Maximally steerable mixed state based on the linear steering inequality and the Clauser-Horne-Shimony-Holt–like steering inequality

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The two-qubit maximally steerable mixed state (MSMS), defined as one that violates to the most degree a steering inequality for any fixed linear entropy, is derived mathematically here based on the two-setting (three-setting) linear steering inequality and the two-setting Clauser-Horne-Shimony-Holt–like steering inequality. Interestingly, the form of the MSMS based on such different steering inequalities obtained here is identical to the two-qubit maximally nonlocal mixed state (MNMS). It is clearly shown that finding any two-qubit state, of which the state mixedness exceeds 2/3, thus violating the three-setting steering inequalities, is impossible. The violation of the inequalities with the Werner state and with the maximally entangled mixed state, respectively, as well as the relations between their optimal violation and the linear entropy, is also discussed comprehensively. In particular, within the range $\varepsilon(\rho) \in [0,2/3]$ of the fixed linear entropy, the Werner state reaches the same violation as the MSMS does of the three-setting linear steering inequality, but not of the two-setting steering inequalities (for the latter inequalities the violation with the Werner state is generally less than that with the MSMS). For the MEMS, the optimal violation is always lower than that of the MSMS for any fixed linear entropy.

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I. INTRODUCTION

In 1935, Einstein, Podolsky, and Rosen (EPR) [1] questioned the completeness of quantum mechanics (QM) by introducing the notion of local realism. Many efforts have since been committed to a deeper understanding of QM, mainly in consideration of three types of quantum correlations: quantum entanglement [2], EPR steering [3], and Bell nonlocality [4]. Of them, the development of quantum entanglement and Bell nonlocality has flourished since 1964 [5]; EPR steering, in contrast, had even lacked a rigorous formulation until the work in 2007 of Wiseman *et al.* [6]. After decades of research, physicists have gradually realized that they are different kinds of quantum correlations. According to the hierarchy of nonlocality, the set of EPR steerable states is a subset of entangled states and a superset of Bell nonlocal states [6]. Nowadays, these concepts have become the center of quantum foundations and have had many practical applications in modern quantum information theory ranging from quantum key distribution [7], communication complexity [8], and random number generation [9].

During decades of investigation, a great number of fruitful results on characterizing the properties of these quantum correlations have been obtained. Most of the results work for the pure states, yet extensions to the mixed state are very limited. This is of concern because the difference between three such quantum correlations can just be exhibited in mixed states, as in the work in Ref. [10]. Hence, a deeper understanding of these quantum correlations in the cases of mixed states is worthwhile and very necessary. Furthermore, the system under consideration is usually in a mixed state since environment-induced noise leading to decoherence is in general unavoidable in real experiments.

It is known that decoherence, which is detrimental to the amount of information in a quantum state, is measured by the purity of the state. To effectively characterize the role of decoherence in information erasing [11], one needs to quantify the purity or its complementary property—the mixedness of the state. Since noise tends to increase the mixedness of a quantum system, it emerges as an intuitive parameter to understand decoherence. A natural question that arises then is how important physical quantities, like quantum correlations, fare against the mixedness of quantum systems. A faithful measure of mixedness is the so-called normalized linear entropy [12–14]. Therefore, it is an intriguing task to obtain the maximum amount of quantum correlation (Bell nonlocality, steering, or entanglement) for a given mixedness.

To date, some interesting examples like maximally entangled mixed states (MEMSs), maximally discordant mixed states (MDMSs), and maximally nonlocal mixed states (MNMSs) have been discussed [15–21], but there have been almost no investigations on maximally steerable mixed states

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(MSMSs) until very recently. McCloskey *et al.* [22] explored the MSMS numerically with respect to some well-known quantifiers of steering, namely, the steering ellipsoid, the steering weight, and the robustness of steering. Except for the above identification criteria of quantum steering, it is well-known that steerability can be measured by the violations of Bell-like EPR steering inequalities. In this paper, we analytically derive the maximally steerable mixed states with the linear steering inequality and the Clauser-Horne-Shimony-Holt (CHSH)–like steering inequality. For comparison, we also investigated the violation of the inequalities with the Werner state and the MEMS.

The paper is organized as follows. In Sec. II, the forms of the MSMSs based on the two-setting linear inequality and the two-setting CHSH-like steering inequality are derived with tight and rigorous mathematical proofs. Interestingly, the forms of the MSMSs based on two such different steering inequalities are identical which is just the MNMS discussed in Ref. [21]. The performance of the maximal violation of such inequalities for the Werner state and the MEMS is thoroughly discussed. In Sec. III, we derive the form of the MSMS based on the three-setting linear steering inequality. It is the same as that in Sec. II. Similarly, the performance of the maximal violation of the three-setting linear steering inequality for the Werner state and the MEMS is also fully discussed and analyzed. Conclusions are drawn in Sec. IV.

II. THE MSMSs BASED ON THE STEERING INEQUALITIES

It is well known that the state of an arbitrary two-qubit system [23] can be reduced, by local unitary equivalence, to [24]

$$\rho = \frac{1}{4}\left(I + \sum_{i}^{3} r_i\sigma_i \otimes I + \sum_{j}^{3} s_j I \otimes \sigma_j + \sum_{k}^{3} \tau_k\sigma_k \otimes \sigma_k\right), \tag{1}$$

where σ_1, σ_2, and σ_3 are the three Pauli matrices, and without loss of generality we suppose $|\tau_1| \geqslant |\tau_2| \geqslant |\tau_3|$. Here we should note that an arbitrary two-qubit state can be converted to such a canonical form while preserving the steerability (or unsteerability), which has been proved in several previous work [25–27].

Now, our aim is to extract the MSMS from Eq. (1). The definition of a MSMS is a state that violates a steering inequality to the most degree for a given linear entropy. Equivalently, it can also be considered that the MSMS has the maximal linear entropy among the states which achieve the same amount of violation of the inequality. Here, we try to find the states which reach the maximal violation bound of a well-known two-setting linear steering inequality (the two-setting CHSH-like steering inequality) and which possess the maximal entropy in the two-qubit system.

In a seminal paper, Cavalcanti *et al.* [28] developed an inequality to diagnose whether a bipartite state is steerable when Alice and Bob are both allowed to measure n observables in their sites. Their inequality is usually called the *linear* steering inequality [28]. For qubits, we can take Bob's kth measurement setting to correspond to the Pauli observable $\hat{\sigma}_k$, along some axis u_k. By denoting Alice's corresponding declared result as the random variable $A_k \in \{+1,-1\}$ for all k, the EPR-steering inequality is of the following form [29]:

$$S_n = \frac{1}{n}\sum_{k=1}^{n}\langle A_k\hat{\sigma}_k\rangle \leqslant C_n. \tag{2}$$

The bound C_n is the maximum value S_n can have if Bob has a preexisting state known to Alice, rather than half of an entangled pair shared with Alice.

A. The MSMS based on the two-setting linear steering inequality

Here, we choose $n = 2$, and the optimal linear steering inequality reads

$$\langle A_1\hat{\sigma}_{1'}\rangle + \langle A_2\hat{\sigma}_{2'}\rangle \leqslant \sqrt{2}, \tag{3}$$

where $\hat{\sigma}_{1'}$ and $\hat{\sigma}_{2'}$ are mutually unbiased bases. This steering inequality, Eq. (3), is the general form of Eq. (19) in Ref. [3], which is considered the optimal steering inequality for the Werner state. Then one may ask: What is the optimal state that violates Eq. (3) maximally? That is, what is the optimal state that has the largest degree of steerability?

To address the problem we use the notion of linear entropy, a measure of state mixedness computed as $\varepsilon(\rho) = \frac{d}{d-1}(1 - \mathrm{Tr}\rho^2)$, with $d = 2^N$. So for the state in Eq. (1) the linear entropy equals

$$\varepsilon(\rho) = \frac{4}{3}(1 - \mathrm{Tr}\rho^2) = 1 - \frac{1}{3}\sum_{i,j,k=1}^{3}\left(r_i^2 + s_j^2 + \tau_k^2\right). \tag{4}$$

The quantum prediction of the left-hand side of Eq. (3) equals $S_2 = \langle\hat{A}_1\hat{\sigma}_{1'}\rangle + \langle\hat{A}_2\hat{\sigma}_{2'}\rangle$, where the random variable A_i is replaced with the observable $\hat{A}_i$. To obtain the maximal value of S_2, let us assume Alice's two observables can be expressed as $A_i \equiv \vec{\sigma}\cdot\vec{n}_i$, $i \in 1,2$. It is worth noticing that these maximal values for state (1) are tight [30]:

$$\mathrm{Max}(S_2) = \sqrt{2}\sqrt{\tau_1^2 + \tau_2^2}, \tag{5}$$

which is obtained by choosing the appropriate measurement directions of $\hat{A}_i$.

It is possible to find many states which simultaneously achieve a given degree of violation. Consider that the MSMS must possess the maximal entropy. According to Eq. (4), to maximize the linear entropy, $\mathrm{Tr}\rho^2$ should be minimized; i.e., all the coefficients of irrelevant terms of Eq. (1) must be chosen as zero: $r_i, s_j = 0$. The factors concerning the violation are τ_i's. Hence, we can write a matrix M which can maximally violate the linear steering inequality:

$$M = \tfrac{1}{4}[\tau_1\sigma_1 \otimes \sigma_1 + \tau_2\sigma_2 \otimes \sigma_2]. \tag{6}$$

Nevertheless, this matrix is not a density matrix, since as we know the trace of a density matrix always equals 1. By referring to Eq. (4), the maximal entropy implies the minimal $\mathrm{Tr}\rho^2$, and so the nonzero entries in the density matrix of the MSMS must be as few as possible.

To this end, we can only add four unknown coefficients in the diagonal elements of matrix M to make it a real density matrix. Let us denote the four unknown diagonal elements as f_1, f_2, f_3, and f_4. The new matrix M' can be written as

$$M' = \begin{pmatrix} \frac{f_1}{4} & 0 & 0 & \frac{\tau_1-\tau_2}{4} \\ 0 & \frac{f_2}{4} & \frac{\tau_1+\tau_2}{4} & 0 \\ 0 & \frac{\tau_1+\tau_2}{4} & \frac{f_3}{4} & 0 \\ \frac{\tau_1-\tau_2}{4} & 0 & 0 & \frac{f_4}{4} \end{pmatrix}. \quad (7)$$

As a real density matrix, M' must satisfy the trace and positive definite properties:

$$f_1 + f_2 + f_3 + f_4 = 4, \quad f_1 f_4 \geqslant (\tau_1 - \tau_2)^2, \\ f_2 f_3 \geqslant (\tau_1 + \tau_2)^2. \quad (8)$$

Further, the MSMS should always reach the maximal linear entropy:

$$\varepsilon(\rho) = \tfrac{4}{3}\big[1 - \tfrac{1}{16}\big(4\tau_1^2 + 4\tau_2^2 + 4\tau_3^2 + f_1^2 + f_2^2 + f_3^2 + f_4^2\big)\big] \\ \leqslant \tfrac{4}{3}\big[1 - \tfrac{1}{2}\big(\tau_1^2 + \tau_2^2\big)\big]. \quad (9)$$

Obviously, the equality sign is only achieved when $f_1 = f_4 = (\tau_1 - \tau_2)$ and $f_2 = f_3 = (\tau_1 + \tau_2)$. Hence, the maximal entropy equals

$$\mathrm{Max}[\varepsilon(\rho)] = \tfrac{4}{3} - \tfrac{2}{3}\big(\tau_1^2 + \tau_2^2\big). \quad (10)$$

According to Eq. (8), when the equality sign is achieved, we get $f_1 = f_4 = \tau_1 - \tau_2$ and $f_2 = f_3 = \tau_1 + \tau_2$, implying that $\tau_1 = 1$ must be satisfied as well. Therefore, the density matrix of the MSMS can be expressed as

$$\rho_{\mathrm{MSMS}} = \begin{pmatrix} \frac{1-\tau_2}{4} & 0 & 0 & \frac{1-\tau_2}{4} \\ 0 & \frac{1+\tau_2}{4} & \frac{1+\tau_2}{4} & 0 \\ 0 & \frac{1+\tau_2}{4} & \frac{1+\tau_2}{4} & 0 \\ \frac{1-\tau_2}{4} & 0 & 0 & \frac{1-\tau_2}{4} \end{pmatrix}. \quad (11)$$

This is the end of the proof of the MSMS based on the two-setting linear steering inequality.

Interestingly, the MSMS, Eq. (11), the proof of which is based on the linear steering inequality, is exactly the same as the MNMS [21], the proof of which is based on the CHSH inequality, reading

$$\rho_{\mathrm{MNMS}} = \frac{1+\tau_2}{2}\rho_1 + \frac{1-\tau_2}{2}\rho_2, \quad (12)$$

where τ_2 is a measure of admixture between the two orthogonal states $\rho_i = |\psi_i\rangle\langle\psi_i|$, with $|\psi_1\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ and $|\psi_2\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$. Hence, the MSMS, or equivalently the MNMS, is of real practice and can be seen as a state that frequently occurs during the experimental state preparation process.

In fact, the state can be considered as an imperfect Bell state of a bipartite spin-1/2 composite system with random spin flipping, where τ_2 represents a flipping coefficient. For this state, the maximum $\mathrm{Max}(S_2)$ equals $\sqrt{2}\sqrt{1+\tau_2^2}$ by choosing proper operators, and the linear entropy equals $\varepsilon(\rho) = (2/3)(1-\tau_2^2)$, with $\varepsilon(\rho) \in [0,2/3]$. Obviously, the relation between $\mathrm{Max}(S_2)$ and the linear entropy $\varepsilon(\rho)$ is $\mathrm{Max}(S_2) = \sqrt{2}\sqrt{2-(3/2)\varepsilon(\rho)}$. Thus, it has been shown that it is impossible to find a state for which the mixedness exceeds 2/3, violating the linear steering inequality (3).

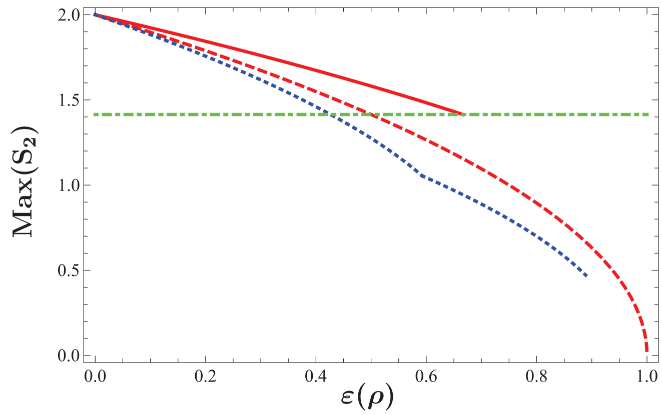

FIG. 1. The maximal violation of the two-setting linear steering inequality versus the linear entropy for the MSMS (red solid line), the MEMS (blue dotted line), and the Werner state (red dashed line).

For comparison, we also investigated the performance of the violation of the linear steering inequality with the Werner state and the MEMS. First, for the Werner state

$$\rho_{\mathrm{W}} = \gamma|\psi_1\rangle\langle\psi_1| + \frac{1-\gamma}{4}\mathrm{I}\otimes\mathrm{I}, \quad (13)$$

where the linear entropy is $\varepsilon(\rho) = (1-\gamma^2)$, it is easy to get $\mathrm{Max}(S_2) = 2\gamma$. Similarly, the relation between $\mathrm{Max}(S_2)$ and the linear entropy $\varepsilon(\rho)$ is $\mathrm{Max}(S_2) = 2\sqrt{1-\varepsilon(\rho)}$, with $\varepsilon(\rho) \in [0,1]$. Obviously, for the Werner state, when the linear entropy is in the range of $\varepsilon(\rho) \in [0,1/2]$, the steering inequality can be violated, which presents steering.

Secondly, the two-qubit MEMS reads

$$\rho_{\mathrm{MEMS}} = \begin{bmatrix} g & 0 & 0 & \frac{\gamma}{2} \\ 0 & 1-2g & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\gamma}{2} & 0 & 0 & g \end{bmatrix}, \quad (14)$$

where $g = 1/3$ for $0 \leqslant \gamma \leqslant 2/3$ and $g = \gamma/2$ for $2/3 < \gamma \leqslant 1$. Specifically, we obtain the following.

(i) For $0 \leqslant \gamma \leqslant 2/3$, the linear entropy is $\varepsilon(\rho) = (\frac{8}{9} - \frac{2}{3}\gamma^2)$, with $\varepsilon(\rho) \in [\frac{16}{27}, \frac{8}{9}]$. The optimal value of S_2 can achieve $\mathrm{Max}(S_2) = \frac{1}{3}\sqrt{2+18\gamma^2}$, and the relation between $\mathrm{Max}(S_2)$ and the linear entropy $\varepsilon(\rho)$ is $\mathrm{Max}(S_2) = \frac{1}{3}\sqrt{26-27\varepsilon(\rho)}$.

(ii) For $2/3 \leqslant \gamma \leqslant 1$, the linear entropy is $\varepsilon(\rho) = \frac{8}{3}(\gamma - \gamma^2)$ with $\varepsilon(\rho) \in [0, \frac{16}{27}]$. The optimal value of S_2 can achieve $\mathrm{Max}(S_2) = \sqrt{2+2\gamma(-4+5\gamma)}$, and the relation between $\mathrm{Max}(S_2)$ and the linear entropy $\varepsilon(\rho)$ is $\mathrm{Max}(S_2) = \frac{1}{2}\sqrt{2[6+\sqrt{4-6\varepsilon(\rho)}]-15\varepsilon(\rho)}$.

In Fig. 1, we plot the maximal violation of the linear steering inequality versus the linear entropy for the MSMS, the MEMS, and the Werner state. The red solid line is the violation for the MSMS. The red dashed line is for the Werner state, when the linear entropy is in the range of $\varepsilon(\rho) \in [0,1/2]$, the steering inequality can be violated which presents steering. The blue dotted line denotes the MEMS. When the linear entropy is larger than 0.4266, the MEMS will not violate the linear steering inequality.

B. The MSMS based on the CHSH-like steering inequality

Apart from the linear steering inequality, several other inequalities to identify Einstein-Podolski-Rosen steering have been proposed and experimentally implemented. Especially, Cavalcanti *et al.* [31] proposed a CHSH-like steering inequality, which is also considered as a very useful criteria of the quantification of EPR steering [30]. Especially, this steering inequality links the correlation between Bell nonlocality and quantum steering [32]. Here we investigate the form of MSMS based on this CHSH-like steering inequality.

Let us consider a simple scenario in which Alice performs two dichotomic measurements while Bob performs two mutually unbiased qubit measurements. The CHSH-like steering inequality can be written as

$$S_{\rm CHSH} = \sqrt{\langle(\hat{A}_1+\hat{A}_2)\hat{B}_1\rangle^2 + \langle(\hat{A}_1+\hat{A}_2)\hat{B}_2\rangle^2} + \sqrt{\langle(\hat{A}_1-\hat{A}_2)\hat{B}_1\rangle^2 + \langle(\hat{A}_1-\hat{A}_2)\hat{B}_2\rangle^2} \leqslant 2, \quad (15)$$

where the outcomes of each measurement are taken as $\{+1,-1\}$. It was shown recently that the maximal value that this inequality can reach is $2\sqrt{2}$ [30], which corresponds to the Cirel'son bound. For an arbitrary two-qubit state, Eq. (1), the maximal value of the right side of Eq. (15) equals

$$\mathrm{Max}(S_{\rm CHSH}) = 2\sqrt{\tau_1^2+\tau_2^2}. \quad (16)$$

Then, our task is to answer the following question: What is the optimal state that violates Eq. (15) maximally and shows the largest degree of steerability?

Following a procedure similar to the one above, we can obtain the form of the MSMS based on the CHSH-like steering inequality. Interestingly, the form we obtain is the same as the one based on the linear steering inequality (12). Not only for these inequalities but also for a general CHSH-like steering inequality defined in Ref. [32], the form of the MSMS obtained is identical to Eq. (12). The general CHSH-like steering inequality has a form identical to that of Eq. (15), except that Bob performs two arbitrary measurements instead. Due to the form of the maximal violation depend on the density matrix is identical to Eq. (16). It is very easy to show that the form of the MSMS is exactly the same as that in Eq. (12). For this state, the maximal amount of $\mathrm{Max}(S_{\rm CHSH})$ is $2\sqrt{1+\tau_2^2}$ by choosing proper operators, and the linear entropy is $\varepsilon(\rho)=(2/3)(1-\tau_2^2)$, with $\varepsilon(\rho)\in[0,2/3]$. The relation between $\mathrm{Max}(S_{\rm CHSH})$ and the linear entropy $\varepsilon(\rho)$ is $\mathrm{Max}(S_{\rm CHSH})=\sqrt{8-6\varepsilon(\rho)}$. Similarly, it is not possible to find a state for which the mixedness exceeds 2/3, violating the CHSH-like steering inequality.

As comparison, we also investigated the performance of the violation of the CHSH-like steering inequality for the Werner state and the MEMS. First, for the Werner state, it is easy to get that the optimal value of $S_{\rm CHSH}$ is $\mathrm{Max}(S_{\rm CHSH})=2\sqrt{2}\gamma$. Similarly, the relation between $\mathrm{Max}(S_{\rm CHSH})$ and the linear entropy $\varepsilon(\rho)$ is $\mathrm{Max}(S_{\rm CHSH})=2\sqrt{2}\sqrt{1-\varepsilon(\rho)}$, with $\varepsilon(\rho)\in[0,1]$. It is obvious that the Werner state achieves a smaller violation of the CHSH-like steering inequality than that with the MSMS, in the range $\varepsilon(\rho)\in[0,2/3]$ of the linear entropy.

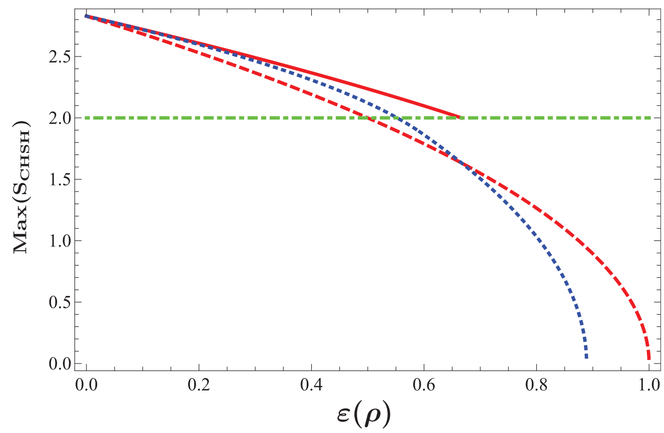

FIG. 2. The maximal violation of the CHSH-like steering inequality versus the linear entropy for the MSMS (red solid line), the MEMS (blue dotted line), and the Werner state (red dashed line).

Secondly, for the MEMS we have the following.

(i) For $0\leqslant\gamma\leqslant 2/3$, the optimal value of $S_{\rm CHSH}$ is $\mathrm{Max}(S_{\rm CHSH})=2\sqrt{2}\gamma$, and the relation between $\mathrm{Max}(S_{\rm CHSH})$ and the linear entropy $\varepsilon(\rho)$ is $\mathrm{Max}(S_{\rm CHSH})=2\sqrt{\frac{8}{3}-3\varepsilon(\rho)}$, with $\varepsilon(\rho)\in[\frac{16}{27},\frac{8}{9}]$.

(ii) For $2/3\leqslant\gamma\leqslant 1$, the optimal value of $S_{\rm CHSH}$ is $\mathrm{Max}(S_{\rm CHSH})=2\sqrt{2}\gamma$, and the relation between $\mathrm{Max}(S_{\rm CHSH})$ and the linear entropy $\varepsilon(\rho)$ is $\varepsilon(\rho)\in[0,\frac{16}{27}]$ and $\mathrm{Max}(S_{\rm CHSH})=\sqrt{2}+\sqrt{2-3\varepsilon(\rho)}$.

In Fig. 2, we plot the maximal violation of the CHSH-like steering inequality versus the linear entropy for the MSMS, the MEMS, and the Werner state. The red solid line is the violation for the MSMS. The red dashed line is for the Werner state, showing clearly that when the linear entropy is larger than 1/2, the Werner state does not violate the inequality. The blue dotted line denotes the MEMS. When the linear entropy is larger than 0.5522, the MEMS does not violate the inequality.

Moreover, this result confirms the previous conclusion in Ref. [32], in which it was shown that all states that are EPR steerable with CHSH-type correlations are also Bell nonlocal, and so the MSMS based on the CHSH-like steering inequality is just the MNMS, which violates the CHSH inequality and the CHSH-like steering inequality in the whole range. It is easy to obtain that for both the Werner state and the MEMS, when $\gamma > 1/\sqrt{2}$, the CHSH inequality and the CHSH-like steering inequality can be violated simultaneously. Despite that the two-setting linear steering inequality (3) is only a sufficient condition for steerability, the form of the MSMS derived from it is the same as the one derived from the CHSH-like steering inequality, implying that the two-setting linear steering inequality is equally effective with the CHSH-like steering inequality for certain steering tests. It is in some sense not strange that the two-setting linear steering inequality is the optimal steering inequality for the Werner state as shown in Ref. [3]. For the MEMS, nevertheless, the two-setting linear steering inequality can be violated when $\gamma > 4/5$, while the CHSH inequality can be violated when $\gamma > 1/\sqrt{2}$.

C. The MSMS based on the three-setting linear steering inequality

Now we choose $n = 3$, and the optimal linear steering inequality reads

$$\langle A_1\hat{\sigma}_{1'}\rangle + \langle A_2\hat{\sigma}_{2'}\rangle + \langle A_3\hat{\sigma}_{3'}\rangle \leqslant \sqrt{3}, \tag{17}$$

where $\hat{\sigma}_{1'}$, $\hat{\sigma}_{2'}$, and $\hat{\sigma}_{3'}$ are mutually unbiased bases. Interestingly, we can find the MSMS form based on this inequality is also the same as the MSMS that we obtained above, even though the constraints are different. Here we give the proof.

Similar to the two-setting case, the quantum prediction of the left-hand side of Eq. (17) equals $S_3 = \langle\hat{A}_1\hat{\sigma}_{1'}\rangle + \langle\hat{A}_2\hat{\sigma}_{2'}\rangle + \langle\hat{A}_3\hat{\sigma}_{3'}\rangle$, where the random variable A_i is replaced with the observable $\hat{A}_i$. To obtain the maximal value of S_3, we assume Alice's three observables can be expressed as $\hat{A}_i \equiv \vec{\sigma}\cdot\vec{n}_i$, $i \in 1,2,3$; it is worth noticing that these maximal values for state (1) are tight [30]:

$$\mathrm{Max}(S_3) = \sqrt{3}\sqrt{\tau_1^2 + \tau_2^2 + \tau_3^2}, \tag{18}$$

which is obtained by choosing the appropriate measurement directions of $\hat{A}_i$.

Again, according to Eq. (4), let us minimize $\mathrm{Tr}\rho^2$ in order to maximize the linear entropy. Namely, all the coefficients of the irrelevant terms of Eq. (1) must be chosen as zero: $r_i, s_j = 0$. The factors concerning the violation are τ_i's. Hence, we can write a matrix M which can maximally violate the linear steering inequality:

$$M = \tfrac{1}{4}[\tau_1\sigma_1\otimes\sigma_1 + \tau_2\sigma_2\otimes\sigma_2 + \tau_3\sigma_3\otimes\sigma_3]. \tag{19}$$

Following the same argument below Eq. (6), the nonzero entries in the density matrix of the MSMS must be as few as possible. To this end, we can only add four unknown coefficients in the diagonal elements of matrix M to make it a real density matrix. Let us denote the four unknown diagonal elements as f_1, f_2, f_3, and f_4. The new matrix M' can be written as

$$M' = \begin{pmatrix} \frac{f_1+\tau_3}{4} & 0 & 0 & \frac{\tau_1-\tau_2}{4} \\ 0 & \frac{f_2-\tau_3}{4} & \frac{\tau_1+\tau_2}{4} & 0 \\ 0 & \frac{\tau_1+\tau_2}{4} & \frac{f_3-\tau_3}{4} & 0 \\ \frac{\tau_1-\tau_2}{4} & 0 & 0 & \frac{f_4+\tau_3}{4} \end{pmatrix}. \tag{20}$$

As a real density matrix, M' must satisfy the trace and positive definite properties:

$$\begin{aligned} &f_1 + f_2 + f_3 + f_4 = 4, \\ &(f_1+\tau_3)(f_4+\tau_3) \geqslant (\tau_1-\tau_2)^2, \\ &(f_2-\tau_3)(f_3-\tau_3) \geqslant (\tau_1+\tau_2)^2. \end{aligned} \tag{21}$$

Further, the MSMS should always reach the maximal linear entropy:

$$\begin{aligned} \varepsilon(\rho) &= \tfrac{4}{3}\big[1 - \tfrac{1}{16}\big[4\tau_1^2 + 4\tau_2^2 + 4\tau_3^2 + f_1^2 + f_2^2 + f_3^2 + f_4^2 \\ &\quad + 2\tau_3(f_1 - f_2 - f_3 + f_4)\big]\big] \\ &\leqslant \tfrac{4}{3}\big[1 - \tfrac{1}{16}\big[4\tau_1^2 + 4\tau_2^2 + 4\tau_3^2 + 2f_1f_4 + 2f_2f_3 \\ &\quad + 2\tau_3(f_1 - f_2 - f_3 + f_4)\big]\big]. \end{aligned} \tag{22}$$

Obviously, the equality sign is only achieved when $f_1f_4 + \tau_3(f_1+f_4) = (\tau_1-\tau_2)^2 - \tau_3^2$ and $f_2f_3 - \tau_3(f_2+f_3) = (\tau_1+\tau_2)^2 - \tau_3^2$. Hence, the maximal entropy equals

$$\mathrm{Max}[\varepsilon(\rho)] = \tfrac{4}{3} - \tfrac{2}{3}\big(\tau_1^2 + \tau_2^2\big). \tag{23}$$

According to Eq. (21), when the equality sign is achieved, we get $f_1 = f_4 = \tau_1 - \tau_2 - \tau_3$ and $f_2 = f_3 = \tau_1 + \tau_2 + \tau_3$, implying that $\tau_1 = 1$ must be satisfied as well. Therefore, the density matrix of the MSMS can be expressed as

$$\rho_{\mathrm{MSMS}} = \begin{pmatrix} \frac{1-\tau_2}{4} & 0 & 0 & \frac{1-\tau_2}{4} \\ 0 & \frac{1+\tau_2}{4} & \frac{1+\tau_2}{4} & 0 \\ 0 & \frac{1+\tau_2}{4} & \frac{1+\tau_2}{4} & 0 \\ \frac{1-\tau_2}{4} & 0 & 0 & \frac{1-\tau_2}{4} \end{pmatrix}. \tag{24}$$

This is the end of the proof of the MSMS based on the three-setting linear steering inequality (17), where the MSMS form of Eq. (24) is just the state defined by Eq. (12). Here, we should note that, as the linear steering inequality in the three-settings steering scenario is only a sufficient condition but not necessary, this form may be a subset of all the MSMSs in the three-setting steering scenario which is still an open question.

For comparison, we also investigated the performance of the violation of the linear steering inequality with the Werner state and the MEMS. First, for the Werner state, it is easy to get $\mathrm{Max}(S_3) = 3\gamma$. Similarly, the relation between $\mathrm{Max}(S_3)$ and the linear entropy $\varepsilon(\rho)$ is $\mathrm{Max}(S_3) = 3\sqrt{1-\varepsilon(\rho)}$, with $\varepsilon(\rho) \in [0,1]$. Interestingly, the Werner state can achieve the same optimal violation of the linear steering inequality as the MSMS with the fixed linear entropy in the range of $\varepsilon(\rho) \in [0,2/3]$. Here we should emphasize that, for a fixed linear entropy, the state which can make the steering inequality be violated maximally is not unique. However, there are no states that can make it be violated higher than the MSMS.

Secondly, for the MEMS we have the following.

(i) For $0 \leqslant \gamma \leqslant 2/3$, the linear entropy is $\varepsilon(\rho) = (\frac{8}{9} - \frac{2}{3}\gamma^2)$, with $\varepsilon(\rho) \in [\frac{16}{27}, \frac{8}{9}]$. The optimal value of S_3 can achieve $\mathrm{Max}(S_3) = \gamma + \frac{1}{3}\sqrt{2+18\gamma^2}$, and the relation between $\mathrm{Max}(S_3)$ and the linear entropy $\varepsilon(\rho)$ is $\mathrm{Max}(S_3) = \sqrt{\frac{4}{3} - \frac{3}{2}\varepsilon(\rho)} + \frac{1}{3}\sqrt{26 - 27\varepsilon(\rho)}$.

(ii) For $2/3 \leqslant \gamma \leqslant 1$, the linear entropy is $\varepsilon(\rho) = \frac{8}{3}(\gamma - \gamma^2)$, with $\varepsilon(\rho) \in [0, \frac{16}{27}]$. The optimal value of S_3 can achieve $\mathrm{Max}(S_3) = \gamma + \sqrt{2 + 2\gamma(-4+5\gamma)}$, and the relation between $\mathrm{Max}(S_3)$ and the linear entropy $\varepsilon(\rho)$ is $\mathrm{Max}(S_3) = \frac{1}{4}\{2 + 2\sqrt{2[6+\sqrt{4-6\varepsilon(\rho)}] - 15\varepsilon(\rho)} + \sqrt{4 - 6\varepsilon(\rho)}\}$.

In Fig. 3, we plot the maximal violation of the linear steering inequality versus the linear entropy for the MSMS, the MEMS, and the Werner state. The red solid line is the violation for the MSMS. For the Werner state, when the linear entropy is in the range of $\varepsilon(\rho) \in [0,2/3]$, the result is identical with that of the MSMS. The red dashed line is for the Werner state when $\varepsilon(\rho) \in [2/3,1]$. The blue dotted line denotes the MEMS. When the linear entropy is larger than 0.5897, the MEMS will not violate the linear steering inequality.

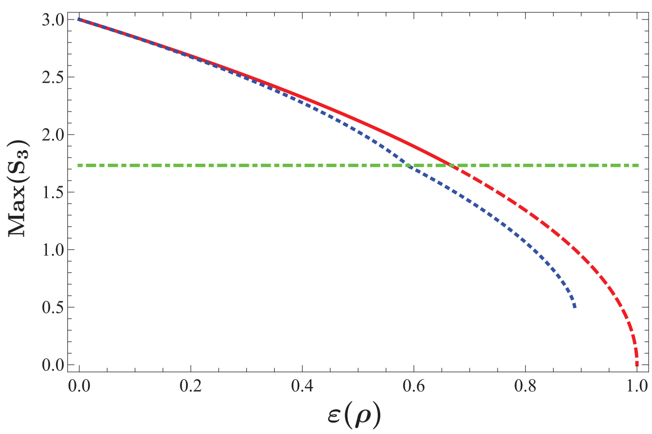

FIG. 3. The maximal violation of the linear steering inequality versus the linear entropy for the MSMS (red solid line), the MEMS (blue dotted line), and the Werner state (red dashed line).

Besides, it is worth comparing the performance between the two-setting steering scenario and the three-setting steering scenario. The forms of the MSMS derived from the two- and the three-setting linear steering inequalities are identical. But for the Werner state in the range of $\varepsilon(\rho) \in [1/2, 2/3]$, the three-setting linear steering inequality can detect steering, while the two-setting linear steering inequality cannot. Similarly, for the MEMS, when $\varepsilon(\rho) \in [0.4266, 0.5897]$, the steerable can be tested by the three-setting linear steering inequality but not the two-setting linear steering inequality. The reason for such a difference is that in the three-setting scenario, the local parties perform more measurements than in the two-setting scenario, and hence can detect more entangled states (see also the blue dotted curves in Figs. 1 and 3).

III. CONCLUSION

In conclusion, we derived the maximal violation of the two-setting (three-setting) linear steering inequality and the CHSH-like steering inequality, respectively, with a fixed state mixedness. Three forms of MSMSs based on such different steering inequalities were derived mathematically. Surprisingly, the three forms are the same as one another and as the MNMS in Ref. [21]. The upper bound of their linear entropy in the range of violating the steering inequalities is $2/3$, which implies it is impossible to find a state other than (1) so that the state mixedness exceeds $2/3$ and violates the inequalities.

As comparison, we also investigated the performance of the maximal violations of such different inequalities for the Werner state and the MEMS. First of all, for the two-setting linear steering inequality and the CHSH-like inequality, when the linear entropy of the Werner state is larger than $1/2$ (i.e., $\gamma = 1/\sqrt{2}$), there is no violation, which is just corresponding to the bound of the Bell nonlocality tested by the CHSH inequality. But for the MEMS, the two-setting linear steering inequality can be violated when $\gamma > 4/5$, while the CHSH-like steering inequality can be violated when $\gamma > 1/\sqrt{2}$, coinciding with the result of the Bell nonlocal test by the violation of the CHSH inequality. It has been clearly shown that the steerable state of the MEMS tested by the two-setting linear inequality is a subset of that of the MEMS tested by the CHSH-like steering inequality. The reason is that the CHSH-like steering inequality is the necessary and sufficient condition of steering in the two-setting steering scenario, but the two-setting linear steering inequality is only a sufficient condition for steerability in such a steering scenario. Second, for the three-setting linear steering inequality, the Werner state with $\varepsilon(\rho) \in [0, 2/3]$ achieves the same amount of maximal violation as the MSMS does. The MEMS will not violate the linear steering inequality when its linear entropy is larger than 0.5897.

Furthermore, it is worth comparing the performance between the two-setting steering scenario and the three-setting steering scenario. For the two- and three-setting linear steering inequalities, identical MSMSs can be derived based on two such steering inequalities. However, this is not true of the Werner state while in the range of $\varepsilon(\rho) \in [1/2, 2/3]$, the steerability of which can be detected by the three-setting linear steering inequality, but cannot be detected by the two-setting linear steering inequality. For the MEMS, similarly, when $\varepsilon(\rho) \in [0.4266, 0.5897]$, the steerability can be detected by the three-setting linear steering inequality but cannot be detected by the two-setting linear steering inequality. The reason for these results is that in the three-setting scenario, the parties perform more measurements than in the two-setting scenario. In the three-setting scenario, there are more entangled states that can violate the steering inequality than in the two-setting scenario. Besides, we should emphasize again that, as the linear steering inequality in the three-settings steering scenario is only a sufficient condition but not necessary, this form may be a subset of all the MSMSs in the three-setting steering scenario which is still an open question.

Moreover, it is necessary to discuss the MSMSs with respect to different steering criteria. In this paper, the forms of MSMSs were derived based on several different steering inequalities, while the previous results in Ref. [22] were derived based on the steering ellipsoid, the steering weight, and the robustness of steering. It is intriguing to note that while in this paper the MSMSs take the same form for various steering inequalities, the MSMSs based on different steering criteria may be different in general. On the other hand, the MSMSs have interesting links with the MNMS or the MDMS. For instance, the MSMSs based on Bell-type steering inequalities are the same as the MNMS; the MSMS based on the steering ellipsoid or the steering weight in Ref. [22] is the same as the MDMS in a large range of $\varepsilon(\rho) \in [0, 2/3]$. Bell-type steering inequalities originate from Bell inequalities by detecting Bell nonlocality, and so the MSMS derived from Bell-type steering inequalities is closely linked with the MNMS, given that the MSMS and the MNMS achieve the same maximal quantum violation. Similarly, the steering ellipsoid is an effective method for detecting quantum discord [33], and so the MSMS derived from the steering ellipsoid or the steering weight is closely linked with the MDMS. Moreover, the MSMS derived from the robustness of steering is, in general, a different family of optimal states.

The method we use in the present paper provides a particularly new perspective to understand the quantum steering for mixed states. It is reasonable to conclude from our results that the two-qubit MSMSs are indeed the most noise-resistant resource for any steering-based quantum information and computation protocols, such as one-sided device-independent quantum key distribution, quantum teleportation and subchannel discrimination, steering-based random number generator, etc.

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