresponding analytical formulas.

To reveal the formula for the boundary between steerable and unsteerable states in Fig. 4 we need the expression for the first-order correlation function $g_{ij}^{(1)}$ in terms of the parameters of the states with double excitation:

$$\begin{aligned} g_{12}^{(1)} &= \frac{C_{011}^* C_{101}}{\sqrt{(P_{101}+P_{110})(P_{011}+P_{110})}}, \\ g_{13}^{(1)} &= \frac{C_{011}^* C_{110}}{\sqrt{(P_{110}+P_{101})(P_{011}+P_{101})}}, \\ g_{23}^{(1)} &= \frac{C_{101}^* C_{110}}{\sqrt{(P_{110}+P_{011})(P_{101}+P_{011})}}. \end{aligned} \tag{35}$$

Considering real probability amplitudes $C_{ijk}^* = C_{ijk}$ of the analyzed states and using relations for first-order correlation function (35), concurrence (25), and steering parameter (8) equal to zero, we arrive at the formula giving the maximal achievable value of the first-order correlation function $g_{ij}^{(1)}$ for the fixed concurrence C_{ij} of unsteerable states:

$$g_{ij}^{(1)} = \frac{1}{2}\sqrt{\frac{6 - 11C_{ij}^2 + \sqrt{36 - 36C_{ij}^2 + C_{ij}^4}}{4 - 5C_{ij}^2}}. \tag{36}$$

For any value of concurrence C_{ij}, there exist steerable states with the maximal first-order correlation function $g_{ij}^{(1)}$, i.e., $g_{ij}^{(1)} = 1$. It can be shown that such states are the pure ones.

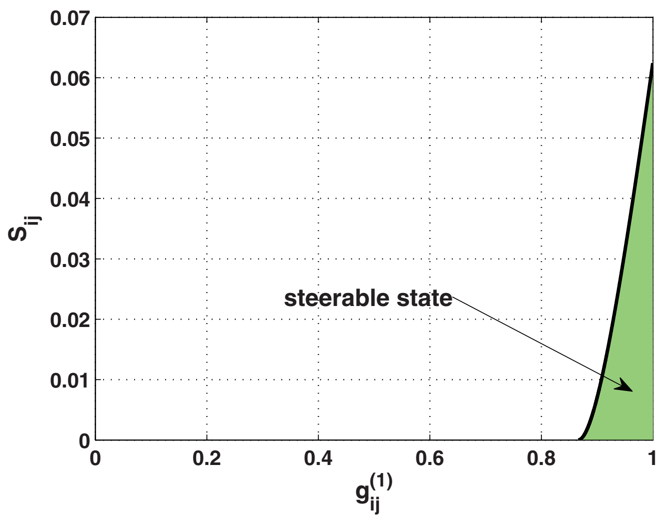

FIG. 5. Steering parameter S_{ij} versus the first-order correlation function $g_{ij}^{(1)}$ for steerable states occupying the green area. Black border curves are plotted following the corresponding analytical formulas.

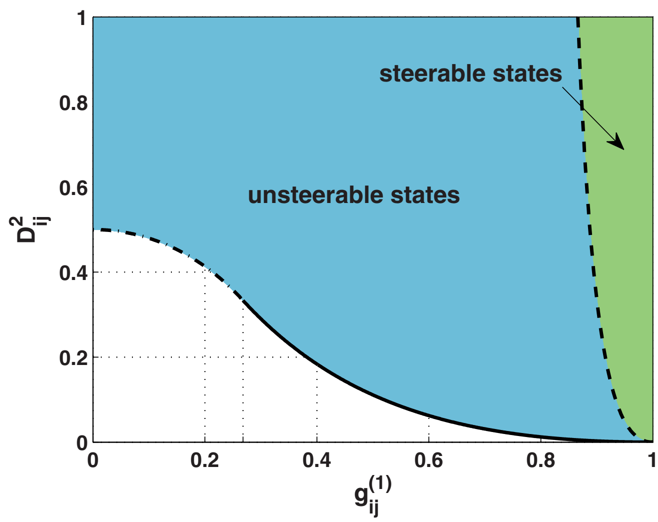

FIG. 6. Degree of coherence D_{ij}^2 versus the first-order correlation function $g_{ij}^{(1)}$ for steerable (green area) and unsteerable (blue area) states. Black border curves are plotted using the corresponding analytical formulas.

On the other hand, the states with the minimal first-order correlation function $g_{ij}^{(1)}$ for the fixed concurrence C_{ij} are the Werner states described by the density matrix $\rho_W^{(II)}$ in Eq. (29) and having their concurrence in Eq. (30). Using Eqs. (8) and (35) we reveal the lower boundary for the first-order correlation function $g_{ij}^{(1)}$ in the diagram of Fig. 4:

$$g_{ij}^{(1)} = \frac{C_{ij}}{\sqrt{4 - 4C_{ij} + C_{ij}^2}}. \tag{37}$$

According to the diagram in Fig. 4, steerable states are endowed with great first-order correlation functions $g_{ij}^{(1)}$ and so there is a question of which values of the steering parameter S_{ij} are compatible with the given value of first-order correlation function $g_{ij}^{(1)}$. The diagram in Fig. 5 answers this question showing that, for the fixed value of the first-order correlation function $g_{ij}^{(1)}$, the steering parameter S_{ij} extends from zero to certain maximal value. The greater the first-order correlation function $g_{ij}^{(1)}$, the greater the achievable maximal value of steering parameter S_{ij}.

This maximal steering parameter S_{ij} is determined along the formula

$$S_{ij}^{\max} = \frac{1}{2\big(g_{ij}^{(1)^2} - 1\big)^2}\Big(-3 + 3g_{ij}^{(1)^2} + 4g_{ij}^{(1)^4} - 2\sqrt{6g_{ij}^{(1)^2} - 22g_{ij}^{(1)^4} + 20g_{ij}^{(1)^6}}\Big) \tag{38}$$

assuming $g_{ij}^{(1)} \in (\frac{\sqrt{3}}{2},1)$. For lower values of the first-order correlation function $g_{ij}^{(1)}$, no steerable states exist. As our analysis revealed, the steerable states with the maximal steering parameter S_{ij} are characterized by a density matrix ρ_{ij},

$$\rho_{ij} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \alpha & \sqrt{\alpha\beta} & 0 \\ 0 & \sqrt{\alpha\beta} & \beta & 0 \\ 0 & 0 & 0 & 1-\alpha-\beta \end{bmatrix}, \tag{39}$$

and one of the probabilities α and β has to be equal to

$$\alpha,\beta = \frac{1}{2\big(1 - 3g_{ij}^{(1)^2} + 2g_{ij}^{(1)^4}\big)}\Big(- 2g_{ij}^{(1)^2} + 4g_{ij}^{(1)^2} - \sqrt{6g_{ij}^{(1)^2} - 22g_{ij}^{(1)^4} + 20g_{ij}^{(1)^6}}\Big) \tag{40}$$

for $g_{ij}^{(1)} > \sqrt{3}/2$. As we already mentioned, the states with the greatest steerable parameter S_{ij} for $g_{ij}^{(1)} = 1$ are pure with the density matrix written in Eq. (23) and α equal to 1/4 or 3/4.

Up to now, we discussed the relationship between coherence on one side and concurrence and steering on the other side by considering independently two manifestations of coherence, internal coherence of subsystems as quantified by the degree D_{ij}^2 of coherence and mutual coherence as quantified by the first-order correlation function $g_{ij}^{(1)}$. Numerical analysis of states with double excitation and different degrees of internal as well as mutual coherence as contained in the diagram in Fig. 6 shows that great values of the first-order correlation function $g_{ij}^{(1)}$ are necessary to have a steerable state. On the other hand, the greater the degree D_{ij}^2 of coherence the better the change to have a steerable state. However, even large degrees D_{ij}^2 of coherence close to 1 do not guarantee steerability of a state as mutual coherence is critical for steering. At the boundary in the plane $(D_{ij}^2, g_{ij}^{(1)})$ between steerable and unsteerable states, unsteerable states with $S_{ij} = 0$ occur and their analysis leaves us with the following formula:

$$D_{ij}^2 = \frac{\left(-1+g_{ij}^{(1)^2}\right)^2\left(45-156g_{ij}^{(1)^2}+136g_{ij}^{(1)^4}\right)}{2\left(3-11g_{ij}^{(1)^2}+10g_{ij}^{(1)^4}\right)^2}. \tag{41}$$

According to Eq. (41), steerable states require the first-order correlation function $g_{ij}^{(1)} > \sqrt{3}/2$. It is worth noting that the analyzed states with double excitation exhibit at least a certain level of coherence, as evidenced in the diagram in Fig. 6. This minimal level of coherence is observed for unsteerable states with minimal achievable values of the steering parameter S_{ij} for which we can derive the following expressions for the minimal degree D_{ij}^2 of coherence:

$$D_{ij}^2 = \frac{1-6g_{ij}^{(1)^2}+g_{ij}^{(1)^4}}{2\left(g_{ij}^{(1)^2}-1\right)^2} \quad \text{for } g_{ij}^{(1)} \leqslant 2-\sqrt{3}, \tag{42}$$

$$D_{ij}^2 = \frac{\left(g_{ij}^{(1)}-1\right)^2}{\left(g_{ij}^{(1)}+1\right)^2} \quad \text{for } g_{ij}^{(1)} \geqslant 2-\sqrt{3}. \tag{43}$$

The unsteerable states giving the boundaries in Eqs. (42) and (43) are described by the density matrix ρ_{ij} in Eq. (39) with one of the probabilities α and β equal to

$$\begin{aligned}\alpha,\beta &= \frac{1}{2\left(2-4g_{ij}^{(1)^2}+2g_{ij}^{(1)^4}\right)}\Big(1-4g_{ij}^{(1)^2}+3g_{ij}^{(1)^4}\\ &\quad \pm\left(g_{ij}^{(1)^2}-1\right)\sqrt{1-14g_{ij}^{(1)^2}+g_{ij}^{(1)^4}}\Big)\\ &\quad \text{for } g_{ij}^{(1)} \leqslant 2-\sqrt{3},\\ \alpha,\beta &= \frac{g_{ij}^{(1)}}{1+g_{ij}^{(1)}} \quad \text{for } g_{ij}^{(1)} \geqslant 2-\sqrt{3}.\end{aligned} \tag{44}$$

V. SECOND-ORDER CORRELATION FUNCTION AND STEERING

Similarly as for the first-order coherence, we elucidate the relationship between the second-order coherence and steering. As we demonstrate below the role of second-order coherence in observing steering is somehow complementary to that of the first-order coherence. Whereas a high level of first-order coherence is needed for steering, only the states with low level of second-order coherence can lead to the steering effects. Second-order coherence is quantified by the second-order correlation function $g_{ij}^{(2)}$ defined as [58,59]

$$g_{ij}^{(2)} = \frac{\langle \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_i \hat{a}_j\rangle}{\langle \hat{a}_i^\dagger \hat{a}_i\rangle\langle \hat{a}_j^\dagger \hat{a}_j\rangle}. \tag{45}$$

As the states with single excitation cannot possess second-order coherence ($g_{ij}^{(2)}=0$) we further analyze only the states with double excitation.

Similarly as in the case of first-order correlation function $g_{ij}^{(1)}$, steerable and unsteerable states contained in the randomly generated ensemble of states separate into distinct, but neighboring, areas in the plane $(g_{ij}^{(2)}, C_{ij})$ spanned by the second-order correlation function and the concurrence (see the diagram in Fig. 7). Contrary to the situation depicted in Fig. 5 plotted in the plane $(g_{ij}^{(1)}, C_{ij})$ the steerable states occupy the area with low values of the second-order correlation function $g_{ij}^{(2)}$. To find the boundary values of the second-order correlation function $g_{ij}^{(2)}$ (as visible in the diagram of Fig. 7), we first express the function $g_{ij}^{(2)}$ in terms of probabilities P_{ijk}, $i,j,k=0,1$ for steerable states:

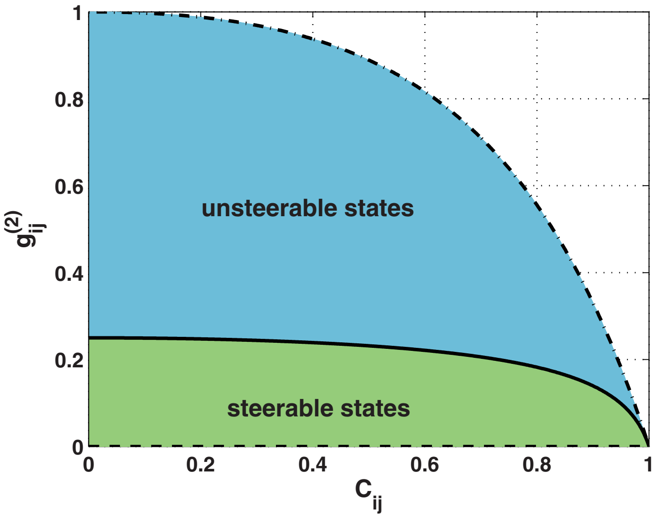

FIG. 7. Second-order correlation function $g_{ij}^{(2)}$ versus concurrence C_{ij} for steerable (green area) and unsteerable (blue area) states. Black border curves are drawn according to the corresponding analytical formulas.

$$\begin{aligned} g_{12}^{(2)} &= \frac{P_{110}}{(P_{101}+P_{110})(P_{011}+P_{110})},\\ g_{13}^{(2)} &= \frac{P_{011}}{(P_{110}+P_{101})(P_{011}+P_{101})},\\ g_{23}^{(2)} &= \frac{P_{101}}{(P_{110}+P_{011})(P_{101}+P_{011})}.\end{aligned} \tag{46}$$

Considering pure states in Eqs. (46) whose probabilities P_{110}, P_{011}, and P_{101}, that appear in the numerators of the fractions in relations (46), are equal to zero, we recognize the states that minimize the second-order correlation function $g_{ij}^{(2)}$ for the fixed concurrence C_{ij}. On the other side, the maximal achievable value of the function $g_{ij}^{(2)}$ for steerable states fixing concurrence C_{ij} arises from the analysis of unsteerable states that have the steering parameter $S_{ij}=0$ at this boundary. This fact, together with relations (46) and (8), gives us the appropriate formula for the boundary:

$$g_{ij}^{(2)} = \frac{1}{2} - \frac{2}{2+C_{ij}^2+\sqrt{36-36C_{ij}^2+C_{ij}^4}}. \tag{47}$$

Unsteerable states with the fixed concurrence C_{ij} are characterized by the second-order correlation functions $g_{ij}^{(2)}$ greater or equal to that given by Eq. (47). It can be shown that the maximal second-order correlation function $g_{ij}^{(2)}$ is reached for the Werner states with the density matrix $\rho_W^{(II)}$ given in Eq. (29) and concurrence C_{ij} determined in Eq. (30). Thus we have the

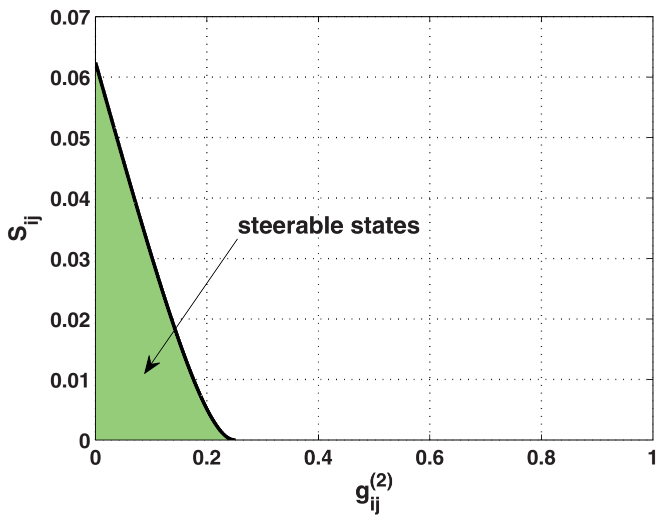

FIG. 8. Steering parameter S_{ij} versus second-order correlation function $g_{ij}^{(2)}$ for steerable set found in the green area. Black border curves are drawn following the corresponding analytical formulas.

following relation for such states, that also gives the upper boundary for unsteerable states in the diagram in Fig. 7:

$$g_{ij}^{(2)} = \frac{4(1-C_{ij})}{(C_{ij}-2)^2}. \tag{48}$$

As the steering is observed only for low values of the second-order correlation function $g_{ij}^{(2)}$, it is interesting to understand what is the restriction implied by a given low value of the second-order correlation function $g_{ij}^{(2)}$ with respect to steering. As the diagram plotted in Fig. 8 and showing steerable states in the plane $(S_{ij}, g_{ij}^{(2)})$ documents, the increasing second-order correlation function $g_{ij}^{(2)}$ dramatically limits the maximal attainable values of the steering parameter S_{ij}. The smaller the second-order correlation function $g_{ij}^{(2)}$ is, the greater the values of the steering parameter S_{ij} can be reached. The maximal achievable value of the steering parameter S_{ij} considered as a function of the second-order correlation function $g_{ij}^{(2)}$ is determined along the formula

$$S_{ij}^{\max} = \frac{1}{2g_{ij}^{(2)^2}}\Big[4 + g_{ij}^{(2)}\big(4g_{ij}^{(2)} - 11\big) - 2\sqrt{4 - 22g_{ij}^{(2)} + 38g_{ij}^{(2)^2} - 20g_{ij}^{(2)^3}}\Big] \tag{49}$$

that is valid for the states described by the density matrix ρ_{ij} written in Eq. (39) and requiring one out of the probabilities α and β taking the following value:

$$\alpha,\beta = \frac{1}{4g_{ij}^{(2)^2} - 2g_{ij}^{(2)}}\Big(2 - 6g_{ij}^{(2)} + 4g_{ij}^{(2)^2} - \sqrt{4 - 22g_{ij}^{(2)} + 38g_{ij}^{(2)^2} - 20g_{ij}^{(2)^3}}\Big). \tag{50}$$

In accord with the formula in Eq. (50), steerable states exist only for $g_{ij}^{(2)} \leqslant 1/4$. As for the limiting case of $g_{ij}^{(2)} = 0$ in the diagram in Fig. 8, the analysis gives us the pure states with the density matrix ρ_{ij} given in Eq. (23) and parameter α equal to $1/4$ or $3/4$ as the states with the maximal steering parameter S_{ij}. We note that these two states also have $g_{ij}^{(1)} = 1$ and give the maximal attainable steering parameter in the diagram in Fig. 5.

VI. SUMMARY

We have discussed EPR steering occurring in a three-qubit system in relation to entanglement and coherence. In our analysis we have considered the states of three qubits with single and double excitations. We have paid attention to different two-qubit states obtained by the state reduction and quantified their entanglement by concurrence. Generating a large ensemble of entangled states, we have analyzed coherence of each state by determining the degree of coherence and first- and second-order correlation function and compare these quantities with the steering parameter. Based on this, we divided all entangled states into the steerable and unsteerable ones. We have shown that the steerable states with double excitation can be uniquely distinguished from the unsteerable ones by being endowed with a high degree of first-order mutual coherence, a low degree of second-order mutual coherence, and having the sum of squared degree of coherence and squared concurrence close to one. The states with maximal steering parameter occur for the degree of coherence around $1/2$. On the other hand, all two-qubit states derived from three-qubit states with single excitation exhibit maximal first-order mutual coherence and no second-order mutual coherence. Moreover, though the steerable states have systematically greater values of the sum of squared degree of coherence and squared concurrence than unsteerable states, the degree of coherence does not allow one to uniquely distinguish both kinds of states. To quantify the relations among entanglement, coherence, and steering, we have derived many boundary (extremal) conditions for steerable and unsteerable states and identify the states found at these boundaries. Doing this we have elucidated the relationship between coherence and steering for the considered entangled states of three qubits.

ACKNOWLEDGMENTS

J.P. was supported by GA ČR (Project No. 18-08874S) and MŠMT ČR (Project No. LO1305).

[1] E. Schrödinger, Discussion of probability relations between separated systems, Math. Proc. Cambridge Philos. Soc. **31**, 555 (1935).

[2] M. D. Reid, Demonstration of the Einstein-Podolsky-Rosen paradox using nondegenerate parametric amplification, Phys. Rev. A **40**, 913 (1989).

[3] Z. Y. Ou, S. F. Pereira, H. J. Kimble, and K. C. Peng, Realization of the Einstein-Podolsky-Rosen Paradox for Continuous Variables, Phys. Rev. Lett. **68**, 3663 (1992).

[4] S. P. Walborn, A. Salles, R. M. Gomes, F. Toscano, and P. H. Souto Ribeiro, Revealing Hidden Einstein-Podolsky-Rosen Nonlocality, Phys. Rev. Lett. **106**, 130402 (2011).

[5] J. Schneeloch, C. J. Broadbent, S. P. Walborn, E. G. Cavalcanti, and J. C. Howell, Einstein-Podolsky-Rosen steering inequalities from entropic uncertainty relations, Phys. Rev. A **87**, 062103 (2013).

[6] P. Chowdhury, T. Pramanik, A. S. Majumdar, and G. S. Agarwal, Einstein-Podolsky-Rosen steering using quantum correlations in non-Gaussian entangled states, Phys. Rev. A **89**, 012104 (2014).

[7] V. Peřinová, A. Lukš, J. Křepelka, and J. Peřina, Quantum correlation and entanglement between an ionizing system and a neighboring atom interacting directly and via a quantized field, Phys. Rev. A **90**, 033428 (2014).

[8] I. I. Arkhipov, J. Peřina, Jr., J. Peřina, and A. Miranowicz, Comparative study of nonclassicality, entanglement, and dimensionality of multimode noisy twin beams, Phys. Rev. A **91**, 033837 (2015).

[9] J. Peřina, Jr., A. Lukš, and W. Leoński, Long-time joint spectra and entanglement of two photoelectrons originating in interacting auto-ionization systems, J. Phys. B: At. Mol. Opt. Phys. **48**, 115007 (2015).

[10] S. Bougouffa and Z. Ficek, Evidence of indistinguishability and entanglement determined by the energy-time uncertainty principle in a system of two strongly coupled bosonic modes, Phys. Rev. A **93**, 063848 (2016).

[11] F. Zernike, The concept of degree of coherence and its application to optical problems, Physica **5**, 785 (1938).

[12] R. H. Brown and R. Q. Twiss, Correlation between photons in two coherent beams of light, Nature (London) **177**, 27 (1956).

[13] R. H. Brown and R. Q. Twiss, A test of a new type of stellar interferometer on Sirius, Nature (London) **178**, 1046 (1956).

[14] R. H. Brown and R. Q. Twiss, Interferometry of the intensity fluctuations in light. II. An experimental test of the theory for partially coherent light, Proc. R. Soc. London A **243**, 291 (1958).

[15] R. J. Glauber, The quantum theory of optical coherence, Phys. Rev. **130**, 2529 (1963).

[16] R. J. Glauber, Coherent and incoherent states of the radiation field, Phys. Rev. **131**, 2766 (1963).

[17] E. C. G. Sudarshan, Equivalence of Semiclassical and Quantum Mechanical Descriptions of Statistical Light Beams, Phys. Rev. Lett. **10**, 277 (1963).

[18] C. L. Mehta and E. C. G. Sudarshan, Relation between quantum and semiclassical description of optical coherence, Phys. Rev. **138**, B274 (1965).

[19] J. Peřina, *Coherence of Light* (Kluwer, Dordrecht, 1985).

[20] J. Peřina, *Quantum Statistics of Linear and Nonlinear Optical Phenomena* (Kluwer, Dordrecht, 1991).

[21] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, UK, 1995).

[22] H.-L. Shi, S.-Y. Liu, X.-H. Wang, W.-L. Yang, Z.-Y. Yang, and H. Fan, Coherence depletion in the Grover quantum search algorithm, Phys. Rev. A **95**, 032307 (2017).

[23] J. Peřina, Jr., Spatial, spectral, and temporal coherence of ultraintense twin beams, Phys. Rev. A **93**, 013852 (2016).

[24] C. Antón, G. Tosi, M. D. Martín, Z. Hatzopoulos, G. Konstantinidis, P. S. Eldridge, P. G. Savvidis, C. Tejedor, and L. Viña, Quantum coherence in momentum space of light-matter condensates, Phys. Rev. B **90**, 081407 (2014).

[25] J. Svozilík, A. Vallés, J. Peřina, and J. P. Torres, Revealing Hidden Coherence in Partially Coherent Light, Phys. Rev. Lett. **115**, 220501 (2015).

[26] L.-H. Sun, G.-X. Li, W.-J. Gu, and Z. Ficek, Generating coherence and entanglement with a finite-size atomic ensemble in a ring cavity, New J. Phys. **13**, 093019 (2011).

[27] L.-H. Sun, G.-X. Li, and Z. Ficek, First-order coherence versus entanglement in a nanomechanical cavity, Phys. Rev. A **85**, 022327 (2012).

[28] J. K. Kalaga, A. Kowalewska-Kudłaszyk, W. Leoński, and A. Barasiński, Quantum correlations and entanglement in a model comprised of a short chain of nonlinear oscillators, Phys. Rev. A **94**, 032304 (2016).

[29] H. M. Wiseman, S. J. Jones, and A. C. Doherty, Steering, Entanglement, Nonlocality, and the Einstein-Podolsky-Rosen Paradox, Phys. Rev. Lett. **98**, 140402 (2007).

[30] S. J. Jones, H. M. Wiseman, and A. C. Doherty, Entanglement, Einstein-Podolsky-Rosen correlations, Bell nonlocality, and steering, Phys. Rev. A **76**, 052116 (2007).

[31] M. K. Olsen, Spreading of entanglement and steering along small Bose-Hubbard chains, Phys. Rev. A **92**, 033627 (2015).

[32] K. Dechoum, P. D. Drummond, S. Chaturvedi, and M. D. Reid, Critical fluctuations and entanglement in the nondegenerate parametric oscillator, Phys. Rev. A **70**, 053807 (2004).

[33] Q. He and Z. Ficek, Einstein-Podolsky-Rosen paradox and quantum steering in a three-mode optomechanical system, Phys. Rev. A **89**, 022332 (2014).

[34] Q. Y. He and M. D. Reid, Genuine Multipartite Einstein-Podolsky-Rosen Steering, Phys. Rev. Lett. **111**, 250403 (2013).

[35] S. Armstrong, M. Wang, R. Y. Teh, Q. Gong, Q. He, J. Janousek, H.-A. Bachor, M. D. Reid, and P. K. Lam, Multipartite Einstein-Podolsky-Rosen steering and genuine tripartite entanglement with optical networks, Nat. Phys. **11**, 167 (2015).

[36] E. G. Cavalcanti, S. J. Jones, H. M. Wiseman, and M. D. Reid, Experimental criteria for steering and the Einstein-Podolsky-Rosen paradox, Phys. Rev. A **80**, 032112 (2009).

[37] D. Cavalcanti, P. Skrzypczyk, G. H. Aguilar, R. V. Nery, P. H. Souto, and S. P. Walborn, Detection of entanglement in asymmetric quantum networks and multipartite quantum steering, Nat. Commun. **6**, 7941 (2015).

[38] A. Máttar, P. Skrzypczyk, G. H. Aguilar, R. V. Nery, P. H. Souto, S. P. Walborn, and D. Cavalcanti, Experimental multipartite entanglement and randomness certification of the W state in the quantum steering scenario, Quantum Sci. Technol. **2**, 015011 (2017).

[39] K. Bartkiewicz, A. Černoch, K. Lemr, A. Miranowicz, and F. Nori, Experimental temporal quantum steering, Sci. Rep. **6**, 38076 (2016).

[40] S.-L. Chen, N. Lambert, C.-M. Li, A. Miranowicz, Y.-N. Chen, and F. Nori, Quantifying Non-Markovianity with Temporal Steering, Phys. Rev. Lett. **116**, 020503 (2016).

[41] S.-L. Chen, N. Lambert, C.-M. Li, G.-Y. Chen, Y.-N. Chen, A. Miranowicz, and F. Nori, Spatio-temporal steering for testing

nonclassical correlations in quantum networks, Sci. Rep. **7**, 3728 (2017).

[42] K. Bartkiewicz, A. Černoch, K. Lemr, A. Miranowicz, and F. Nori, Temporal steering and security of quantum key distribution with mutually unbiased bases against individual attacks, Phys. Rev. A **93**, 062345 (2016).

[43] Q. He, L. Rosales-Zárate, G. Adesso, and M. D. Reid, Secure Continuous Variable Teleportation and Einstein-Podolsky-Rosen Steering, Phys. Rev. Lett. **115**, 180502 (2015).

[44] S. Kocsis, M. J. W. Hall, A. J. Bennet, D. J. Saunders, and G. J. Pryde, Experimental measurement-device-independent verification of quantum steering, Nat. Commun. **6**, 5886 (2015).

[45] M. Hillery, V. Bužek, and A. Berthiaume, Quantum secret sharing, Phys. Rev. A **59**, 1829 (1999).

[46] Y.-A. Chen, A.-N. Zhang, Z. Zhao, X.-Q. Zhou, C.-Y. Lu, C.-Z. Peng, T. Yang, and J.-W. Pan, Experimental Quantum Secret Sharing and Third-Man Quantum Cryptography, Phys. Rev. Lett. **95**, 200502 (2005).

[47] B. Opanchuk, L. Arnaud, and M. D. Reid, Detecting faked continuous-variable entanglement using one-sided device-independent entanglement witnesses, Phys. Rev. A **89**, 062101 (2014).

[48] A. Acín, N. Brunner, N. Gisin, S. Massar, S. Pironio, and V. Scarani, Device-Independent Security of Quantum Cryptography Against Collective Attacks, Phys. Rev. Lett. **98**, 230501 (2007).

[49] D. Mondal, T. Pramanik, and A. K. Pati, Nonlocal advantage of quantum coherence, Phys. Rev. A **95**, 010301 (2017).

[50] Z. Ficek and R. Tanaś, Entangled states and collective non-classical effects in two-atom systems, Phys. Rep. **372**, 369 (2002).

[51] J. Fei, D. Zhou, Y.-P. Shim, S. Oh, X. Hu, and M. Friesen, Mediated gates between spin qubits, Phys. Rev. A **86**, 062328 (2012).

[52] E. G. Cavalcanti, Q. Y. He, M. D. Reid, and H. M. Wiseman, Unified criteria for multipartite quantum nonlocality, Phys. Rev. A **84**, 032115 (2011).

[53] J. K. Kalaga and W. Leoński, Quantum steering borders in three-qubit systems, Quantum Inf. Process. **16**, 175 (2017).

[54] J. K. Kalaga, W. Leoński, and R. Szczęśniak, Quantum steering and entanglement in three-mode triangle Bose–Hubbard system, Quantum Inf. Process. **16**, 265 (2017).

[55] J. K. Kalaga, W. Leoński, and R. Szczęśniak, Quantum steering in an asymmetric chain of nonlinear oscillators, Photon. Lett. Poland **9**, 97 (2017).

[56] S. Hill and W. K. Wootters, Entanglement of a Pair of Quantum Bits, Phys. Rev. Lett. **78**, 5022 (1997).

[57] W. K. Wootters, Entanglement of Formation of an Arbitrary State of Two Qubits, Phys. Rev. Lett. **80**, 2245 (1998).

[58] C. C. Gerry and P. L. Knight, *Introductory Quantum Optics* (Cambridge University Press, Cambridge, UK, 2005).

[59] D. F. Walls and G. J. Milburn, *Quantum Optics*, 2nd ed. (Springer-Verlag, Berlin Heidelberg, 2008).