Entanglement witnesses from mutually unbiased bases

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We provide a class of entanglement witnesses constructed in terms of mutually unbiased bases (MUBs). This construction reproduces many well-known examples such as the celebrated reduction map and the Choi map together with its generalizations. We illustrate our construction by a detailed analysis of the three-dimensional case: In this case, one obtains a family of entanglement witnesses parametrized by an L-dimensional torus ($L = 2,3,4$ being a number of MUBs used in the construction).

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I. INTRODUCTION

Quantum entanglement is one of the most fundamental resources for modern quantum technologies and quantum information processing [1–3]. It is therefore clear that the characterization of entanglement and other quantum correlations [4] is of great importance for quantum information science (actually, the problem of determining whether or not a given state is entangled is NP-hard [5]). For low-dimensional $\mathbb{C}^2\otimes\mathbb{C}^2$ (qubit-qubit) and $\mathbb{C}^2\otimes\mathbb{C}^3$ (qubit-qutrit) systems the problem is solved due to the celebrated Peres-Horodecki partial transposition criterion [6,7]. However, for more complex systems there are states passing the partial transposition criterion [so-called positive partial transposition (PPT) states] which are entangled as was first shown in [8] for a qutrit-qutrit system. Hence, one needs more refined methods to check for separable or entanglement.

The most general approach to the separability problem is based on the concept of a positive map [9–12] and the directly related concept of an entanglement witness [13]: A linear map $\Phi : \mathcal{B}(\mathcal{H}_1) \longrightarrow \mathcal{B}(\mathcal{H}_2)$ is positive if $\Phi X \geqslant 0$ for any $X \geqslant 0$. In what follows $\mathcal{B}(\mathcal{H})$ denotes a vector space (even a C^* algebra) of bounded operators in $\mathcal{H}$. In this paper we consider only the finite-dimensional case and hence $\mathcal{B}(\mathcal{H})$ may be viewed as a matrix algebra $M_d(\mathbb{C})$ with $d = \dim\mathcal{H}$. A bipartite state ρ is separable if and only if $(\mathbb{1}\otimes\Phi)\rho \geqslant 0$ for all positive maps [2]. A Hermitian operator W acting on $\mathcal{H}_1\otimes\mathcal{H}_2$ is an entanglement witness if and only if $\langle\psi_1\otimes\psi_2|W|\psi_1\otimes\psi_2\rangle \geqslant 0$, but W is not a positive operator. The property $\langle\psi_1\otimes\psi_2|W|\psi_1\otimes\psi_2\rangle \geqslant 0$ is much weaker than the standard positivity of W, which is equivalent to $\langle\psi|W|\psi\rangle \geqslant 0$ for all $\psi \in \mathcal{H}_1\otimes\mathcal{H}_2$. One often calls such operators block positive. Hence, an entanglement witness (EW) is a block-positive operator but not a positive operator (actually, block positivity implies that W is Hermitian). A bipartite state ρ is separable if and only if $\mathrm{Tr}(\rho W) \geqslant 0$ for all entanglement witnesses. Any entangled state may be detected by an appropriate positive (but not completely positive) map or by an appropriate entanglement witness [2] (see also [14] for a review on entanglement witnesses).

An entanglement witness W is called decomposable if $W = A + B^\Gamma$, where $B^\Gamma = (\mathbb{1}\otimes T)B$ defines partial transposition and $A,B \geqslant 0$. Such witnesses, however, cannot detect PPT entangled states, that is, entangled states with positive partial transposition $\rho^\Gamma \geqslant 0$. To deal with PPT entangled states one needs nondecomposable witnesses, which are much harder to construct. Actually, there is no general construction of such objects. Moreover, given an EW, it is in general very hard to check whether it is decomposable or not.

Another important issue is how effective a given witness is in detecting entangled states. One calls W an optimal EW [15,16] if $W - A$ is no longer block positive for arbitrary $A \geqslant 0$, that is, W cannot be improved by subtracting a positive operator. A witness W is called nd-optimal [15] if $W - D$ is no longer block positive for an arbitrary decomposable operator D. Clearly, an nd-optimal witness is necessarily optimal. It turns out [15,16] that W is nd-optimal if W and W^Γ are optimal [Ha and Kye [17] call W (i) cooptimal if W^Γ is optimal and (ii) bioptimal if both W and W^Γ are optimal]. An EW has a *spanning property* if there exists a set of product vectors $|\psi_k\rangle\otimes|\phi_k\rangle$ such that $\langle\psi_k\otimes\phi_k|W|\psi_k\otimes\phi_k\rangle = 0$ spans the Hilbert space $\mathcal{H}_1\otimes\mathcal{H}_2$. Lewenstein *et al.* [15] proved that any witness with a spanning property is optimal. It should be stressed, however, that there are optimal witnesses which do not have a spanning property.

Finally, since the set of block-positive operators is convex, one may consider its extremal elements. The witness W is extremal if it satisfies the following property: If $W - A$ is block positive for some block-positive operator A, then $A = aW$ with $a \leqslant 1$. Among extremal elements there is a dense set of so-called exposed elements. An extremal W is exposed if it satisfies the following property: Suppose there exists a separable state $\rho_{\rm sep}$ such that $\mathrm{Tr}(W\rho_{\rm sep}) = 0$ and let $\mathrm{Tr}(W'\rho_{\rm sep}) = 0$ for some block-positive operator W'; then $W' = aW$, with $a > 0$. Interestingly, the spanning property is sufficient (but not necessary) for optimality and necessary (but not sufficient) for exposedness (cf. Kye's review [18] devoted to geometric structures related to the set of entanglement witnesses and the work of Hansen *et al.* [19] for a review of extremal entanglement witnesses).

In this paper we analyze a class of EWs constructed in terms of mutually unbiased bases (MUBs). It turns out that our construction reproduces many well-known examples of witnesses or equivalently positive maps such as the celebrated reduction map and the Choi map together with its generalizations. We discuss the issue of optimality and extremality. This problem is very hard and only partial results are presented.

II. POSITIVE MAPS FROM MUBS

Let us recall that two orthonormal bases $|\psi_k\rangle$ and $|\phi_l\rangle$ in $\mathbb{C}^d$ define MUBs if and only if for any k and l the following condition is satisfied:

$$|\langle\psi_k|\phi_l\rangle|^2 = \frac{1}{d}. \tag{1}$$

Moreover, it is well known [20] that the number $N(d)$ of MUBs in $\mathbb{C}^d$ is bounded by $N(d) \leqslant d+1$ [21] (see [22] for the review). If $d = p^r$ with p being a prime number, one has $N(d) = d+1$. In this case, explicit constructions are known [20,21]. If $d = d_1 d_2$, then $N(d) \geqslant \min\{N(d_1), N(d_2)\}$ [23]. Moreover, Grassl [24] provided a construction of three MUBs in an arbitrary dimension. Mutually unbiased bases have already found many important applications in quantum tomography [20,25,26], quantum cryptography [27,28], the mean king's problem [29,30], and entropic uncertainty relations [31–33]. They have been used to witness entangled quantum states in [34,35] as well as in [36] (the result in the latter work was recently generalized to the multipartite scenario [37]).

Now we provide the construction of a large class of trace-preserving positive maps in $M_d(\mathbb{C})$. Let $\{|\psi_1^{(\alpha)}\rangle, \ldots, |\psi_d^{(\alpha)}\rangle\}$, with $\alpha = 1, \ldots, L$, denote L MUBs [clearly $L \leqslant N(d) \leqslant d+1$]. Let us define the corresponding rank-1 projectors as $P_l^{(\alpha)} = |\psi_l^{(\alpha)}\rangle\langle\psi_l^{(\alpha)}|$. Moreover, let $\mathcal{O}^{(\alpha)}$ be a set of orthogonal rotation in $\mathbb{R}^d$ around the axis $\mathbf{n}_* = (1,1,\ldots,1)/\sqrt{d}$, that is, $\mathcal{O}^{(\alpha)}\mathbf{n}_* = \mathbf{n}_*$.

Theorem 1. The map

$$\Phi X = \Phi_* X - \frac{1}{d-1}\sum_{\alpha=1}^{L}\sum_{k,l=1}^{d}\mathcal{O}_{kl}^{(\alpha)}\mathrm{Tr}\big(\tilde{X}P_l^{(\alpha)}\big)P_k^{(\alpha)}, \tag{2}$$

where $\tilde{X} = X - \Phi_* X$ defines the traceless part of X ($\Phi_* X = \frac{1}{d}\mathbb{I}_d \mathrm{Tr} X$ defines the completely depolarizing channel) is positive and trace preserving.

Proof. Denote by $\mathcal{D}(d)$ the space of $d \times d$ density matrices. Recall that in $\mathcal{D}(d)$ one may inscribe a maximal ball $B_\star$ center at the maximally mixed state $\rho_\star = \frac{1}{d}\mathbb{I}$ [38]: $\rho \in B_\star \subset \mathcal{D}(d)$ if and only if

$$\mathrm{Tr}\rho^2 \leqslant \frac{1}{d-1}. \tag{3}$$

To prove positivity of Φ we show that for any rank-1 projector $P = |\psi\rangle\langle\psi|$ one has

$$\mathrm{Tr}(\Phi P)^2 \leqslant \frac{1}{d-1}, \tag{4}$$

that is, Φ maps any rank-1 projector into the ball $B_\star$. One finds

$$\begin{aligned}\mathrm{Tr}(\Phi P)^2 &= \mathrm{Tr}\Bigg\{\frac{1}{d^2}\mathbb{I} - \frac{2}{d-1}\sum_{\alpha=1}^{L}\sum_{k,l=1}^{d}\mathcal{O}_{kl}^{(\alpha)}\mathrm{Tr}\big(\tilde{P}P_l^{(\alpha)}\big)P_k^{(\alpha)} + \frac{1}{(d-1)^2}\sum_{\alpha,\beta=1}^{L}\sum_{k,l,m,n=1}^{d}\mathcal{O}_{kl}^{(\alpha)}\mathrm{Tr}\big(\tilde{P}P_l^{(\alpha)}\big)P_k^{(\alpha)}\mathcal{O}_{mn}^{(\beta)}\mathrm{Tr}\big(\tilde{P}P_n^{(\beta)}\big)P_m^{(\beta)}\Bigg\}\\ &= \frac{1}{d} - \frac{2}{d-1}\sum_{\alpha=1}^{L}\sum_{k,l=1}^{d}\mathcal{O}_{kl}^{(\alpha)}\mathrm{Tr}\big(\tilde{P}P_l^{(\alpha)}\big) + \frac{1}{(d-1)^2}\sum_{\alpha=1}^{L}\sum_{k,l,m,n=1}^{d}\mathcal{O}_{kl}^{(\alpha)}\mathcal{O}_{mn}^{(\alpha)}\mathrm{Tr}\big(\tilde{P}P_l^{(\alpha)}\big)\mathrm{Tr}\big(\tilde{P}P_n^{(\alpha)}\big)\delta_{km}\\ &\quad+\frac{1}{d}\frac{1}{(d-1)^2}\sum_{\alpha\neq\beta=1}^{L}\sum_{k,l,m,n=1}^{d}\mathcal{O}_{kl}^{(\alpha)}\mathcal{O}_{mn}^{(\beta)}\mathrm{Tr}\big(\tilde{P}P_l^{(\alpha)}\big)\mathrm{Tr}\big(\tilde{P}P_n^{(\beta)}\big).\end{aligned}$$

Now let us observe that

$$\sum_{k,l=1}^{d}\mathcal{O}_{kl}^{(\alpha)}\mathrm{Tr}\big(\tilde{P}P_l^{(\alpha)}\big) = 0,$$

due to $\mathcal{O}^{(\alpha)}\mathbf{n}_* = \pm\mathbf{n}_*$, and hence

$$\mathrm{Tr}(\Phi P)^2 = \frac{1}{d} + \frac{1}{(d-1)^2}\sum_{\alpha=1}^{L}\sum_{l=1}^{d}\big[\mathrm{Tr}\big(\tilde{P}P_l^{(\alpha)}\big)\big]^2,$$

where we have used $\sum_k \mathcal{O}_{kl}^{(\alpha)}\mathcal{O}_{km}^{(\alpha)} = \delta_{lm}$. Now

$$\big[\mathrm{Tr}\big(\tilde{P}P_l^{(\alpha)}\big)\big]^2 = \big[\mathrm{Tr}\big(PP_l^{(\alpha)}\big)\big]^2 + \frac{1}{d^2} - \frac{2}{d}\mathrm{Tr}\big(PP_l^{(\alpha)}\big),$$

and using the inequality [33,34]

$$\sum_{\alpha=1}^{L}\sum_{l=1}^{d}\big[\mathrm{Tr}\big(PP_l^{(\alpha)}\big)\big]^2 \leqslant 1 + \frac{L-1}{d}, \tag{5}$$

we finally arrive at

$$\mathrm{Tr}(\Phi P)^2 \leqslant \frac{1}{d} + \frac{1}{(d-1)^2}\left(1 + \frac{L-1}{d} + \frac{L}{d} - \frac{2L}{d}\right) = \frac{1}{d-1},$$

which ends the proof of positivity. The proof of trace preservation is elementary. ■

Note that the formula for Φ may be rewritten as

$$\Phi X = \frac{1}{d-1}\left\{\frac{d+L-1}{d}\mathbb{I}\,\mathrm{Tr}X - \sum_{\alpha=1}^{L}\sum_{k,l=1}^{d}\mathcal{O}_{kl}^{(\alpha)}\mathrm{Tr}\big(XP_l^{(\alpha)}\big)P_k^{(\alpha)}\right\} \tag{6}$$

and hence it simplifies for $L = d+1$ to

$$\Phi X = \frac{1}{d-1}\left\{2\,\mathbb{I}\,\mathrm{Tr}X - \sum_{\alpha=1}^{d+1}\sum_{k,l=1}^{d}\mathcal{O}_{kl}^{(\alpha)}\mathrm{Tr}\big(XP_l^{(\alpha)}\big)P_k^{(\alpha)}\right\}. \tag{7}$$

Recall that if $L = d + 1$, then one may perform complete tomography of ρ,

$$\rho = \frac{1}{d}\mathbb{I}_d + \sum_{\alpha=1}^{d+1}\sum_{k=1}^{d} a_k^{(\alpha)} P_k^{(\alpha)}, \tag{8}$$

with real parameters

$$a_k^{(\alpha)} = \mathrm{Tr}\big(\tilde{\rho} P_k^{(\alpha)}\big) = \mathrm{Tr}\big(\rho P_k^{(\alpha)}\big) - \frac{1}{d}. \tag{9}$$

Hence, for each $\alpha = 1,\dots,d+1$ one has

$$\sum_{k=1}^{d} a_k^{(\alpha)} = 0, \tag{10}$$

which means that the vector $\mathbf{a}^{(\alpha)} = (a_1^{(\alpha)},\dots,a_d^{(\alpha)})$ is orthogonal to the vector $\mathbf{n}_*$. Having performed complete tomography of ρ, one may simplify the proof of Theorem 1. Note that the map Φ may be rewritten in terms of $a_l^{(\alpha)}$ as

$$\Phi\rho = \Phi_*\rho - \frac{1}{d-1}\sum_{\alpha=1}^{d+1}\sum_{k=1}^{d}\sum_{l=1}^{d} \mathcal{O}_{kl}^{(\alpha)} a_l^{(\alpha)} P_k^{(\alpha)}. \tag{11}$$

Now using $\mathrm{Tr}\rho^2 \leqslant 1$ one finds

$$\mathrm{Tr}\rho^2 = \frac{1}{d} + \sum_{\alpha=1}^{d+1}\sum_{k=1}^{d} \big|a_k^{(\alpha)}\big|^2 = \frac{1}{d} + \sum_{\alpha=1}^{d+1} |\mathbf{a}^{(\alpha)}|^2 \leqslant 1, \tag{12}$$

which implies $\sum_{\alpha=1}^{d+1} |\mathbf{a}^{(\alpha)}|^2 \leqslant \frac{d-1}{d}$. Finally, using (12) and $|\mathcal{O}^{(\alpha)}\mathbf{a}^{(\alpha)}| = |\mathbf{a}^{(\alpha)}|$, one finds

$$\mathrm{Tr}(\Phi\rho)^2 = \frac{1}{d} + \frac{1}{(d-1)^2}\sum_{\alpha=1}^{d+1} |\mathbf{a}^{(\alpha)}|^2 \leqslant \frac{1}{d-1}, \tag{13}$$

which proves that $\Phi\rho \in \mathbf{B}_* \subset \mathcal{D}(d)$. Finally, the corresponding entanglement witness

$$W_\Phi = (d-1)\sum_{i,j=1}^{d} |i\rangle\langle j| \otimes \Phi|i\rangle\langle j|$$

reads

$$W_\Phi = \frac{d+L-1}{d}\mathbb{I}_d \otimes \mathbb{I}_d - \sum_{\alpha=1}^{L}\sum_{k,l=1}^{d} \mathcal{O}_{kl}^{(\alpha)} \overline{P}_l^{(\alpha)} \otimes P_k^{(\alpha)}, \tag{14}$$

which simplifies for $L = d+1$ to

$$W_\Phi = 2\,\mathbb{I}_d \otimes \mathbb{I}_d - \sum_{\alpha=1}^{d+1}\sum_{k,l=1}^{d} \mathcal{O}_{kl}^{(\alpha)} \overline{P}_l^{(\alpha)} \otimes P_k^{(\alpha)}. \tag{15}$$

Remark 1. For the maximal set of MUBs, that is, $L = d + 1$, the inequality (16) is replaced by [39,40]

$$\sum_{\alpha=1}^{d+1}\sum_{l=1}^{d} \big[\mathrm{Tr}\big(P P_l^{(\alpha)}\big)\big]^2 = 2 \tag{16}$$

and hence any rank-1 projector P is mapped via Φ onto the sphere $S_\star$, being the boundary of $B_\star$.

III. SPECIAL CLASSES: PERMUTATIONS

The special class of orthogonal $d \times d$ matrices with the additional property $\mathcal{O}\mathbf{n}_* = \mathbf{n}_*$ is provided by permutations: If Π is a permutation matrix then clearly $\Pi\mathbf{n}_* = \mathbf{n}_*$. Taking the simplest case corresponding to $\mathcal{O}^{(\alpha)} = \mathbb{I}_d$, one finds

$$\Phi[X] = \Phi_*[X] - \frac{1}{d-1}\sum_{\alpha=1}^{d+1}\sum_{k=1}^{d} \mathrm{Tr}\big(\tilde{X} P_k^{(\alpha)}\big) P_k^{(\alpha)}. \tag{17}$$

Now, one easily proves

$$\sum_{\alpha=1}^{d+1}\sum_{k=1}^{d} \mathrm{Tr}\big(A P_k^{(\alpha)}\big) P_k^{(\alpha)} = A + d\Phi_*[A]$$

and hence

$$\Phi[X] = \frac{1}{d-1}(\mathbb{I}_d \mathrm{Tr} X - X), \tag{18}$$

which is the well-known reduction map.

Consider now $\mathcal{O}^{(1)} = S$, where S is the permutation defined by $S|i\rangle = |i+1\rangle$. Let $\mathcal{O}^{(2)} = \cdots = \mathcal{O}^{(d+1)} = \mathbb{I}_d$. One finds

$$\Phi[X] = \frac{1}{d-1}\left(2\varepsilon[X] + \sum_{i=2}^{d-1} \varepsilon[S^i X S^{\dagger i}] - X\right), \tag{19}$$

where

$$\varepsilon[X] = \sum_{i=1}^{d} P_i^{(1)} X P_i^{(1)} = \sum_{i=1}^{d} |i\rangle\langle i|X|i\rangle\langle i|.$$

The map (19) belongs to the family of positive maps

$$\tau_{d,k}[X] = \frac{1}{d-1}\left((d-k)\varepsilon[X] + \sum_{i=1}^{k} \varepsilon[S^i X S^{\dagger i}] - X\right), \tag{20}$$

developed by Ando [41,42]. Actually, (19) is dual to $\tau_{d,d-2}$.

This construction may be immediately generalized if one considers $d+1$ permutations $\pi^{(\alpha)}$ and defines the corresponding entanglement witnesses by

$$W_\Phi = 2\mathbb{I}_d \otimes \mathbb{I}_d - \sum_{\alpha=1}^{d+1}\sum_{k=1}^{d} \overline{P}_{\pi^{(\alpha)}(k)}^{(\alpha)} \otimes P_k^{(\alpha)}.$$

These witnesses were analyzed by Hiesmayr and Rutkowski [43].

IV. CASE STUDY: $d = 3$

For $d = 3$ one has four MUBs $\mathcal{B}_1,\dots,\mathcal{B}_4$ defined as $\mathcal{B}_1 = \{\psi_1^{(1)} = |1\rangle, \psi_2^{(1)} = |2\rangle, \psi_3^{(1)} = |3\rangle\}$, where $|1\rangle,|2\rangle,|3\rangle$ define a computational basis in $\mathbb{C}^3$. The remaining three MUBs are defined as

$$\big|\psi_k^{(\alpha)}\big\rangle = U_\alpha|k\rangle, \tag{21}$$

where the unitary matrices U_α read

$$U_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^* & \omega \\ 1 & \omega & \omega^* \end{pmatrix}, \quad U_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^* \\ \omega^* & \omega & 1 \end{pmatrix},$$

and $U_4 = U_3^*$ (with $\omega = e^{2i\pi/3}$). One finds, for $\mathcal{B}_2$, $\mathcal{B}_3$, and $\mathcal{B}_4$,

$$\left\{\frac{|1\rangle+|2\rangle+|3\rangle}{\sqrt{3}},\frac{|1\rangle+\omega^*|2\rangle+\omega|3\rangle}{\sqrt{3}},\frac{|1\rangle+\omega|2\rangle+\omega^*|3\rangle}{\sqrt{3}}\right\},$$
$$\left\{\frac{|1\rangle+|2\rangle+\omega^*|3\rangle}{\sqrt{3}},\frac{|1\rangle+\omega|2\rangle+\omega|3\rangle}{\sqrt{3}},\frac{|1\rangle+\omega^*|2\rangle+|3\rangle}{\sqrt{3}}\right\},$$
$$\left\{\frac{|1\rangle+|2\rangle+\omega|3\rangle}{\sqrt{3}},\frac{|1\rangle+\omega^*|2\rangle+\omega^*|3\rangle}{\sqrt{3}},\frac{|1\rangle+\omega|2\rangle+|3\rangle}{\sqrt{3}}\right\}.$$

A general proper rotation in $\mathbb{R}^3$ preserving the direction $\mathbf{n} = (n_1,n_2,n_3)$, with $|\mathbf{n}| = 1$, is given by the Rodrigues formula

$$R(\mathbf{n},\varphi) = \begin{pmatrix} \cos\varphi + n_1^2(1-\cos\varphi) & n_1n_2(1-\cos\varphi) - n_3\sin\varphi & n_1n_3(1-\cos\varphi) + n_2\sin\varphi \\ n_1n_2(1-\cos\varphi) + n_3\sin\varphi & \cos\varphi + n_2^2(1-\cos\varphi) & n_2n_3(1-\cos\varphi) - n_1\sin\varphi \\ n_3n_1(1-\cos\varphi) - n_2\sin\varphi & n_3n_2(1-\cos\varphi) + n_1\sin\varphi & \cos\varphi + n_3^2(1-\cos\varphi) \end{pmatrix}. \tag{22}$$

Hence, taking $\mathbf{n} = \mathbf{n}_* = (1,1,1)/\sqrt{3}$, one finds

$$\mathcal{O}(\varphi) := R(\mathbf{n}_*,\varphi) = \begin{pmatrix} c_1(\varphi) & c_2(\varphi) & c_3(\varphi) \\ c_3(\varphi) & c_1(\varphi) & c_2(\varphi) \\ c_2(\varphi) & c_3(\varphi) & c_1(\varphi) \end{pmatrix}, \tag{23}$$

where

$$c_1(\varphi) = \frac{2}{3}\cos\varphi + \frac{1}{3},$$
$$c_2(\varphi) = \frac{2}{3}\cos\left(\varphi - \frac{2\pi}{3}\right) + \frac{1}{3}, \tag{24}$$
$$c_3(\varphi) = \frac{2}{3}\cos\left(\varphi + \frac{2\pi}{3}\right) + \frac{1}{3}.$$

One has $\mathcal{O}(0) = \mathbb{I}_3$. Note that $c_1(\varphi) + c_2(\varphi) + c_3(\varphi) = 1$.

In what follows we consider our construction corresponding to $L = 4,3,2$ (the case $L = 1$ is trivial since one always gets $W \geqslant 0$; actually, in this case there is no sense to use the term MUBs). Interestingly, the corresponding class of maps and witnesses is parametrized by the L-dimensional torus.

A. The case $L = 4$

One finds, for the corresponding entanglement witness parametrized by four angles $\{\varphi_1,\varphi_2,\varphi_3,\varphi_4\}$,

$$W = \left(\begin{array}{ccc|ccc|ccc} a & \cdot & \cdot & \cdot & p^* & \cdot & \cdot & \cdot & p \\ \cdot & b & \cdot & \cdot & \cdot & q^* & q & \cdot & \cdot \\ \cdot & \cdot & c & r^* & \cdot & \cdot & \cdot & r & \cdot \\ \hline \cdot & \cdot & r & c & \cdot & \cdot & \cdot & r^* & \cdot \\ p & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & p^* \\ \cdot & q & \cdot & \cdot & \cdot & b & q^* & \cdot & \cdot \\ \hline \cdot & q^* & \cdot & \cdot & \cdot & q & b & \cdot & \cdot \\ \cdot & \cdot & r^* & r & \cdot & \cdot & \cdot & c & \cdot \\ p^* & \cdot & \cdot & \cdot & p & \cdot & \cdot & \cdot & a \end{array}\right), \tag{25}$$

where to make the figure more transparent we replaced all 0 by dots. The parameters $\{a,b,c,p,q,r\}$ are defined as

$$a = \frac{2}{3}(1-\cos\varphi_1),$$
$$b = \frac{2}{3}\left(\frac{\sqrt{3}}{2}\sin\varphi_1 + \frac{1}{2}\cos\varphi_1 + 1\right),$$
$$c = \frac{2}{3}\left(-\frac{\sqrt{3}}{2}\sin\varphi_1 + \frac{1}{2}\cos\varphi_1 + 1\right),$$
$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = -\frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^* & \omega \\ 1 & \omega & \omega^* \end{pmatrix}\begin{pmatrix} e^{i\varphi_2} \\ e^{-i\varphi_3} \\ e^{i\varphi_4} \end{pmatrix}. \tag{26}$$

Example 1. Taking $\varphi_2 = \varphi_3 = \varphi_4 = 0$ one finds

$$p = -1, \quad q = 0, \quad r = 0.$$

This reproduces the family of maps analyzed in [44], being a generalization of the celebrated Choi positive nondecomposable extremal maps [45] (see also [46]). It is well known [47–49] that in this case (i) W is decomposable if and only if $b = c$, which means that $\varphi_1 = 0$ or $\varphi_1 = \pi$; (ii) W is nd-optimal if and only if $\varphi_1 \in [-2\pi/3,2\pi/3]$ (for $\varphi_1 = \pm 2\pi/3$ one recovers the celebrated Choi maps); (iii) W has a bispanning property if and only if $\varphi_1 \in (-2\pi/3,2\pi/3)$; (iv) W is extremal if and only if $\varphi_1 \in [-2\pi/3,0) \cup (0,2\pi/3]$; and (v) W is exposed if and only if it is extremal and $\varphi_1 \neq 0$. For further analysis see also [50,51].

Example 2. Taking $\varphi_2 = -\varphi_3 = \varphi_4 = \theta$, one arrives at

$$W = \left(\begin{array}{ccc|ccc|ccc} a & \cdot & \cdot & \cdot & -e^{i\theta} & \cdot & \cdot & \cdot & -e^{-i\theta} \\ \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & c & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & c & \cdot & \cdot & \cdot & \cdot & \cdot \\ -e^{-i\theta} & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & -e^{i\theta} \\ \cdot & \cdot & \cdot & \cdot & \cdot & b & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c & \cdot \\ -e^{i\theta} & \cdot & \cdot & \cdot & -e^{-i\theta} & \cdot & \cdot & \cdot & a \end{array}\right), \tag{27}$$

which was analyzed in [52].

B. The case $L = 3$

Now one has three orthogonal rotations parametrized by $\{\varphi_1,\varphi_2,\varphi_3\}$. The corresponding operator W has again the structure (25) with the same parameters a,b,c and the remaining off-diagonal parameters p',q',r' read

$$\begin{pmatrix} p' \\ q' \\ r' \end{pmatrix} = -\frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & \omega^* \\ 1 & \omega \end{pmatrix}\begin{pmatrix} e^{i\varphi_2} \\ e^{-i\varphi_3} \end{pmatrix}. \tag{28}$$

Example 3. Taking $\varphi_2 = \varphi_3 = 0$, one finds

$$p = -\tfrac{2}{3}, \quad q = \tfrac{1}{3}\omega, \quad r = q^*.$$

Note that in this case taking

$$a = \tfrac{4}{3}, \quad b = c = \tfrac{1}{3},$$

one finds $W \geqslant 0$, which means that the map Φ is completely positive and hence cannot be used to detect quantum entanglement.

C. The case $L = 2$

Now one has two orthogonal rotations parametrized by $\{\varphi_1,\varphi_2\}$. Again, W is given by (25) with the same parameters a,b,c and the remaining off-diagonal parameters p'',q'',r'' read

$$p'' = q'' = r'' =: z = -\tfrac{1}{3}e^{i\varphi_2}. \tag{29}$$

One finds

$$W = \left(\begin{array}{ccc|ccc|ccc} a & \cdot & \cdot & \cdot & z^* & \cdot & \cdot & \cdot & z \\ \cdot & b & \cdot & \cdot & \cdot & z^* & z & \cdot & \cdot \\ \cdot & \cdot & c & z^* & \cdot & \cdot & \cdot & z & \cdot \\ \hline \cdot & \cdot & z & c & \cdot & \cdot & \cdot & z^* & \cdot \\ z & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & z^* \\ \cdot & z & \cdot & \cdot & \cdot & b & z^* & \cdot & \cdot \\ \hline \cdot & z^* & \cdot & \cdot & \cdot & z & b & \cdot & \cdot \\ \cdot & \cdot & z^* & z & \cdot & \cdot & \cdot & c & \cdot \\ z^* & \cdot & \cdot & \cdot & z & \cdot & \cdot & \cdot & a \end{array}\right), \tag{30}$$

which is an analog of (27). Now, depending upon φ_1, one may have $W \geqslant 0$ or W is a proper entanglement witness.

D. The PPT entangled state detected by W

Consider the $3\otimes 3$ state

$$\rho = \frac{1}{15}\left(\begin{array}{ccc|ccc|ccc} 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & 2 & \cdot & \cdot & \cdot & -1 & -1 & \cdot & \cdot \\ \cdot & \cdot & 2 & -1 & \cdot & \cdot & \cdot & -1 & \cdot \\ \hline \cdot & \cdot & -1 & 2 & \cdot & \cdot & \cdot & -1 & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & -1 & \cdot & \cdot & \cdot & 2 & -1 & \cdot & \cdot \\ \hline \cdot & -1 & \cdot & \cdot & \cdot & -1 & 2 & \cdot & \cdot \\ \cdot & \cdot & -1 & -1 & \cdot & \cdot & \cdot & 2 & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \end{array}\right). \tag{31}$$

One easily checks that ρ is a PPT. Now taking $\varphi_1 = \varphi_2 = \pi$ and $\varphi_3 = \varphi_4 = 0$, one finds, from (25),

$$W = \frac{1}{3}\left(\begin{array}{ccc|ccc|ccc} 4 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & -1 \\ \cdot & 1 & \cdot & \cdot & \cdot & 2 & 2 & \cdot & \cdot \\ \cdot & \cdot & 1 & 2 & \cdot & \cdot & \cdot & 2 & \cdot \\ \hline \cdot & \cdot & 2 & 1 & \cdot & \cdot & \cdot & 2 & \cdot \\ -1 & \cdot & \cdot & \cdot & 4 & \cdot & \cdot & \cdot & -1 \\ \cdot & 2 & \cdot & \cdot & \cdot & 1 & 2 & \cdot & \cdot \\ \hline \cdot & 2 & \cdot & \cdot & \cdot & 2 & 1 & \cdot & \cdot \\ \cdot & \cdot & 2 & 2 & \cdot & \cdot & \cdot & 1 & \cdot \\ -1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & 4 \end{array}\right) \tag{32}$$

and

$$\mathrm{Tr}(\rho W) = -\tfrac{2}{15} < 0, \tag{33}$$

which proves that ρ being a PPT is entangled. Interestingly, entanglement of this state is not detected by witnesses from the well-known family corresponding to $\varphi_2 = \varphi_3 = \varphi_4 = 0$ (cf. Example 1). Similarly, one easily checks that ρ is not detected by the other three families of witnesses corresponding to $\varphi_1 = \varphi_3 = \varphi_4 = 0$, $\varphi_1 = \varphi_2 = \varphi_4 = 0$, and $\varphi_1 = \varphi_2 = \varphi_3 = 0$. These are direct generalizations of [44] obtained by permuting MUBs. Finally, the realignment test is not conclusive, giving the value of realignment $R = 1$ (recall that if $R > 1$, then a state is entangled [53]). To conclude, one cannot detect entanglement of (31) using either partial transposition and realignment tests or previously known entanglement witnesses.

V. PRIME DIMENSIONS AND WEYL OPERATORS AND SPECTRA

Let us recall the construction of Weyl operators [54–56]

$$U_{kl} = \sum_{k,l=0}^{d-1} \omega^{kl}|m\rangle\langle m+l|,$$

which satisfy the well-known relations

$$U_{kl}U_{rs} = \omega^{ks}U_{k+r,l+s}, \quad U_{kl}^\dagger = \omega^{kl}U_{-k,-l}.$$

Bertlmann and Krammer [56] provided the following theorem.

Theorem 2. Let W be a Hermitian operator defined by

$$W = a\sum_{k,l=0}^{d-1} c_{kl}U_{kl}\otimes U_{-k,l}, \tag{34}$$

with $a > 0$ and $c_{00} = d-1$. If the remaining c_{kl} satisfy $|c_{kl}| \leqslant 1$, then W is a block-positive operator, that is, $\langle x\otimes y|W|x\otimes y\rangle$ for arbitrary $x,y \in \mathbb{C}^d$.

It is well known that if d is prime then $d+1$ MUBs are directly related to Weyl operators. In this case the set of d^2-1 Weyl operators U_{kl} with $(k,l) \neq (0,0)$ splits into $d+1$ sets of mutually commuting operators, that is, $[U_{kl},U_{ij}] = 0$ if and only if $kj = il \pmod{d}$. These $d+1$ families correspond to $d+1$ MUBs. Consider as an example $d = 3$. One has $U_{00} = \mathbb{I}_3$ and

$$U_{01} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad U_{02} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$U_{10} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad U_{11} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega \\ \omega^2 & 0 & 0 \end{pmatrix},$$

$$U_{12} = \begin{pmatrix} 0 & 0 & 1 \\ \omega & 0 & 0 \\ 0 & \omega^2 & 0 \end{pmatrix}, \quad U_{20} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix},$$

$$U_{21} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega^2 \\ \omega & 0 & 0 \end{pmatrix}, \quad U_{22} = \begin{pmatrix} 0 & 0 & 1 \\ \omega^2 & 0 & 0 \\ 0 & \omega & 0 \end{pmatrix},$$

with $\omega = e^{2\pi i/3}$. One has $d+1 = 4$ families of commuting operators

$$\{U_{10},U_{20}\}, \quad \{U_{11},U_{22}\}, \quad \{U_{12},U_{21}\}, \quad \{U_{01},U_{02}\}.$$

One finds that (34) and (25) have the same structure and they are related via $a = 1/3$ together with

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^* & \omega \\ 1 & \omega & \omega^* \end{pmatrix}\begin{pmatrix} c_{00} \\ c_{10} \\ c_{20} \end{pmatrix} \tag{35}$$

and

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = -\frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^* & \omega \\ 1 & \omega & \omega^* \end{pmatrix}\begin{pmatrix} c_{01} \\ c_{11} \\ c_{21} \end{pmatrix}. \tag{36}$$

Note that Hermiticity of W implies that $c_{kl}^* = c_{-k,-l}$ and hence $c_{02} = c_{01}^*$, $c_{11} = c_{22}^*$, and $c_{12} = c_{21}^*$. One has therefore

$$c_{10} = e^{i\varphi_1}, \quad c_{01} = -e^{i\varphi_2}, \quad c_{22} = -e^{i\varphi_2}, \quad c_{21} = -e^{i\varphi_4},$$

that is, $|c_{kl}| = 1$ for all pairs $(k,l) \neq (0,0)$.

VI. CONCLUSION

We provided a class of entanglement witnesses and positive maps constructed in terms of mutually unbiased bases. Interestingly, this construction reproduces many well-known examples such as the celebrated reduction map and the Choi map together with its generalizations but also gives rise to completely different witnesses and maps. In the three-dimensional case we obtain a family of witnesses parametrized by a four-dimensional torus. As an example we provided a $3\otimes3$ entangled state (31) such that one cannot detect its entanglement either using partial transposition and realignment tests or using previously known entanglement witnesses.

It is clear that further analysis is needed in order to investigate the issues of optimality and extremality. Such analysis is known only for the special class of Choi-like witnesses (cf. Example 1). Also the problem of a spanning property deserves further studies.

Note that if $d = d_1 d_2$ we may consider Φ as a positive map acting on the matrix algebra of a composite system $M_d(\mathbb{C}) = M_{d_1}(\mathbb{C})\otimes M_{d_2}(\mathbb{C})$. Interestingly, all density matrices satisfying

$$\mathrm{Tr}\rho^2 \leqslant \frac{1}{d_1 d_2 - 1}$$

are separable [57] (actually, they are *superseparable*, i.e., separable with respect to an arbitrary partition of $\mathbb{C}^{d_1 d_2}$ into a tensor product of $\mathbb{C}^{d_1}$ and $\mathbb{C}^{d_2}$). Hence, they belong to a class of maps analyzed in [58] that have the additional property that when applied to any state (or a given entanglement class) result in a separable state or, more generally, a state of another certain entanglement class (e.g., Schmidt number less than or equal to k). Another interesting research program may be devoted to further analysis of a more general construction of maps in terms of MUBs in the spirit of [58].

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