Tensor products of process matrices with indefinite causal structure

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Theories with indefinite causal structure have been studied from both the fundamental perspective of quantum gravity and the practical perspective of information processing. In this paper we point out a restriction in forming tensor products of objects with indefinite causal structure in certain models: there exist both classical and quantum objects the tensor products of which violate the normalization condition of probabilities, if all local operations are allowed. We obtain a necessary and sufficient condition for when such unrestricted tensor products of multipartite objects are (in)valid. This poses a challenge to extending communication theory to indefinite causal structures, as the tensor product is the fundamental ingredient in the asymptotic setting of communication theory. We discuss a few options to evade this issue. In particular, we show that the sequential asymptotic setting does not suffer the violation of normalization.

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I. INTRODUCTION

Modern studies of indefinite causal structure in operational probabilistic theories began with Hardy's seminal works of the causaloid framework [1,2]. The main motivation to study indefinite causal structures is the expectation that in quantum gravity causal structures (dynamical degrees of freedom of gravity) should subject to indefiniteness (as other dynamical degrees of freedom do in ordinary quantum theory). The causaloid framework includes complex Hilbert-space theory and other probabilistic theories such as real Hilbert-space theory as special cases. This is to take into account the possibility that the ultimate theory of quantum gravity is based on mathematical structures beyond complex Hilbert space. Apart from interest in fundamental physics, Hardy also pointed out that the new framework suggests useful applications for practical information processing—a quantum computer that takes advantage of indefinite causal structure may outperform a quantum computer that does not [3].

The program Hardy initiated more than ten years ago has since then seen a boom across different areas of physics and information science. Chiribella, D'Ariano, and Perinotti [4] developed an important framework for quantum networks in complex Hilbert space for general purposes from both a constructive and a neat axiomatic perspective (a framework based on similar mathematical content was developed to study quantum games previously [5]). Although the framework of [4] still assumes a definite causal order (represented by a directed acyclic graph) among the elementary circuits, the mathematical elements that enable indefinite causal order are already present. The framework is causally neutral in the sense that all objects (including both channels and states) are represented through the Choi isomorphism [6] as an operator, and a composition rule (the link product) is given to specify how the operators compose and offer predictions of probabilities. Given this general setup one could already talk about an operational probabilistic theory without specifying a definite causal order for the elementary circuits. Indeed, in [7] (see further developments in [8]), the original framework of [4] was developed to include indefinite causal order, and a computation protocol that cannot be reproduced by definite causal order computation is given. This explicit protocol confirms Hardy's previous suggestion that "quantum gravity computers" outperform ordinary quantum computers [3], and has attracted much attention from both theoreticians and experimentalists.

Another important framework in the study of indefinite causal structure is the process matrix framework by Oreshkov, Costa, and Brukner [9] (see also [10,11]). This framework is devised from the outset to incorporate indefinite causal structure into complex Hilbert-space quantum theory. As in [4], the process matrix framework represents objects such as channels and states as operators through the Choi isomorphism. A series of works based on this framework were carried out to study the new features indefinite causal structure brings to quantum theory (e.g., [12–14]), and we are gathering an increasingly better understanding of "quantum causality" (see [15] for a review of related works and other frameworks including the duotensor [16] and quantum conditional states [17]). The introduction of indefinite causal structure into quantum theory not only brought in new understandings but also supplied some new questions. In particular, the subtlety in taking tensor products of objects with indefinite causal structure was known to some researchers in the community and was explicitly mentioned in [18] and implicitly encompassed in the construction of the combs formulation in [8]. In this paper, we study this issue in more detail. We provide a necessary and sufficient condition (Theorem 1) to characterize objects the tensor products of which need qualification. We show that for

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this set of objects (which includes both classical and quantum ones) the straightforward application of tensor product can lead to violations of the normalization condition for probability. We discuss some possible alternate ways to take tensor products that avoid such a conflict. In addition, we pose it as an important open question to the indefinite causal structure community to clarify whether the need to qualify tensor products imposes restrictions on realizing objects with indefinite causal structure in the laboratory.

A main motivation of the present paper is to pave the way to develop communication theory for theories with indefinite causal structure. In the usual Shannon asymptotic setting one takes multiple copies of the communication resource such as a channel or a state to define capacity. The result in this paper shows that this setting cannot be extended to quantum theory with indefinite causal structure with straightforward application of tensor products that imposes no restrictions on the allowed local operations. We show that if one carefully distinguishes the parallel and sequential asymptotic settings then the sequential asymptotic setting can be extended to objects with indefinite causal structure.

II. PROCESSES

The major result of this paper (Theorem 1) is based on a lemma (Proposition 1 below) the content of which is phrased and proved by Oreshkov and Giarmatzi [11] in the process matrix framework. We note that the present paper can equally be carried out in other frameworks such as those in [4,7]. For convenience of directly applying the lemma we based the study in the process matrix framework. In this section we very briefly recall the relevant part of the process matrix framework [9–11] and introduce some nomenclatures. It is postulated that local physics is described by ordinary quantum theory with definite causal structure, while the global (possibly indefinite) causal structure is described by processes. A process is a linear map from CP maps describing local physics to real numbers describing probabilities of observation outcomes. Both channels and states are special cases of processes.

We use $A,B,C,\cdots$ to denote the parties where local physics takes place. A party A is associated with an input system a_1 with Hilbert space $\mathcal{H}^{a_1}$ and an output system a_2 with Hilbert space $\mathcal{H}^{a_2}$. Through the Choi isomorphism [6] processes can be represented as linear operators. A process W associated with parties $A,B,C,\cdots$ is represented as a linear operator $W^{a_1a_2b_1b_2c_1c_2\cdots} \in L(\mathcal{H})$, where $\mathcal{H} := \mathcal{H}^{a_1}\otimes\mathcal{H}^{a_2}\otimes\mathcal{H}^{b_1}\otimes\mathcal{H}^{b_2}\otimes\mathcal{H}^{c_1}\otimes\mathcal{H}^{c_2}\otimes\cdots$. Sometimes we write the process as $W^{abc\cdots}$ for simplicity.

It is assumed that processes yield positive and normalized outcome probabilities for physical reasons, and can also act on subsystems of local parties. These imply that

$$W \geqslant 0, \tag{1}$$

$$\mathrm{Tr}W = \dim(\mathcal{H}^{a_2}\otimes\mathcal{H}^{b_2}\otimes\mathcal{H}^{c_2}\otimes\cdots) =: d_O, \tag{2}$$

and in addition the following proposition.

Proposition 1 (Oreshkov and Giarmatzi [11], reformulated using the language of this paper). A multipartite process obeys the normalization of probability condition if and only if in addition to the identity term it contains at most terms which are type a_1 on some party A.

To understand this proposition we need to introduce the notion of types. A process $W^{ab\cdots}$ can be expanded in the Hilbert-Schmidt basis $\{\sigma_i^x\}_{i=0}^{d_x^2-1}$ of the x subsystem operators $L(\mathcal{H}^x)$ as

$$W^{ab\cdots} = \sum_{i,j,k,l,\cdots} w_{ijkl\cdots}\sigma_i^{a_1}\otimes\sigma_j^{a_2}\otimes\sigma_k^{b_1}\otimes\sigma_l^{b_2}\otimes\cdots, \quad \times\, w_{ijkl\cdots}\in\mathbb{R}. \tag{3}$$

We set the convention to take σ_0^x to be 11 for any subsystem x. We refer to terms of the form $\sigma_i^x\otimes 11^{\text{rest}}$ for $i\geqslant 1$ as a type x term, $\sigma_i^x\otimes\sigma_j^y\otimes 11^{\text{rest}}$ for $i,j\geqslant 1$ as a type xy term, etc. The identity term is referred to as a trivial term to be of trivial type.

Restricting attention to some party A, we say that a term is "in-type" ("out-type") on A if it is a_1 (a_2) type on the system a_1a_2, regardless of what type it is on the systems of other parties. We say that it "includes the in-type" ("includes the out-type") on A if it is a_1 or a_1a_2 (a_2 or a_1a_2) type on the system a_1a_2, regardless of what type it is on the systems of other parties. On two parties A and B, terms of type a_2b_1 and $a_1a_2b_1$ are called A to B signaling terms, since A's output is correlated with B's input. Similarly terms of type a_1b_2 and $a_1b_1b_2$ are called B to A signaling terms.

In the following section we introduce process products and prove our main results.

III. CONDITIONS FOR FORMING VALID AND INVALID PRODUCTS

A party $\{A',a_1',a_2'\}$ and a party $\{A'',a_1'',a_2''\}$ can be combined into a new party $\{A,a_1,a_2\}=\{A'A'',a_1'a_1'',a_2'a_2''\}$ if all channels from a_1 to a_2 can be applied by A. Here $A'A''$ is a shorthand notation for combining parties A' and A'', and xy is a system the Hilbert space of which is $\mathcal{H}_x\otimes\mathcal{H}_y$.

Such products of parties are implicitly used when one forms tensor products of channels. A channel M is a two-party resource that mediates information between some party A' and some party B'. Given any other channel N mediating information between A'' and B'', the tensor product $M\otimes N$ is a channel associated with $A=A'A''$ and $B=B'B''$, where all channels from a_1 to a_2 can be applied by A and all channels from b_1 to b_2 can be applied by B.

Such tensor products are crucial in information theory, as one often studies tasks in the asymptotic setting, where the same resource is used arbitrarily many times. Out of interest, for example in quantum gravity and in particular quantum black holes, we want to study information communication theory of processes with indefinite causal structure [19]. In order to consider the asymptotic setting for processes we need to define products of processes and check if they are valid processes. Analogous to channel products, for two processes $W^{a'b'\cdots}$ and $Z^{a''b''\cdots}$ with the same number of parties, we tentatively define their product as $P^{ab\cdots}=W^{a'b'\cdots}\otimes Z^{a''b''\cdots}$. It takes in channels of parties $A,B,\cdots$ and outputs probabilities where $A=A'A''$, $B=B'B''$, $\cdots$. The situation for two parties is illustrated in Fig. 1. Following this construction, the asymptotic setting of a two-party process W^{ab} would require a process $W^{ab\otimes n}=W^{a'b'}\otimes W^{a''b''}\cdots$ that is an n-fold tensor product.

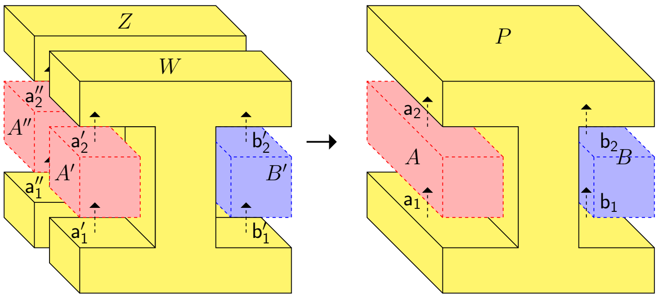

FIG. 1. An example of a process product for two bipartite processes.

The product parties $A = A'A''\cdots$ and $B = B'B''\cdots$ each represent a localized region of spacetime where all channels are allowed.

This asymptotic setting without restriction on the allowed local operations holds without problems for quantum theory with definite causal structure. However, a simple example shows that if arbitrary local operations are allowed, process products are not always valid processes. Consider the process

$$W^{xy} = \frac{d_O}{2}(\omega^{x_1} \otimes \rho_{\sim}^{x_2 y_1} \otimes \omega^{y_2} + \omega^{x_2} \otimes \rho_{\sim}^{x_1 y_2} \otimes \omega^{y_1}), \quad (4)$$

where d_O is the dimension of the outputs, ω is the maximally mixed state, and $\rho_{\sim} := (\mathbb{1} + \sigma_3 \otimes \sigma_3)/4$ is a maximally correlating state that can represent a classical identity channel in the $\{|0\rangle, |1\rangle\}$ basis. This process can be viewed as an equal-weight classical mixture of a channel from x to y and another one from y to x. Suppose A' and B' share a process $W^{a'b'}$ of this form, and A'' and B'' also share a process $W^{a''b''}$ of the same form. The operator $W^{ab} := W^{a'b'} \otimes W^{a''b''}$ for the two parties $A = A'A''$ and $B = B'B''$ is not a valid process. $\rho_{\sim}^{a'_2 b'_1} \otimes \rho_{\sim}^{a''_1 b''_2}$ includes a term $\sigma_3^{a'_2} \otimes \sigma_3^{a''_1} \otimes \sigma_3^{b'_1} \otimes \sigma_3^{b''_2}$, which leads to a type $a_1a_2b_1b_2$ term and according to Proposition 1 renders the process W^{ab} invalid. Intuitively, $\rho_{\sim}^{a'_2 b'_1} \otimes \rho_{\sim}^{a''_1 b''_2}$ creates a causal loop and violates the normalization of probability condition.

Although $W^{a'b'}$ and $W^{a''b''}$ cannot be composed directly, it is possible to have a global process P^{ab} that reduces to the two individual processes upon partial tracing. For example, let A and B have a process of the same form as (4). This is a process on the combined parties. The reduced processes $\mathrm{Tr}_{a''b''} W^{ab}$ and $\mathrm{Tr}_{a'b'} W^{ab}$ are exactly $W^{a'b'}$ and $W^{a''b''}$.

The restriction of process products has an analogy with the "nonseparability" of entangled states in quantum theory. If ρ^{xy} is entangled, then $\rho^{xy} \neq \rho^x \otimes \rho^y$, and similarly for some processes $W^{ab} \neq W^{a'b'} \otimes W^{a''b''}$. The difference is that for processes tensor products not only may not recover the original process but may even be invalid.

Note that the processes in the example can be viewed as classical because one can regard it as a classical mixture of classical resources. One can also substitute Choi states of quantum channels for those of classical channels to obtain an example of quantum process that is restricted in forming products. The invalidity of arbitrary products is a feature of quantum as well as classical resources.

We also note that even for channels there exists a similar subtlety in forming products [18,20–22]. Given a channel M from A' to B' and another one N from A'' and B'', the product $M \otimes N$ from $A'B''$ and $B'A''$ allows a causal loop and does not preserve probability. Because channels exist in ordinary quantum theory with definite causal structure, one may be tempted to say that the subtlety in forming products is not a new issue brought about by indefinite causal structure theories. It is, however, debatable whether the above construction is allowed by a theory with definite causal structure. For example, in the general framework of quantum networks [4] this kind of construction is explicitly forbidden by the first quantum combs tensor product rule. The rule requires the preservation of relative ordering of the original systems, and is well motivated in the context of definite causal structure. In any case the restriction of tensor products is more manifest for processes than for channels. For M and N although one cannot combine local parties as $A'B''$ and $B'A''$, one can always combine them as $A'A''$ and $B'B''$ to form a valid tensor product channel. On the other hand, it will be clear from the results below that there are processes W and Z for which neither way of combination leads to a valid process.

The first main result of the paper is the following necessary and sufficient condition characterizing when two general multipartite processes cannot (and can) be composed into a valid product process when arbitrary local operations are allowed.

Theorem 1. The product $P = W \otimes Z$ of two processes W and Z is not a valid process if and only if there exist a nontrivial term of W and a nontrivial term of Z that obey the following.

(1) On any party where one term is trivial, the other is either trivial or includes the out-type.

(2) On any party where one term is the in-type, the other term includes the out-type.

Proof. Suppose W and Z satisfy the conditions and consider the tensor product of the two nontrivial terms. By Proposition 1, to prove that P is invalid we need to show that P contains a nontrivial term that is not type a_1 for any party A. Conditions 1 and 2 guarantee that this is satisfied for the product term we consider.

Conversely, suppose P is not a valid process. By Proposition 1, P contains a nontrivial term that is not type a_1 for any party A. This term must arise out of a tensor product of nontrivial terms over subparties A', A''. On any A this term of P is type trivial, a_1a_2 or a_2. We consider each case in turn. For any A where this term is trivial, it must come from a tensor product of terms that are trivial on A. Write this kind of tensor product as (0,0), where zero denotes the trivial type and (.,.) denote unordered term pairs over A', A''. Next, for any A where this term is type a_1a_2, it must come from a tensor product of the kind (0,12), (1,12), (1,2), (2,12), or (12,12), where 1 and 2 denote in- and out-type, respectively. Finally, for any A where this term is type a_2, it must come from a tensor product of the kind (0,2) or (2,2). To sum up, on any A, this term of P comes from a tensor product of the kind (0,0), (0,12), (0,2), (1,12), (1,2), (2,12), (2,2), or (12,12). This implies conditions 1 and 2. ■

A useful special case is the condition on two-party processes. Intuitively, the fulfillment of the two conditions in the corollary below gives rise to causal loops, which violate the normalized probability condition for processes and hence leads to invalid products.

Corollary 1. A product $P^{ab} = W^{a'b'} \otimes Z^{a''b''}$ of two-party processes is not a valid process if and only if the following are true.

(1) Both W and Z have signaling terms.

(2) The Hilbert-Schmidt terms of W and Z put together contain signaling terms of both directions.

Proof. Suppose W and Z obey the two conditions. Then we can pick a signaling term from W of one direction and a signaling term from Z of the other direction. We show that this pair of terms satisfies conditions 1 and 2 in Theorem 1, and hence the product is not a valid process. Neither term is trivial on either of the two parties, so condition 1 of Theorem 1 is fulfilled. Condition 2 is also fulfilled because the terms signal to different directions.

Conversely, suppose P is not valid. By Theorem 1, there is a nontrivial term from W and a nontrivial term from Z that obey conditions 1 and 2 of Theorem 1. By Proposition 1, both terms are in-type on some party. By condition 2 of Theorem 1 they must be in-type on different parties, and they include the out-type on the parties where they are not in-type. In other words, they are signaling terms to different directions. This proves conditions 1 and 2 of the statement. ■

A product of more than two processes can be constructed iteratively, and the validity of the product process must be checked at each step. If a set of processes cannot form a valid product in one sequence of construction, changing the sequence of construction will not make it valid. This is because the invalid term will always be present. Corollary 1 allows us to straightforwardly identify those two-party processes for which the Shannon asymptotic setting without restriction on local operations is (in)valid.

Corollary 2. The n-fold tensor product $W^{ab\otimes n} = W^{a'b'} \otimes W^{a''b''} \cdots$ of a process W^{ab} with itself is not a valid process if and only if it contains signaling terms of both directions.

The above results show that the asymptotic setting without restriction on local operations does not hold for all processes. They suggest two ways to make sense of the asymptotic setting. One can either restrict attention to those processes that have valid products (as characterized by Theorem 1) or try to find a restricted set of local operations for which the products do not violate the normalization condition of the framework. One option is to only allow nonsignaling channels within each product party [20]. We show below that there is another perhaps more justified option, which allows more general local operations and has a clear physical interpretation. This is the sequential asymptotic setting.

In general, the asymptotic setting can correspond to at least two physical settings. The first is the parallel setting, where two parties share many copies of a resource at the same time (Fig. 1 depicts this type of tensor product, when time is taken to point upwards, and W and Z are taken to exist at "the same time step"). The second is the sequential setting, where two parties share one copy of a resource at many time steps. In the sequential setting, the local operation a party performs decomposes into operations at different time steps, and these operations follow a definite time sequence. This physical interpretation imposes a natural restriction on the local operations, which can be generalized to processes if different copies of the process appear in a definite temporal order. In this sequential setting, the tensor products of processes obey the normalization condition.

Suppose n copies of a process W appear in a definite temporal order $W^{a'b'} \prec W^{a''b''} \prec \cdots \prec W^{a^{(n)}b^{(n)}}$. A can apply local operations to systems $a',a'',\cdots,a^{(n)}$, and B can apply them to systems $b',b'',\cdots,b^{(n)}$ that obey this temporal order. The local operations that A and B can apply to compose with and close all open systems to obtain probabilities are the so-called n-combs [4]. For A, such an n-comb takes the form $M^{a_2'a_2''\cdots a_2^{(n)}}_{a_1'a_1''\cdots a_1^{(n)}}$ and for B it takes the form $N^{b_2'b_2''\cdots b_2^{(n)}}_{b_1'b_1''\cdots b_1^{(n)}}$, where the systems obey the temporal order $a_1' \prec a_2' \prec a_1'' \prec a_2'' \prec \cdots \prec a_1^{(n)} \prec a_2^{(n)}$ and $b_1' \prec b_2' \prec b_1'' \prec b_2'' \prec \cdots \prec b_1^{(n)} \prec b_2^{(n)}$.

According to the "universality of quantum memory channels" theorem, the combs M and N can be decomposed into a sequence of memory channels, e.g., $M = M(1)^{e_1a_2'}_{a_1'} M(2)^{e_2a_2''}_{e_1a_1''} \cdots\ M(n)^{a_2^{(n)}}_{e_{n-1}a_1^{(n)}}$, where $M(i)$ are channels at time steps i with e_i as memory systems that correlate the channels. Similarly, $N = N(1)^{f_1b_2'}_{b_1'} N(2)^{f_2b_2''}_{f_1b_1''} \cdots N(n)^{b_2^{(n)}}_{f_{n-1}b_1^{(n)}}$. Then probability from composing the copies of W with M and N obeys the normalization condition

$$\begin{aligned} M(\otimes W)N = &\left[M(1)^{e_1a_2'}_{a_1'} W^{a_1'b_1'}_{a_2'b_2'} N(1)^{f_1b_2'}_{b_1'}\right] \\ &\times \left[M(2)^{e_2a_2''}_{e_1a_1''} W^{a_1''b_1''}_{a_2''b_2''} N(2)^{f_2b_2''}_{f_1b_1''}\right]\cdots \\ &\times \left[M(n)^{a_2^{(n)}}_{e_{n-1}a_1^{(n)}} W^{a_1^{(n)}b_1^{(n)}}_{a_2^{(n)}b_2^{(n)}} N(n)^{b_2^{(n)}}_{f_{n-1}b_1^{(n)}}\right] = 1. \end{aligned} \quad (5)$$

Within each square bracket there is a channel (including states and deterministic effects as special cases) operating on the memory systems, because a process composed with local channels with memory yields a channel [9]. In the end the composition of channels yields the number 1. If one substitutes subnormalized operators in place of the combs to represent quantum instruments, then it is easy to see that the probabilities must be in the interval [0,1] and sum to 1. Therefore the sequential asymptotic setting generalizes to quantum theory with indefinite causal structure. Intuitively, the sequential setting avoids the "causal loop" in (4) that violates the normalization condition by not allowing signaling from a system at a future time step to a system at a past time step.

IV. DISCUSSIONS

We showed that for processes we cannot take tensor products unrestrictedly, if arbitrary channels are allowed as local operations in a product party. Is this a defect of the process matrix framework itself? One interpretation is that the processes are descriptions of the environment of an entire family of parties [18,22], and the need to take tensor products of arbitrary processes with indefinite causal structure do not actually arise. Another option is to allow for tensor products of arbitrary parties but restrict the allowed operations in the local parties such that the normalization condition of processes is preserved [20]. The sequential asymptotic setting we presented above is an example of this kind. A further option is to adopt a more general framework that does not impose a normalization condition for the matrices and/or operators that carry the information of the indefinite causal structure. Oreshkov and

Cerf's operational quantum theory without predefined time provides an example [23]. We think an important question is to clarify whether these perspectives are compatible with attempts to create processes in the laboratory, because if one can create a process with signaling Hilbert-Schmidt terms in both directions it is conceivable that one can create more to act jointly on them, and there is no apparent reason why local operations on the joint parties must be restricted.

For communication theory, our results imply that communication tasks defined in the asymptotic limit are not meaningful for processes characterized by Theorem 1 when local operation is unrestricted for the combined parties $A = A'A'' \cdots$, $B = B'B'' \cdots$, $\cdots$. Similarly caution needs to be taken for asymptotic entanglement theory of processes [19]. On the other hand, such issues do not affect one-shot capacities, or asymptotic capacities in the sequential setting. It is possible that some other physically motivated restrictions on local operations yield additional well-defined capacities.

The restriction induces some interesting questions for further research. To what extent does the restriction generalize to indefinite causal structure theories in general [1,2]? For the particular example we used to demonstrate the restriction, there exists a global process that reduces to the two individual processes. When is this true in general?

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