Editors' Suggestion

# Second sound in a two-dimensional Bose gas: From the weakly to the strongly interacting regime

Miki Ota* and Sandro Stringari
*INO-CNR BEC Center and Dipartimento di Fisica, Università di Trento, 38123 Povo, Italy*

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Using Landau's theory of two-fluid hydrodynamics, we investigate first and second sounds propagating in a two-dimensional (2D) Bose gas. We study the temperature and interaction dependence of both sound modes and show that their behavior exhibits a deep qualitative change as the gas evolves from the weakly interacting to the strongly interacting regime. Special emphasis is placed on the jump of both sounds at the Berezinskii-Kosterlitz-Thouless transition, caused by the discontinuity of the superfluid density. We find that the excitation of second sound through a density perturbation becomes weaker and weaker as the interaction strength increases as a consequence of the decrease in the thermal expansion coefficient. Our results could be relevant for future experiments on the propagation of sound on the Bose-Einstein condensate (BEC) side of the BCS-BEC crossover of a 2D superfluid Fermi gas.

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## I. INTRODUCTION

Superfluidity is one of the most remarkable manifestations of quantum physics at the macroscopic level occurring in diverse systems, from cold atomic gases [1–4] to neutron stars [5]. Below the critical temperature $T_c$ at which the phase transition occurs, the system exhibits two-fluid behavior [6,7], characterized by a mixture of a normal component, behaving as a viscous fluid, and a superfluid component, moving without friction. In these systems, the superfluid density plays a key role in the understanding of related phenomena, such as the frictionless flow of superfluid [8,9] and the formation of quantized vortices [10–12]. In a weakly interacting three-dimensional (3D) Bose gas the superfluid density is directly related to the experimentally accessible Bose-Einstein condensate (BEC) fraction. However, this is no longer true for strongly interacting systems such as $^4$He or for the unitary Fermi gas, where one does not have a straight correspondence between the superfluid and the condensate densities. The situation is even more challenging in two-dimensional (2D) systems, where Bose-Einstein condensation is ruled out at finite temperature, as a direct consequence of the Hohenberg-Mermin-Wagner theorem [13,14]. For these systems, a promising way to investigate superfluidity and to identify the value of the superfluid density concerns the measurement of second sound [15,16]. This phenomenon arises from the two-fluid nature of the system and corresponds to a wave propagation of the normal and superfluid components of opposite phase, with a speed of sound directly related to the superfluid density. Experimentally, the way to probe second sound depends in a crucial way on the nature of the system. While in $^4$He or in the unitary Fermi gas second sound is essentially an entropy oscillation, and is conveniently excited through a thermal perturbation [17–19], the situation drastically changes for a weakly interacting Bose gas, where the coupling between entropy and density oscillations becomes important, because of the large value of the thermal expansion coefficient, allowing for the excitation of second sound through a density perturbation [20,21]. Recently, second sound was observed in the unitary Fermi gas, yielding information on the temperature dependence of the superfluid density [22]. The first experiment on the propagation of second sound in a weakly interacting 2D Bose gas has also become available recently [23].

In this paper, we study the nature and experimental accessibility of first and second sounds in 2D Bose gases, exploring the transition from the weakly interacting to the strongly interacting regimes. The former case was investigated in [24], which points out the occurrence of discontinuities of both sound modes at the Berezinskii-Kosterlitz-Thouless (BKT) transition [25,26], as a direct consequence of the jump of the superfluid density. In the present work we extend the investigation to the strongly interacting case, which corresponds experimentally to a 2D Bose gas loaded in an optical lattice [27] or the BEC regime of a 2D Fermi gas [28,29]. In particular, we show that the discontinuity of the first sound velocity becomes less and less pronounced in the strongly interacting regime, while it remains sizable in the case of second sound. Since in 2D systems the thermodynamic quantities derivable from the equation of state do not show any discontinuity at the phase transition [30], the experimental measurement of second sound would also provide a unique way to observe directly the BKT phase transition. This is particularly interesting for 2D Fermi gases, where the recent observation of the BKT jump, based on the measurement of the pair momentum distribution [31], stimulated a debate in the literature [32].

## II. LANDAU'S TWO-FLUID EQUATIONS

Throughout this paper we consider a 2D gas, where the third direction is assumed to be frozen. In practice this condition is well satisfied in experiments [33–39]. We also set $\hbar = k_B = 1$ for simplicity. We start our investigation from Landau's two-fluid hydrodynamic equations, describing the finite-temperature dynamics of a uniform system in the

*miki.ota@unitn.it

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superfluid phase. The equations assume local thermodynamic equilibrium, ensured by collisions. In the limit of low-amplitude oscillations, the linearized Landau equations take the form

$$\frac{\partial^2 n}{\partial t^2} = \nabla^2 P, \tag{1}$$

$$\frac{\partial^2 \bar{s}}{\partial t^2} = \frac{n_s \bar{s}^2}{n_n}\nabla^2 T, \tag{2}$$

where $n = n_n + n_s$ is the total atom density, given by the sum of the normal density $n_n$ and the superfluid density $n_s$. $P$ is the pressure, and $\bar{s}$ and $T$ are the entropy at constant volume per particle and the temperature, respectively. By looking for plane-wave solutions and using general thermodynamic relations, Eqs. (1) and (2) give rise to the quartic equation,

$$c^4 - \left[\frac{1}{mn\kappa_s} + \frac{n_s T \bar{s}^2}{mn_n \bar{c}_v}\right]c^2 + \frac{n_s T \bar{s}^2}{mn_n \bar{c}_v}\frac{1}{mn\kappa_T} = 0, \tag{3}$$

for the sound velocity, where $m$ is the mass of the atom, $\bar{c}_v$ is the specific heat at constant volume per particle, and $\kappa_s$ and $\kappa_T$ are the adiabatic and thermal compressibilities, respectively. Below the critical temperature Eq. (3) possesses two positive solutions, corresponding to first and second sounds.

In this work, all the thermodynamic quantities are calculated using the universal relations (URs) for a weakly interacting 2D Bose gas derived in [34,35,40–42]. The theory provides dimensionless universal functions $f_n(x,g)$ and $f_P(x,g)$ depending on the variable $x = \mu/T$, with $\mu$ the chemical potential, and on the dimensionless coupling constant $g$. These functions are related to the density and to the pressure of the gas according to

$$f_n(x,g) = \lambda_T^2 n, \quad f_P(x,g) = \frac{\lambda_T^2}{T}P, \tag{4}$$

where $\lambda_T = \sqrt{2\pi/mT}$ is the thermal de Broglie wavelength, and are related to each other by the thermodynamic relation $f_n = \partial f_P/\partial x$. Starting from these functions one can then derive expressions for all the quantities appearing in Eq. (3) [24], namely,

$$\begin{aligned} \bar{s} &= 2\frac{f_P}{f_n} - x, \quad \kappa_T = \frac{1}{nT}\frac{f_n'}{f_n}, \quad \kappa_s = \frac{1}{nT}\frac{f_n}{2f_P} \\ \bar{c}_v &= 2\frac{f_P}{f_n} - \frac{f_n}{f_n'}, \quad \bar{c}_p = \left(2\frac{f_P}{f_n} - \frac{f_n}{f_n'}\right)2\frac{f_P f_n'}{f_n^2}, \end{aligned} \tag{5}$$

where $f_n' = \partial f_n/\partial x$, and $\bar{c}_p$ is the specific heat at constant pressure, per particle. Universal relations further provide another dimensionless function, $f_s(x,g) = \lambda_T^2 n_s$, from which one can evaluate the superfluid density. In our work, we use the analytical expression for the dimensionless functions provided in [41]. We note that Ref. [41] also provides Monte Carlo values for $f_n$ and $f_P$ and we have verified that both approaches give practically the same results.

Figure 1(a) shows the ratio of thermal and adiabatic compressibilities as a function of the temperature, for different values of the coupling constant. The figure shows that this ratio, which also fixes the value of the thermal expansion coefficient [see Eq. (6) below], decreases as the repulsive interaction between bosons becomes stronger. However, from thermodynamic principles, the ratio $\kappa_T/\kappa_s$ cannot be lower than 1, and Fig. 1(a) shows a clear failure of the predictions

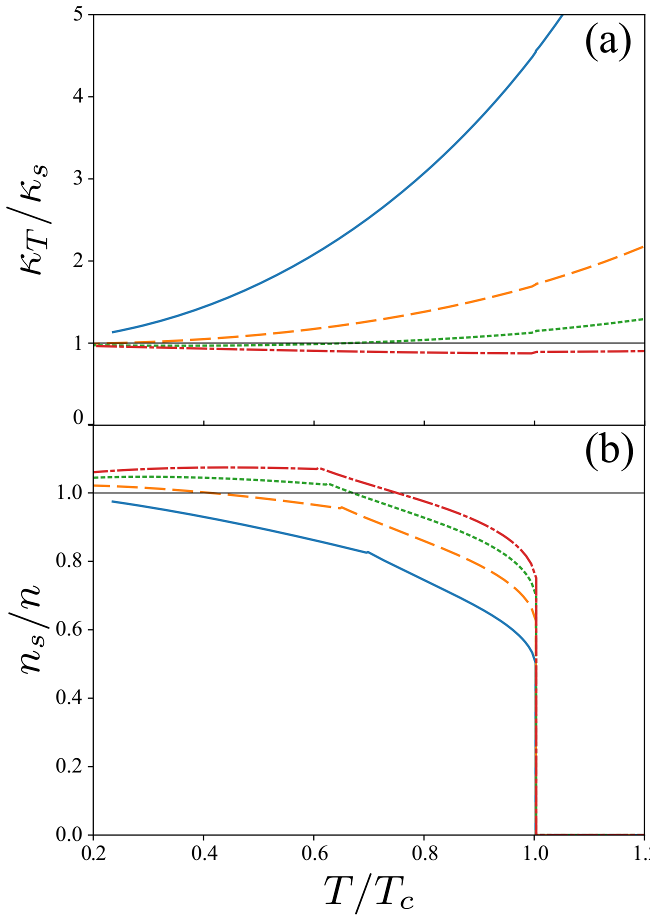


FIG. 1. (a) Ratio of isothermal and adiabatic compressibilities $\kappa_T/\kappa_s$ for different values of $g$. From top to bottom, $g = 0.1$ (solid line), $g = 0.5$ (dashed line), $g = 1$ (dotted line), and $g = 1.5$ (dashed-dotted line). (b) Superfluid density fraction $n_s/n$ for different values of $g$. The values of $g$ are the same as in (a). The unphysical kinks observed at $T \simeq 0.6T_c$ in the superfluid density are due to the analytical treatment of the dimensionless functions in the universal relations approach [43]. The solid black line is a guide for the eye: (a) $\kappa_T/\kappa_s = 1$ and (b) $n_s/n = 1$.

based on the UR for $g \geqslant 1$. In Fig. 1(b) we show the superfluid density fraction $n_s/n$ for the same values of the coupling constant. Again, we see another failure of the universal relation, which predicts a value for the ratio $n_s/n$ higher than 1 at low temperatures if the coupling constant $g$ is large enough. This failure is the consequence of the fact that the URs correctly describe only the fluctuating region near the critical point [40,41,43]. As the interaction increases this region around $T_c$ shrinks, reducing the region of applicability of the URs approach, although it allows for a good estimate of $T_c$ also for large values of $g$, as confirmed by the comparison with *ab initio* quantum Monte Carlo calculations [44]. For the above reasons in the following we limit our theoretical analysis, based on the predictions of the URs approach, to values of $g \leqslant 1$. We briefly note that $g \simeq 0.1$ is a typical value of the coupling constant for a dilute 2D Bose gas [36], and values of $g \lesssim 2$ correspond to a strongly interacting Bose gas [27] or the BEC regime of a 2D Fermi gas [37,38], where the system is expected to behave physically like a gas of bosonic dimers [45].

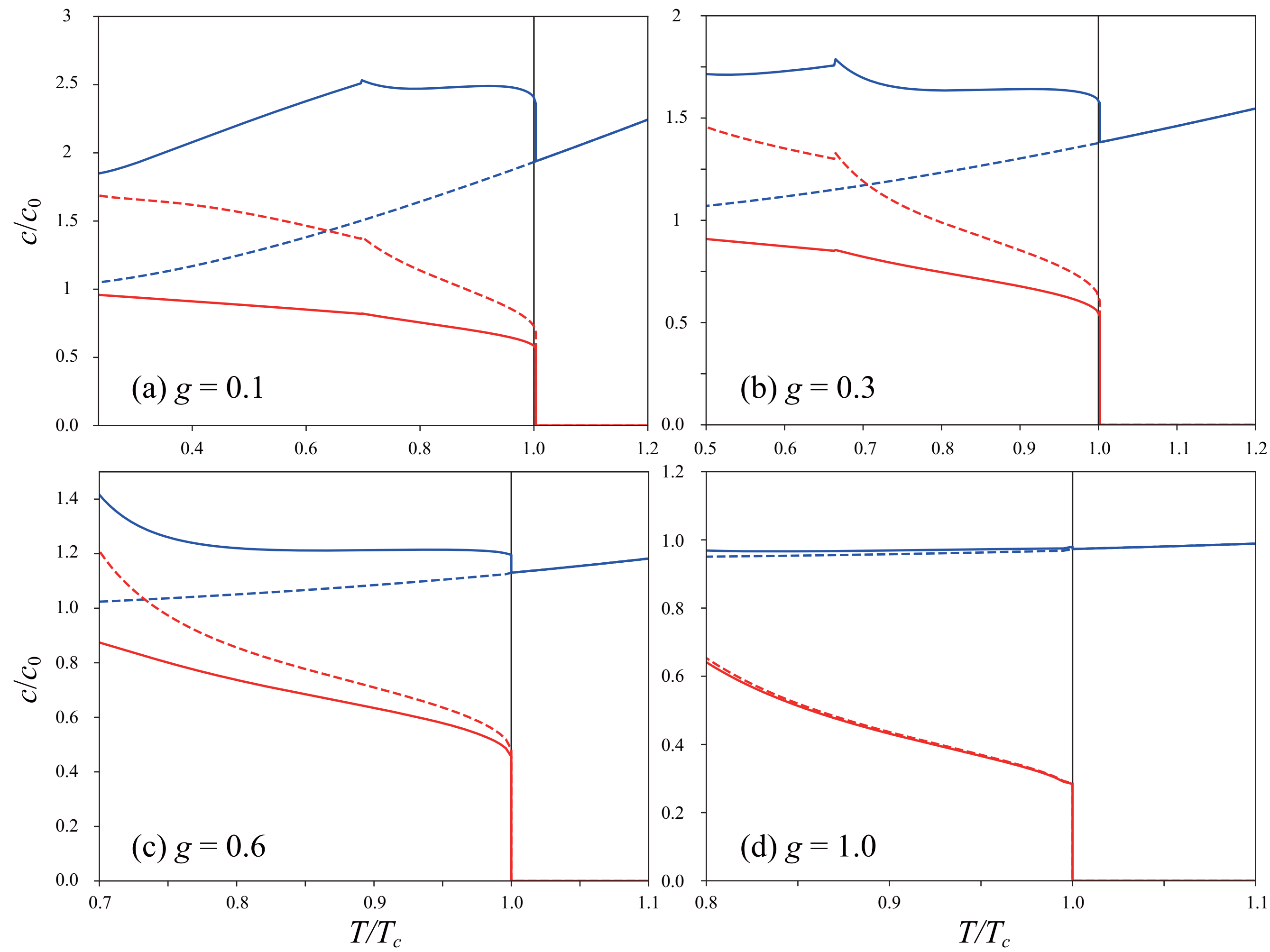


FIG. 2. First and second sound as a function of temperature for different values of $g$. The upper blue and lower red solid lines correspond to first and second sound calculated from Eq. (3), respectively. The upper blue and lower red dashed lines are the approximated forms of the first and second sound for a small thermal expansion coefficient, given by Eq. (7). The unphysical kinks observed at $T \simeq 0.6T_c$ in the sound velocities originate from the superfluid density (see Fig. 1).

### III. SOUND PROPAGATION IN A 2D BOSE GAS

Figure 2 shows the first and second sounds obtained by solving Eq. (3) (solid line), for different values of $g$ (the results for $g = 0.1$ are reported in [24]). The velocities are calculated for a fixed value of the total density and are expressed in units of the zero-temperature Bogoliubov sound velocity $c_0 = \sqrt{gn}/m$. As one can see, both sound velocities show a jump at the transition temperature. This behavior, originating from the BKT universal jump of the superfluid density, is studied in detail in the following. In order to understand the evolution of the sound modes with the coupling constant one notes that if the thermal expansion coefficient $\alpha = -\frac{1}{n}\frac{\partial n}{\partial T}|_P$ satisfies the condition

$$\alpha T = \left(\frac{\kappa_T}{\kappa_s} - 1\right) \ll 1, \tag{6}$$

the two solutions of Eqs. (1) and (2) take the form of wave equations for the density and for the entropy, respectively, the corresponding sound velocities being given by

$$c_{10}^2 = \frac{1}{mn\kappa_s}, \quad c_{20}^2 = \frac{n_s T \bar{s}^2}{mn_n \bar{c}_p}. \tag{7}$$

Figures 2(a) and 2(b) show that the calculated velocities strongly deviate from Eq. (7) (shown as the dashed line), revealing the strong coupling between the density and the entropy modes in the highly compressible regime where the condition $\alpha T \ll 1$ is violated. Figure 3 shows that as the coupling constant increases, the gas evolves from weakly interacting to strongly interacting behavior, becoming less compressible. As a consequence, Eq. (7) becomes more and more accurate, as shown in Figs. 3(c) and 3(d). The transition between the weakly interacting and the strongly interacting regimes is then expected to take place for values of the 2D coupling constant corresponding to $g \sim 0.5$. This regime can be reached in a 2D Bose gas with Feshbach resonance [27] or in the BEC side of the BEC-BCS crossover in 2D superfluid Fermi gases [38]. It is noteworthy that the already mentioned unphysical

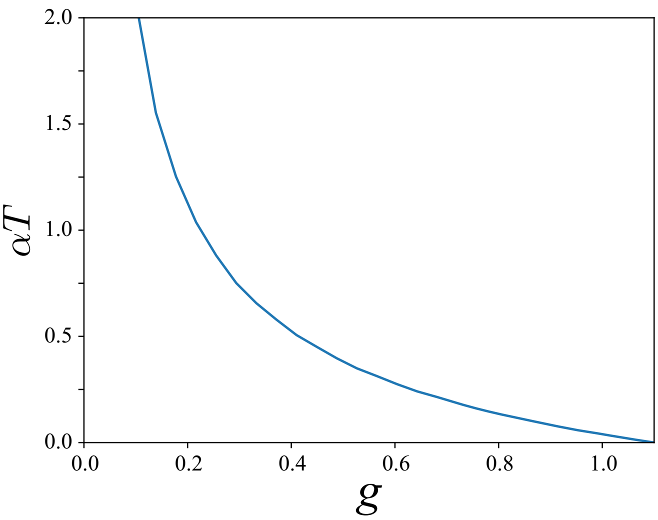


FIG. 3. Thermal expansion coefficient $\alpha T$ at $T = 0.8T_c$ as a function of the 2D coupling constant $g$.

violation of the thermodynamic relation $\kappa_T/\kappa_s \geqslant 1$ predicted by the use of URs for large values of the coupling constant has little effect on the sound speeds, while violation of the condition $n_s \leqslant n$ has dramatic unphysical consequences due to the resulting negativity of the normal density. The proper estimate of the sound velocities in the strongly interacting regime should then be based on more realistic estimates of the superfluid density. Accurate calculations of the superfluid density as well as of the relevant thermodynamic functions of 2D Fermi gases, based on quantum Monte Carlo simulations [44,47] or many-body theories [48–50], would, in particular, allow for a safer evaluation of the sound velocities along the whole BCS-BEC crossover.

In the case of very dilute Bose gases, an accurate approximated solution of Eq. (3) is obtained by replacing all the relevant thermodynamic quantities, except the isothermal compressibility and the superfluid density, with the values predicted by the ideal Bose gas [15]. Then Eq. (3) gives

$$c_{1,\mathrm{WI}}^2 = \frac{nT\bar{s}^2}{n_n m\bar{c}_v}, \quad c_{2,\mathrm{WI}}^2 = \frac{n_s}{n}\frac{1}{mn\kappa_T}. \tag{8}$$

Figure 4 shows, again, the sound velocities for the same values of the coupling constant but compared, this time, to Eq. (8) (dotted lines). As expected, the approximation successfully

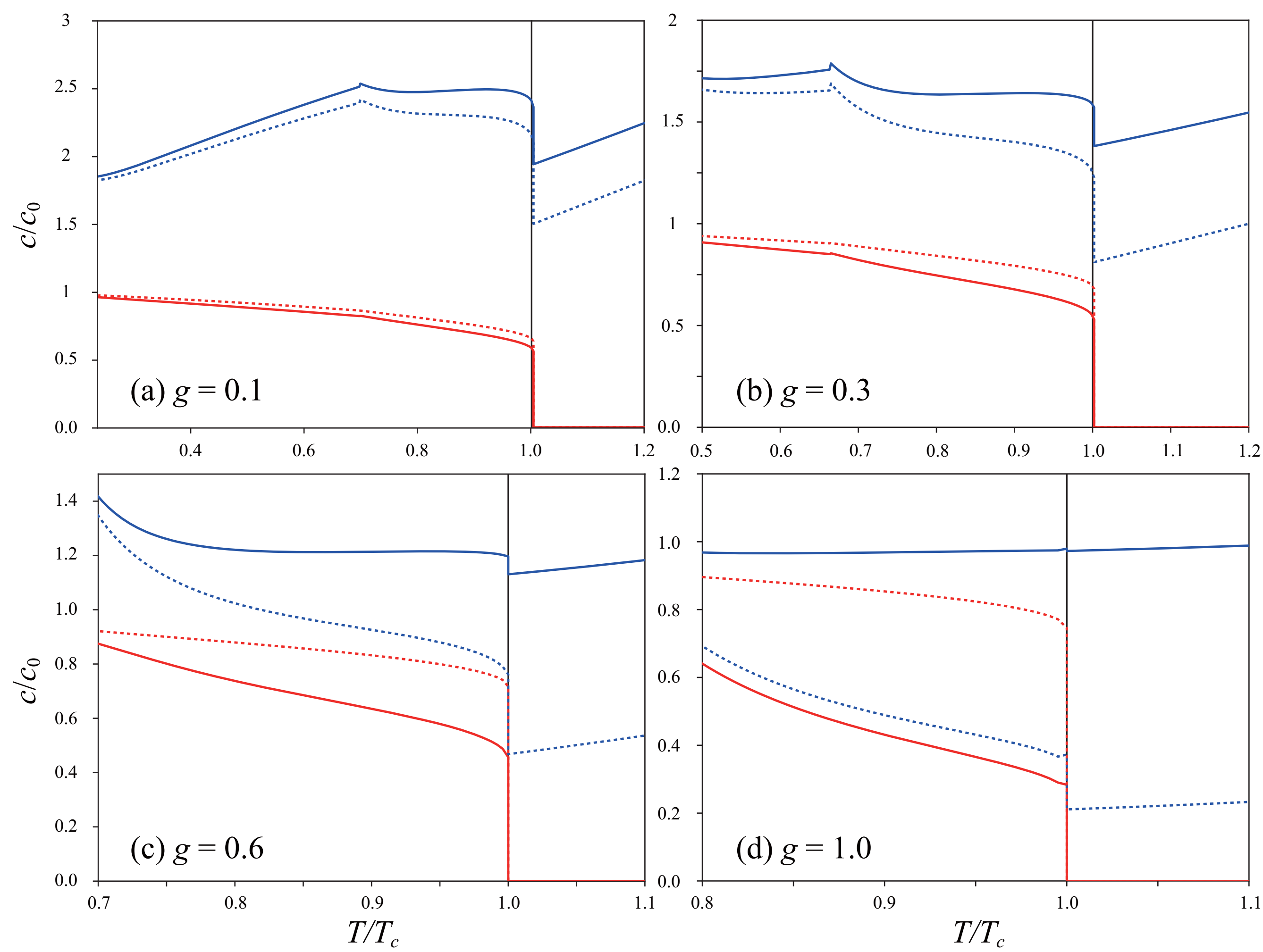


FIG. 4. First and second sounds as a function of the temperature for different values of $g$. Solid lines are the same as in Fig. 2. The upper blue and lower red dotted lines are the approximated forms of the first and second sounds for a weakly interacting Bose gas, given by Eq. (8).

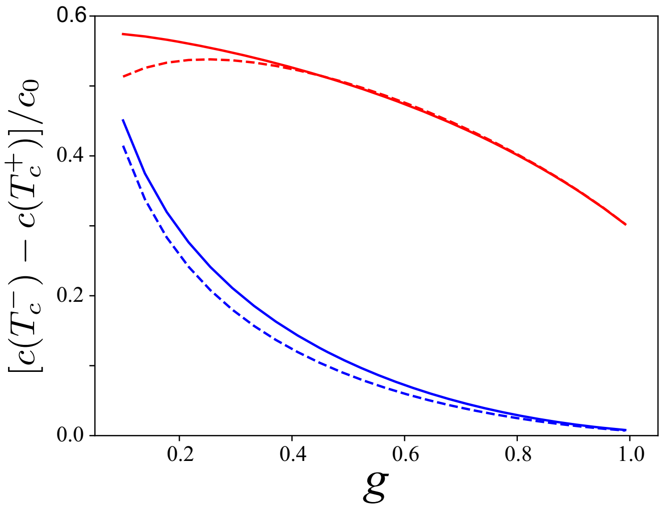


FIG. 5. BKT jump in sound velocities $c^-_{\rm BKT} - c^+_{\rm BKT}$ as a function of $g$. The velocity jumps for first sound (lower, solid blue line) and second sound (upper, solid red line) obtained from Eq. (3) are compared to the approximated expression, Eq. (9) (dashed blue and red lines for first and second sounds, respectively).

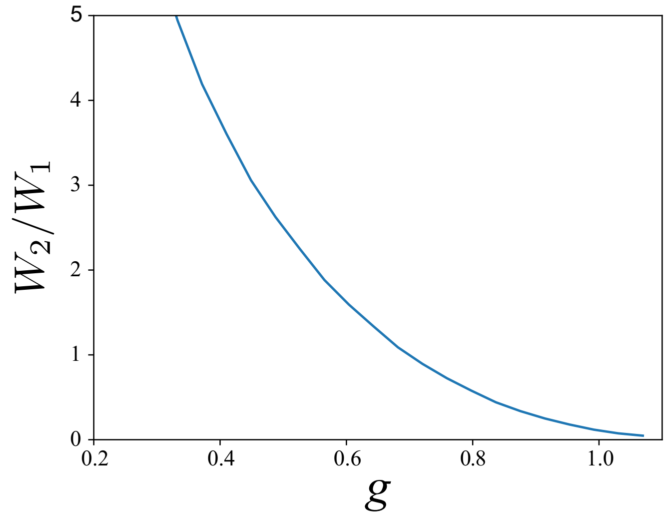


FIG. 6. Ratio of compressibility sum-rule contribution $W_2/W_1$ at $T = 0.8T_c$. $W_1$ ($W_2$) is the relative contribution of the first (second) sound mode to the compressibility sum rule, Eq. (11).

describes the exact sound speeds for small $g$ and becomes less and less accurate as one increases the value of $g$.

While in the 3D case the sound velocities near $T_c$ can be estimated by putting $n_s \to 0$, leading to Eq. (7), this assumption cannot be used in two dimensions because of the presence of the gap. One can, however, derive a first-order correction to the values of $c_{10}$ and $c_{20}$ resulting from the solution of Eq. (3), by assuming $\alpha T c_{20}^2/c_{10}^2 \ll 1$. One finds

$$c^2_{1,\rm BKT} = c^2_{10}\left(1 + \alpha T\frac{c^2_{20}}{c^2_{10}}\right), \quad c^2_{2,\rm BKT} = c^2_{20}\left(1 - \alpha T\frac{c^2_{20}}{c^2_{10}}\right). \tag{9}$$

Results of (9) are expected to be valid near $T_c$, and in particular, they correctly describe the jump $c(T_c^-) - c(T_c^+)$ of the first and second sound velocities when one crosses the critical temperature for a wide range of values of the coupling constant, as explicitly shown in Fig. 5. According to Eq. (9), the deviation of the sound velocities from $c_{10}$ and $c_{20}$ near $T_c$ is characterized by the factor $\alpha T c^2_{20}/c^2_{10}$, revealing the crucial role played by the difference between the thermal and the adiabatic compressibilities. This is explicitly shown in Fig. 2(d), where, for large values of $g$, the jump of the first sound disappears due to the vanishingly small value of the thermal expansion coefficient $\alpha$.

### IV. EXCITATION OF SECOND SOUND IN A 2D BOSE GAS

As briefly mentioned in the introductory part, it is of great interest to understand whether second sound can be excited using a density probe. Experimentally, this can be achieved using a sudden laser perturbation applied to the center of the trap, or through a sudden modification of the confining potential in the case of a box potential [51,52]. By assuming that the perturbation acts on macroscopic length scales, in the linear approximation the induced density fluctuations are determined by the static polarizability, fixed by the compressibility sum rule [1]

$$\lim_{\mathbf{q}\to 0}\int_{-\infty}^{\infty} d\omega\frac{1}{\omega}S(\mathbf{q},\omega) = \frac{1}{2}n\kappa_T, \tag{10}$$

where $S(\mathbf{q},\omega)$ is the dynamical structure factor with wave vector $\mathbf{q}$ and frequency $\omega$. On the other hand, the energy-weighted momentum also satisfies the energy-weighted sum rule $\int_{-\infty}^{\infty} d\omega\omega S(\mathbf{q},\omega) = q^2/(2m)$. Since in the macroscopic limit of small $\mathbf{q}$ one expects that the two sum rules are exhausted by the two sound modes, one can express the relative contribution of each sound mode to the compressibility sum rule, Eq. (10), in the form [53]

$$W_1 = \frac{1 - mn\kappa_T c_2^2}{2m\left(c_1^2 - c_2^2\right)}, \quad W_2 = \frac{mn\kappa_T c_1^2 - 1}{2m\left(c_1^2 - c_2^2\right)}, \tag{11}$$

where we have naturally chosen $c_1 > c_2$. If the ratio $W_2/W_1$ is not too low, second sound can be excited through a density perturbation. We also note that, under the assumption $c_1 \geqslant c_{10}$, the thermal expansion coefficient sets the lower bound

$$\frac{W_2}{W_1} \geqslant \alpha T. \tag{12}$$

Figure 6 shows the ratio of the relative contributions of second and first sounds to the compressibility sum rule, Eq. (11), calculated by solving the Landau equation, (3). From comparison with Fig. 3 we can see that, as expected from Eq. (12), the ratio $W_2/W_1$ follows the same evolution as $\alpha T$. This observation explicitly reveals that the excitation of second sound via a density probe becomes more and more difficult as one increases the value of the coupling constant.

Since the BEC regime of a 2D Fermi gas can be described in terms of an interacting molecular Bose gas, our results provide valuable information for the description of this system in a useful range of, hopefully, experimentally accessible parameters. From this point of view, the most interesting region to explore experimentally would be around $g \simeq 0.5$, where the ratio $W_2/W_1 \simeq 2$ is still high to allow for the excitation of second sound via a density probe. Such experiments would provide unique information on the value of the superfluid density and on the applicability of the universal relations for 2D Bose gases beyond the weakly interacting regime.

## V. SUMMARY

In conclusion, we have provided a systematic investigation of the behavior of second sound in a 2D interacting Bose gas, exploring the transition between the weakly interacting limit and the regime characterized by larger values of the 2D coupling constant $g$. Second sound is sensitive to the behavior of the superfluid density and its measurement can then provide unique information on the effects of superfluidity, a phenomenon of great interest, especially in two dimensions, where the system is characterized by the Berezinskii-Kosterlitz-Thouless transition. We have shown that the nature of second sound exhibits a great change as a function of the coupling constant. For small values of $g$ second sound can be identified as a density wave, of easy experimental detection. For larger values of $g$, second sound loses its density character and takes the form of a temperature, or entropy wave, in analogy with the behavior exhibited by superfluid helium and by the 3D Fermi gas at unitarity.

A challenging open question is to understand whether the collisional regime, required to apply the Landau two-fluid hydrodynamic approach, is guaranteed under the experimentally available conditions. A recent experiment [23] on the propagation of sound in a weakly interacting Bose gas confined in a 2D box potential has shown that, differently from the predictions of two fluid hydrodynamic equations, a density wave can propagate at low velocity even above the critical temperature, thereby suggesting that the collisional regime is not guaranteed in this experiment. Due to the finite size $L$ of the box, the frequency of the lowest mode, of order $v/L$, where $v$ is the velocity of sound, may not in fact be low enough compared to the collisional frequency, thereby violating the hydrodynamic condition. More theoretical work is then needed to better understand whether sound can propagate in a 2D Bose gas in the absence of collisions.

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$$g = -4\pi \frac{1}{2\ln(k_{\mathrm{F}}a_{2\mathrm{D}}) + \ln(\gamma^2/4\pi)}, \tag{13}$$

where $k_{\mathrm{F}}a_{2\mathrm{D}}$ is the interaction parameter for a 2D Fermi gas [28], with $k_{\mathrm{F}}$ and $a_{2\mathrm{D}}$ the Fermi wave vector and the 2D $s$-wave scattering length, respectively. $\gamma = a_B/a_{2\mathrm{D}} \simeq 0.55$ [46] is a constant relating the effective bosonic scattering length $a_B$ to that for fermions, $a_{2\mathrm{D}}$.
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