

Decay of homogeneous two-dimensional quantum turbulence

Andrew W. Baggaley and Carlo F. Barenghi

Joint Quantum Centre Durham-Newcastle, School of Mathematics, Statistics and Physics, Newcastle University, Newcastle upon Tyne NE1 7RU, England, United Kingdom



(Received 20 November 2017; published 1 March 2018)

We numerically simulate the free decay of two-dimensional quantum turbulence in a large, homogeneous Bose-Einstein condensate. The large number of vortices, the uniformity of the density profile, and the absence of boundaries (where vortices can drift out of the condensate) isolate the annihilation of vortex-antivortex pairs as the only mechanism which reduces the number of vortices, N_v , during the turbulence decay. The results clearly reveal that vortex annihilation is a four-vortex process, confirming the decay law $N_v \sim t^{-1/3}$ where t is time, which was inferred from experiments with relatively few vortices in small harmonically trapped condensates.

DOI: [10.1103/PhysRevA.97.033601](https://doi.org/10.1103/PhysRevA.97.033601)

I. MOTIVATION

Quantum turbulence (the chaotic motion of quantum vortices in superfluid helium [1] and cold gases [2]) has become a prototype problem of nonlinear statistical physics. The absence of viscosity and the nature of vorticity distinguish quantum turbulence from ordinary turbulence: in quantum fluids, in fact, vorticity is not a continuous field of arbitrary shape and strength (as in ordinary fluids) but is concentrated on the nodal points (in two dimensions) or lines (in three dimensions) of a complex wave function ψ . Around these points or lines where $\psi = 0$, the phase of ψ changes [3] by 2π . The large scale properties of quantum turbulence thus depend on the interactions of discrete vortices, which induce effects such as Kelvin waves [4–7], vortex reconnections [8–10], and phonon emission [11,12]. At temperatures sufficiently close to the critical temperatures, the interaction of vortices with thermal excitations [13,14] induces friction effects [15].

In turbulence, the study of free decay is fruitful because it removes the arbitrariness of the forcing which is necessary to sustain a statistical steady state. In three dimensions, experiments [16,17] and numerical simulations [18] of the decay of quantum turbulence in superfluid helium have revealed the existence of two turbulent regimes: a quasiclassical (or Kolmogorov) regime, which decays as $L(t) \sim t^{-3/2}$, and an ultraquantum (or Vinen) regime, which decays as $L(t) \sim t^{-1}$, where the vortex line density L (defined as the length of vortex lines per unit volume) measures the turbulence's intensity. Physically, the Kolmogorov regime is characterized by a cascade of kinetic energy from large to small eddies (similar to what happens in ordinary turbulence), whereas the Vinen regime lacks a cascade and is more akin to a random flow [19]. Recent studies of three-dimensional (3D) turbulence in atomic condensates have identified these two regimes [20], despite uncertainties due to the small number of vortices in the system compared to liquid helium experiments.

In two dimensions, quantum turbulence takes the form of a chaotic configuration of quantized point vortices. Since no direct vortex visualization is available in superfluid helium films, all relevant two-dimensional (2D) experiments have been performed in trapped atomic Bose-Einstein condensates

where vortices can be easily imaged. The 2D context has unique features (absent in three dimensions) associated to the possibility of nonthermal fixed points [21], an inverse energy cascade [22], and the emergence of vortex clusters [23,24]. In this paper we are concerned with a simpler question: in analogy with three dimensions, what is the law governing the free decay of a random vortex configuration consisting of an equal number of positive and negative vortices? This question was experimentally addressed in a harmonically trapped condensate by Kwon *et al.* [25]: they found that the time evolution of the number of vortices, $N_v(t)$ [the 2D equivalent of the vortex line density $L(t)$], is fairly well described by the logistic equation

$$\frac{dN_v}{dt} = -\Gamma_1 N_v - \Gamma_2 N_v^2. \quad (1)$$

In analogy with the kinetic theory of gases, Kwon *et al.* [25] argued that the rate coefficients Γ_1 and Γ_2 represent one-vortex and two-vortex processes, respectively: the drift of vortices out of the condensate, and annihilations of vortex-antivortex pairs (the 2D analog of 3D reconnections). Stagg *et al.* [26] modeled numerically the experiment of Kwon *et al.* [25], analyzed the results using Eq. (1), and determined that annihilations increase with temperature (see also Ref. [14]). Cidrim *et al.* [27] attempted to generalize Eq. (1) to the case of net polarization $P = (N_v^+ - N_v^-)/(N_v^+ + N_v^-) \neq 0$ (where N_v^+ and N_v^- are the numbers of positive and negative vortices, respectively, and $N_v = N_v^+ + N_v^-$). They noticed that the original interpretation of Γ_1 and Γ_2 as one-vortex and two-vortex processes cannot be correct, as negative values of Γ_1 were required to fit decays during which no vortices visibly entered the condensate. Using different model equations for N_v^+ and N_v^- , they obtained a better fit to the observed decay. In the case $P = 0$ (corresponding to the experiment of Kwon *et al.* [25]), the model of Cidrim *et al.* [27] reduces to

$$\frac{dN_v}{dt} = -\Gamma_1 N_v^{3/2} - \Gamma_2 N_v^4, \quad (2)$$

where the $N_v^{3/2}$ and N_v^4 dependences of the drift and the annihilation terms were derived using physical arguments. In particular, the quartic nature of the annihilation term in Eq. (2) agrees with the observation of Groszek *et al.* [28] that the

annihilation of a vortex-antivortex pair is a four-vortex process, not a two-vortex process (hence N_v^4 rather than N_v^2). Briefly, the argument is the following. Without dissipation, a vortex and antivortex alone would be a stable configuration which travels at constant velocity. A third vortex is necessary to bring the two vortices together, destroying the circulation and creating a stable nonlinear wave; this wave, called “crescent-shaped” by Kwon *et al.* [25] and “vortexonium” by Groszek *et al.* [28], was identified as a soliton by Nazarenko and Onorato [29]. The fourth vortex is necessary to destroy the nonlinear wave upon collision, radiating phonons away. Groszek *et al.* [28] also highlighted the role played by the trapping potential; in particular, they found that vortex clustering is energetically less likely in harmonically trapped condensates compared to recently developed box traps [30,31].

In contrast, in the presence of dissipation, vortices of opposite circulation move towards one another and annihilate directly. Hence, it is natural to expect that in the presence of dissipation the decay of two-dimensional quantum turbulence follows a two-vortex process. Indeed, our results will verify that this is the case.

Unfortunately the number $N_v(t)$ of point vortices in the cited studies is relatively small due to the constraints of current experimentally available condensates. The decay curves $N_v(t)$ are therefore noisy, and it is difficult to determine with precision the exponents of the two effects—vortex drift and vortex annihilation. Moreover, the drift of vortices out of the condensate is likely to depend on the steepness of the trapping potential, which now is not necessarily harmonic [30,31].

To make progress towards understanding the law of 2D turbulence decay, we concentrate on the annihilation process which here we study in the absence of vortex drift by performing numerical simulations in a uniform condensate without boundaries. In other words, we want to determine accurately the exponent k of the rate equation

$$\frac{dN_v}{dt} = -\Gamma_1 N_v^k, \quad (3)$$

when the only mechanism responsible for decreasing $N_v(t)$ is annihilations of vortex-antivortex pairs. For large times, the solution of Eq. (3) scales as $N_v \sim t^{1/(1-k)}$, if $k > 1$. A

precise measurement of the exponent k will help future works to determine the decay in finite-sized, nonuniform condensates, where the decay depends also on vortices drifting out of the boundaries.

II. MODEL

Our model is the 2D Gross-Pitaevskii equation for an atomic condensate:

$$(i - \gamma)\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) + g|\psi|^2\psi - \mu\psi, \quad (4)$$

where $\psi(x, y, t)$ is the wave function, m is the boson mass, g is the interaction strength, μ is the chemical potential, $\hbar = h/(2\pi)$, and h is Planck’s constant. The phenomenological dissipation coefficient γ [32] is used in some of our numerical simulations to mimic the interaction of the condensate with the thermal cloud, in particular the loss of energy (i.e., the reduction in size) of vortex-antivortex pairs.

Equation (4) is made dimensionless using the length scale $\xi = \hbar/\sqrt{m\mu}$, the time scale \hbar/μ , and the density scale $|\psi|^2 = \mu/g$, and solved in the (dimensionless) periodic domain $-D \leq x, y \leq D$ with $D = 512\xi$. The large size of the domain (compared to the vortex core size which is of the order of ξ) and the absence of boundaries allow us to track the evolution and annihilations of thousands of vortices, a number which is larger than in the typical experiments and previous numerical simulations. Space is discretized onto a $N = 2048^2$ uniform Cartesian mesh, spatial derivatives are approximated by a sixth-order finite-difference scheme, and a third-order Runge-Kutta scheme is used for time evolution.

The initial conditions of our simulations consist of a large number N_v of vortices with approximately net zero polarization ($N_v^+ \approx N_v^-$). To create this condition modeling an experimentally feasible manner, we initialize the system with the nonequilibrium state [33,34], $\psi(\mathbf{x}, 0) = \sum_k a_k \exp(i\mathbf{k} \cdot \mathbf{x})$, where $\mathbf{k} = (k_x, k_y)$ is the wave vector, and the coefficients a_k are uniform and the phases are distributed randomly. By taking $k_x, k_y \in \mathbb{Z}$ we ensure our initial configuration satisfies the periodic boundaries we impose. We perform three sets of

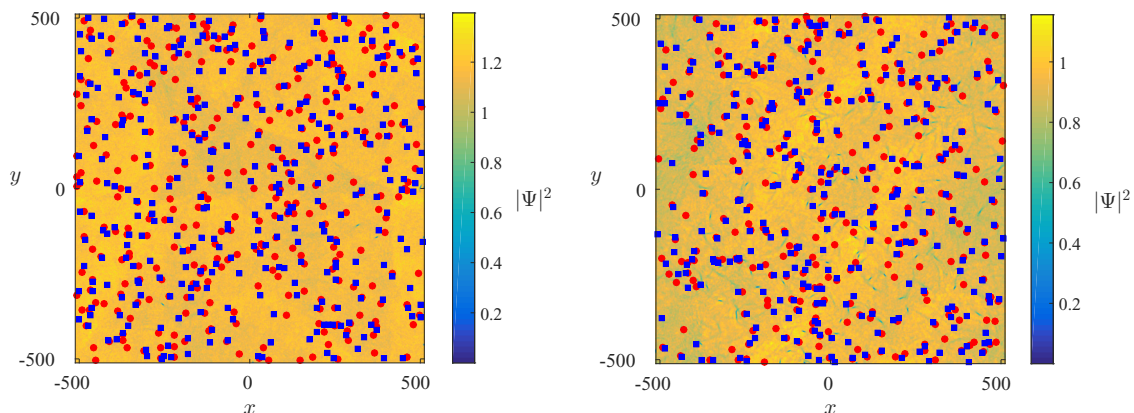


FIG. 1. The condensate’s density $|\psi|^2$ vs x, y during the decay of quantum turbulence without dissipation ($\gamma = 0$, left) and with dissipation ($\gamma = 0.01$, right). Vortex locations are inferred by an algorithm which identifies the 2π phase winding and the associated density depletion. We mark the location of vortices with a positive circulation with a red circle and those with a negative circulation using a blue square.

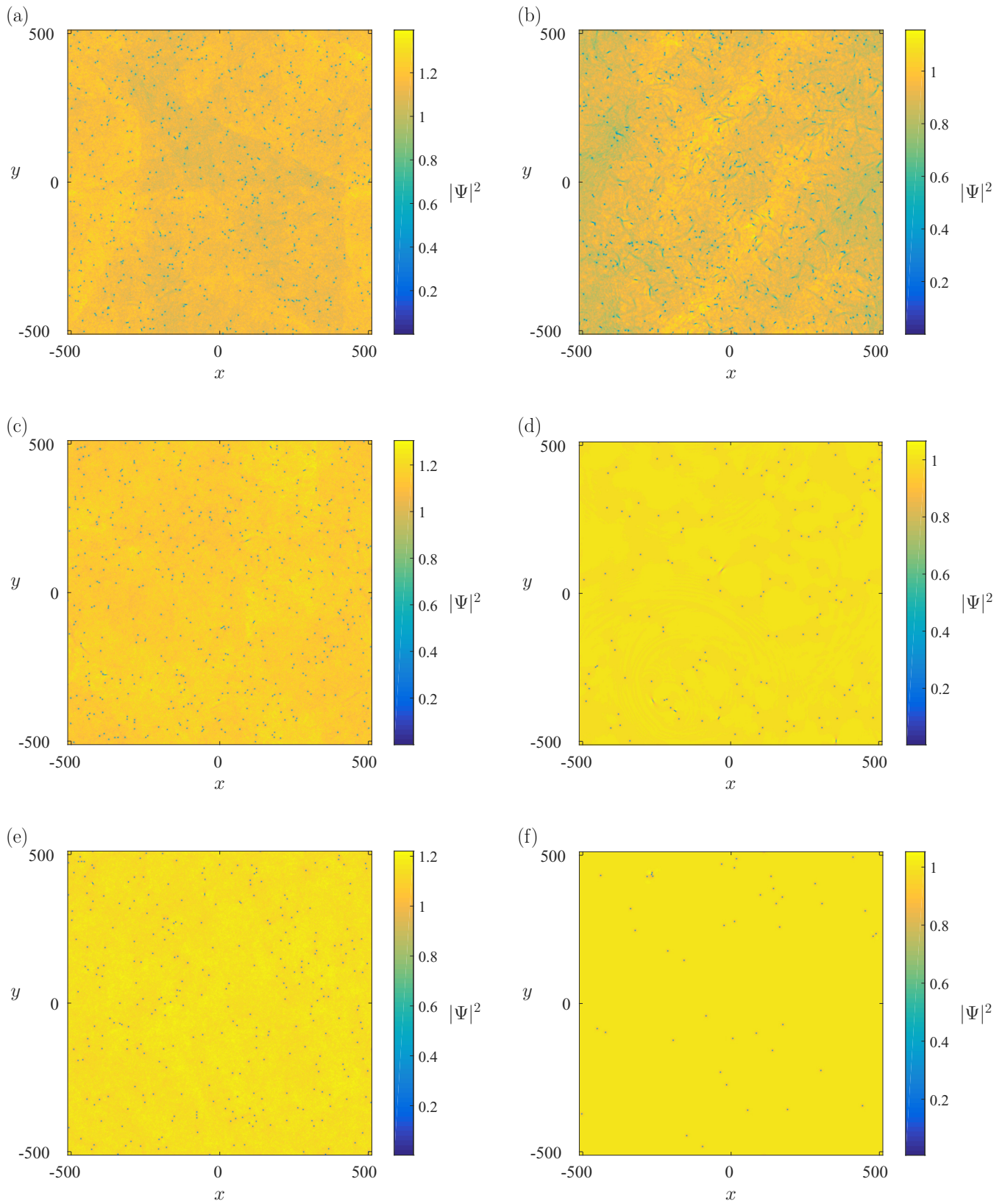


FIG. 2. The condensate’s density $|\psi|^2$ vs x, y during the decay of quantum turbulence without dissipation ($\gamma = 0$, left) and with dissipation ($\gamma = 0.01$, right) at times (a) $t = 7.5 \times 10^3$, (c) $t = 2.5 \times 10^4$, (e) $t = 1 \times 10^5$, (b) $t = 1 \times 10^3$, (d) $t = 1 \times 10^4$, and (f) $t = 5 \times 10^4$. The small holes in these density plots are the vortices.

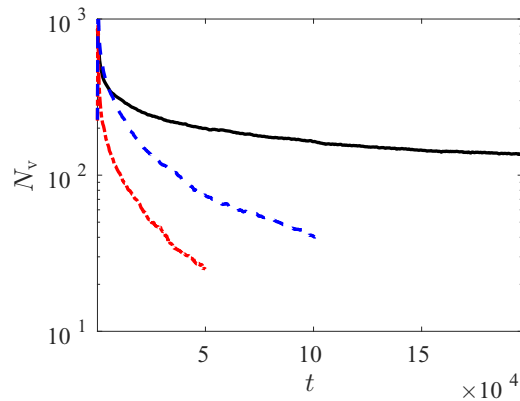


FIG. 3. The total vortex number, N_v , plotted vs time, t . The solid black curve, dashed blue curve, and dot-dashed red curve correspond to ensemble-averaging ten simulations without dissipation ($\gamma = 0$) and with dissipation ($\gamma = 0.0025$ and $\gamma = 0.01$), respectively. Notice the more rapid decay induced by increasing dissipation.

simulations: without dissipation ($\gamma = 0$) and at two different levels of dissipation, $\gamma = 0.01$ and 0.0025 . To ensure our results are independent of the initial conditions we make use of ensemble averaging and all results presented are averaged over simulations from ten different initial nonequilibrium states.

During the time evolution we compute the total number of vortices based on a previously tested algorithm [26] which identifies locations where the condensate possesses a 2π winding of the phase, and an associated density depletion (see Fig. 1).

III. RESULTS

Figure 2 displays the evolution of the condensate density on the x, y plane at different times t for the nondissipative ($\gamma = 0$, left column) case and a dissipative ($\gamma = 0.01$, right column) case. As vortices move chaotically (accelerate) in each other's velocity fields, they radiate sound waves [11], turning part of their kinetic energy into acoustic energy (phonons). Two vortices of opposite signs which collide annihilate, radiating more sound energy [8,12]. It is apparent from the figure that dissipation damps out density oscillations and removes vortices more quickly. The number of vortices N_v versus time t

(ensemble-averaged over ten simulations) is displayed in Fig. 3 for both nondissipative and dissipative cases. As expected, the decay of vortices is much faster in the presence of (larger) dissipation.

Figure 4 analyzes the decay in a quantitative way. The left panel of Fig. 4 shows that, in the absence of dissipation, the vortex number decays as $N_v(t) \sim t^{-0.3}$ in agreement with the $k = 4$ scaling in Eq. (3) of a four-vortex process [27,28] which would yield $N_v \sim t^{-1/3}$ (red dashed line). The blue dot-dashed line of this panel shows that the exponent $k = 2$ of the two-vortex process would not be a good fit.

The central and right panels of Fig. 4 show that, with $\gamma = 0.0025$ and 0.01 , the final part of the decay is steeper ($N_v \sim t^{-1}$) and more similar (particularly for $t > 2 \times 10^4$) to the prediction $N_v \sim t^{-1}$ (red dashed line) of the two-vortex process. Clearly, the $N_v \sim t^{-0.3}$ decay associated with the four-vortex process would not be a good fit at large times.

We also observe that dissipation introduces a transient $N_v \sim t^{-1/2}$ regime (clearly visible in the central and right panels) before the final $N_v \sim t^{-1}$ regime is achieved. This transient regime is the predicted outcome of a three-vortex process. It seems reasonable to assume that early in the simulations, when the vortex density is large, the annihilation of two vortices is predominantly induced by vortex dynamics (i.e., the presence of a third vortex), and not by dissipation. The four-vortex scaling is not seen if the soliton that emerges from the annihilation is strongly damped by the dissipation. However, once the vortex density becomes sufficiently small ($N_v < 100$ in these simulations), the dissipation becomes the dominant mechanism which brings vortices together and annihilates them, hence the two-vortex scaling emerges.

IV. CONCLUSION

We have performed numerical simulations of the free decay of 2D vortex configurations, which initially contain thousands of vortices. The very large homogeneous condensate and the absence of boundary effects have clearly confirmed that vortex annihilation is a four-vortex process which is described by the rate equation $dN_v/dt = -\Gamma_1 N_v^4$ proposed by Cidrim *et al.* [27] and Groszek *et al.* [28]. The presence of dissipation adds additional complexity. Initially the decay follows the three-vortex rate equation $dN_v/dt = -\Gamma_1 N_v^3$, as dissipation eliminates the need for a fourth vortex to dissipate the resulting

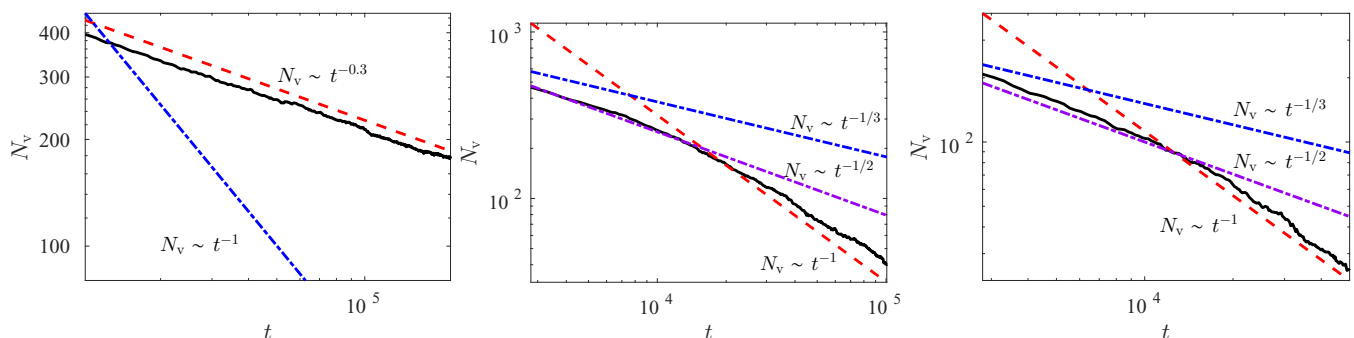


FIG. 4. Log-log plots of the data presented in Fig. 3, with corresponding best fits plotted as red dashed lines (position adjusted for clarity). The left panel corresponds to the case $\gamma = 0$, the central panel corresponds to $\gamma = 0.0025$, and the right panel corresponds to $\gamma = 0.01$. Alternative theoretical fits (discussed in the text) are plotted as dot-dashed lines.

soliton. However, at small vortex densities dissipation brings the vortex and the closest antivortex together without the need of the presence of other vortices, and the late-time decay is faster, in qualitative agreement with the rate equation $dN_v/dt = -\Gamma_1 N_v^2$ first proposed by Kwon *et al.* [25].

Having established the contribution to the turbulence decay arising from the annihilation of vortices with antivortices, it will be easier in future experiments to find the contribution

from vortices drifting out of the condensate (an effect which likely depends on the steepness of the confining potential).

ACKNOWLEDGMENT

This paper was supported by Engineering and Physical Sciences Research Council Grant No. EP/R005192/1.

-
- [1] C. F. Barenghi, L. Skrbek, and K. R. Sreenivasan, *Proc. Natl. Acad. Sci. USA* **111**, 4647 (2014).
- [2] M. C. Tsatsos, P. E. S. Tavares, A. Cidrim, A. R. Fritsch, M. A. Caracanhas, F. E. A. dos Santos, C. F. Barenghi, and V. S. Bagnato, *Phys. Rep.* **622**, 1 (2016).
- [3] More generally, the phase change is $2\pi q$ where q is an integer, but vortices with $q > 1$ are unstable.
- [4] D. Kivotides, J. C. Vassilicos, D. C. Samuels, and C. F. Barenghi, *Phys. Rev. Lett.* **86**, 3080 (2001).
- [5] T. P. Simula, T. Mizushima, and K. Machida, *Phys. Rev. Lett.* **101**, 020402 (2008).
- [6] E. Fonda, D. P. Meichle, N. T. Ouellette, S. Hormozf, and D. P. Lathropa, *Proc. Natl. Acad. Sci. USA* **111**, 4707 (2014).
- [7] P. Clark di Leoni, P. D. Mininni, and M. E. Brachet, *Phys. Rev. A* **94**, 043605 (2016).
- [8] S. Zuccher, M. Caliari, A. W. Baggaley, and C. F. Barenghi, *Phys. Fluids* **24**, 125108 (2012).
- [9] S. Serafini, L. Galantucci, E. Iseni, T. Bienaimé, R. N. Bisset, C. F. Barenghi, F. Dalfovo, G. Lamporesi, and G. Ferrari, *Phys. Rev. X* **7**, 021031 (2017).
- [10] A. Vilhois, D. Proment, and G. Krstulovic, *Phys. Rev. Fluids* **2**, 044701 (2017).
- [11] M. Leadbeater, D. C. Samuels, C. F. Barenghi, and C. S. Adams, *Phys. Rev. A* **67**, 015601 (2003).
- [12] M. Leadbeater, T. Winiecki, D. C. Samuels, C. F. Barenghi, and C. S. Adams, *Phys. Rev. Lett.* **86**, 1410 (2001).
- [13] B. Jackson, N. P. Proukakis, C. F. Barenghi, and E. Zaremba, *Phys. Rev. A* **79**, 053615 (2009).
- [14] J. H. Kim, W. J. Kwon, and Y. Shin, *Phys. Rev. A* **94**, 033612 (2016).
- [15] C. F. Barenghi, W. F. Vinen, and R. J. Donnelly, *J. Low Temp. Phys.* **52**, 189 (1982).
- [16] P. M. Walmsley and A. I. Golov, *Phys. Rev. Lett.* **100**, 245301 (2008).
- [17] D. E. Zmeev, P. M. Walmsley, A. I. Golov, P. V. E. McClintock, S. N. Fisher, and W. F. Vinen, *Phys. Rev. Lett.* **115**, 155303 (2015).
- [18] A. W. Baggaley, C. F. Barenghi, and Y. A. Sergeev, *Phys. Rev. B* **85**, 060501(R) (2012).
- [19] C. F. Barenghi, Y. A. Sergeev, and A. W. Baggaley, *Scientific Reports* **6**, 35701 (2016).
- [20] A. Cidrim, A. C. White, A. J. Allen, V. S. Bagnato, and C. F. Barenghi, *Phys. Rev. A* **96**, 023617 (2017).
- [21] B. Nowak, J. Schole, D. Sexty, and T. Gasenzer, *Phys. Rev. A* **85**, 043627 (2012).
- [22] M. T. Reeves, T. P. Billam, B. P. Anderson, and A. S. Bradley, *Phys. Rev. Lett.* **110**, 104501 (2013).
- [23] T. P. Billam, M. T. Reeves, B. P. Anderson, and A. S. Bradley, *Phys. Rev. Lett.* **112**, 145301 (2014).
- [24] T. Simula, M. J. Davis, and K. Helmersson, *Phys. Rev. Lett.* **113**, 165302 (2014).
- [25] W. J. Kwon, G. Moon, J.-Y. Choi, S. W. Seo, and Y.-I. Shin, *Phys. Rev. A* **90**, 063627 (2014).
- [26] G. W. Stagg, A. J. Allen, N. G. Parker, and C. F. Barenghi, *Phys. Rev. A* **91**, 013612 (2015).
- [27] A. Cidrim, F. E. A. dos Santos, L. Galantucci, V. S. Bagnato, and C. F. Barenghi, *Phys. Rev. A* **93**, 033651 (2016).
- [28] A. J. Groszek, T. P. Simula, D. M. Paganin, and K. Helmersson, *Phys. Rev. A* **93**, 043614 (2016).
- [29] S. Nazarenko and M. Onorato, *J. Low Temp. Phys.* **146**, 31 (2007).
- [30] A. L. Gaunt, T. F. Schmidutz, I. Gotlibovych, R. P. Smith, and Z. Hadzibabic, *Phys. Rev. Lett.* **110**, 200406 (2013).
- [31] N. Navon, A. L. Gaunt, R. P. Smith, and Z. Hadzibabic, *Science* **347**, 167 (2015).
- [32] M. Tsubota, K. Kasamatsu, and M. Ueda, *Phys. Rev. A* **65**, 023603 (2002).
- [33] N. G. Berloff and B. V. Svistunov, *Phys. Rev. A* **66**, 013603 (2002).
- [34] G. W. Stagg, N. G. Parker, and C. F. Barenghi, *Phys. Rev. A* **94**, 053632 (2016).