Rigorous quantum limits on monitoring free masses and harmonic oscillators

S. M. Roy*
HBCSE, Tata Institute of Fundamental Research, Mumbai 400 088, India

(Received 30 November 2017; published 13 March 2018)

There are heuristic arguments proposing that the accuracy of monitoring position of a free mass m is limited by the standard quantum limit (SQL): $\sigma^2(X(t)) \geqslant \sigma^2(X(0)) + (t^2/m^2)\sigma^2(P(0)) \geqslant \hbar t/m$, where $\sigma^2(X(t))$ and $\sigma^2(P(t))$ denote variances of the Heisenberg representation position and momentum operators. Yuen [Phys. Rev. Lett. **51**, 719 (1983)] discovered that there are contractive states for which this result is incorrect. Here I prove universally valid rigorous quantum limits (RQL), viz. rigorous upper and lower bounds on $\sigma^2(X(t))$ in terms of $\sigma^2(X(0))$ and $\sigma^2(P(0))$, given by Eq. (12) for a free mass and by Eq. (36) for an oscillator. I also obtain the maximally contractive and maximally expanding states which saturate the RQL, and use the contractive states to set up an Ozawa-type measurement theory with accuracies respecting the RQL but beating the standard quantum limit. The contractive states for oscillators improve on the Schrödinger coherent states of constant variance and may be useful for gravitational wave detection and optical communication.

DOI: 10.1103/PhysRevA.97.032108

I. INTRODUCTION

A quantum system is prepared, for example, by a measurement, in an initial state. Subsequent monitoring or measurements of an observable A may be useful to detect any external disturbances additional to the intrinsic change in the uncertainty of the observable due to the system evolving by its own Hamiltonian. Much before the actual discovery of gravitational waves [1] it was realized that accurate monitoring of position of an oscillator and of a free mass, including quantum effects, are important for gravitational wave interferometers [2].

For an arbitrary initial state of a free mass or an oscillator, I shall obtain rigorous quantum limits (RQL) on the intrinsic uncertainty after time t.

For any observable with Schrödinger operator A (e.g., position $A = X$ or momentum $A = P$), and any Hamiltonian H, the Heisenberg operator $A(t)$ at time t and its variance $\sigma^2(A(t))$ are defined by

$$A(t) \equiv \exp(iHt/\hbar)\ A\ \exp(-iHt/\hbar), \tag{1}$$

$$\sigma^2(A(t)) \equiv \langle\psi(0)|(\Delta A(t))^2|\psi(0)\rangle, \tag{2}$$

$$\Delta A(t) \equiv A(t) - \langle A(t)\rangle, \tag{3}$$

$$\langle A(t)\rangle \equiv \langle\psi(0)|A(t)|\psi(0)\rangle, \tag{4}$$

where $|\psi(0)\rangle$ is the initial state.

II. HEURISTIC STANDARD QUANTUM LIMIT ON MONITORING POSITION OF A FREE MASS

There are heuristic arguments proposing that the accuracy of position monitoring is limited by the standard quantum limit (SQL) [3,4] on the variance of the position operator $X(t)$:

$$\sigma^2(X(t)) \geqslant \sigma^2(X(0)) + (t^2/m^2)\sigma^2(P(0)) \tag{5}$$

$$\geqslant 2(t/m)\sigma(X(0))\sigma(P(0)) \geqslant \hbar t/m. \tag{6}$$

For the free mass, $H = P^2/(2m)$. The inequality (5) is actually an equality for Gaussian states,

$$\langle p|\psi(t)\rangle = (\pi\alpha)^{-1/4} \exp\left[-\frac{(p-\beta)^2}{2\alpha} - it\frac{p^2}{2m}\right],$$

$$\sigma^2(P(t)) = \frac{\alpha}{2},\quad \sigma^2(X(t)) = \hbar^2\frac{1+(\alpha t/(m\hbar))^2}{2\alpha}. \tag{7}$$

One heuristic argument for the SQL [3,4], Eq. (5) starts from $H = P^2/(2m), \Delta X(t) = \Delta X(0) + (t/m)\Delta P(0)$,

$$\begin{aligned}\sigma^2(X(t)) &= \sigma^2(X(0)) + (t^2/m^2)\sigma^2(P(0))\\ &\quad + (t/m)\langle\psi(0)|\Delta X(0)\Delta P(0)\\ &\quad + \Delta P(0)\Delta X(0)|\psi(0)\rangle.\end{aligned} \tag{8}$$

One obtains the SQL if one assumes that the third term on the right-hand side is non-negative.

In a seminal paper, Yuen [5] noted that there are contractive states for which this assumption is incorrect. In an interesting and correct argument for the SQL, valid in certain measurement models, Caves [6] noted that in some models, resolution of the meter $\geqslant\sigma(X(0))$ may entail that the variance of the position measurement at time t is $\geqslant\sigma^2(X(0)) + \sigma^2(X(t))$, which is $\geqslant\hbar t/m$ by the uncertainty principle. Yuen [5] and Ozawa [7] (see also [9]) point out the existence of other measurement models for which the imperfect resolution correction can be much smaller than $\sigma^2(X(0))$. I address myself first to finding a rigorous version of the heuristic SQL Eq. (5) on $\sigma^2(X(t))$ and optimum contractive states. I then briefly discuss how the Ozawa [7] measurement model and the contractive states may be used for repeated measurements on oscillators or free masses over finite times, respecting, of course, the RQL presented here, but beating the SQL.

*smroy@hbcse.tifr.res.in

III. RIGOROUS QUANTUM LIMIT ON MONITORING POSITION OF A FREE MASS

We start from Eq. (8) and find exact limits on the third term on the right-hand side. Using

$$[\Delta X(0),\Delta P(0)] = i\hbar, \tag{9}$$

we have,

$$\begin{aligned}&\langle\psi(0)|\Delta X(0)\Delta P(0)+\Delta P(0)\Delta X(0)|\psi(0)\rangle + i\hbar\\&=2\langle\psi(0)|\Delta X(0)\Delta P(0)|\psi(0)\rangle.\end{aligned}\tag{10}$$

The Cauchy-Schwarz inequality on the right-hand side yields

$$\begin{aligned}&(\langle\psi(0)|\Delta X(0)\Delta P(0)+\Delta P(0)\Delta X(0)|\psi(0)\rangle)^2\\&\leqslant 4\sigma^2(X(0))\sigma^2(P(0))-\hbar^2,\end{aligned}\tag{11}$$

which is a rearrangement of the usual uncertainty relation on the product of variances of X and P [8].

Substituting this into Eq. (8) I have the rigorous quantum limits:

$$\begin{aligned}&\sigma^2(X(0))+(t/m)^2\sigma^2(P(0))\\&\quad-(t/m)\sqrt{4\sigma^2(X(0))\sigma^2(P(0))-\hbar^2}\\&\leqslant\sigma^2(X(t))\\&\leqslant\sigma^2(X(0))+(t/m)^2\sigma^2(P(0))\\&\quad+(t/m)\sqrt{4\sigma^2(X(0))\sigma^2(P(0))-\hbar^2}.\end{aligned}\tag{12}$$

It must be stressed that the bounds are fundamental quantum limits valid for arbitrary states. The only states saturating the inequalities are those for which the Schwarz inequalities are equalities, i.e., $\Delta P(0)|\psi(0)\rangle$ is a complex constant times $\Delta X(0)|\psi(0)\rangle$. Hence the RQL, Eq. (12) are equalities if and only if

$$\Delta P(0)|\psi(0)\rangle = i\lambda\Delta X(0)|\psi(0)\rangle, \tag{13}$$

$$\begin{aligned}&\langle X'|\psi(0)\rangle\\&=\left(\frac{\mathrm{Re}\lambda}{\pi\hbar}\right)^{1/4}\exp\left(\frac{i\langle P(0)\rangle X'}{\hbar}-\frac{\lambda(X'-\langle X(0)\rangle)^2}{2\hbar}\right),\end{aligned}\tag{14}$$

with $\mathrm{Re}\lambda > 0$,

$$\begin{aligned}&|\mathrm{Im}\lambda| = \frac{1}{2\sigma^2(X(0))}\sqrt{4\sigma^2(X(0))\sigma^2(P(0))-\hbar^2},\\&\sigma^2(X(0)) = \hbar/(2\,\mathrm{Re}\lambda),\quad \sigma^2(P(0)) = \hbar|\lambda|^2/(2\,\mathrm{Re}\lambda),\end{aligned}\tag{15}$$

and,

$$\begin{aligned}&\langle\psi(0)|\Delta X(0)\Delta P(0)+\Delta P(0)\Delta X(0)|\psi(0)\rangle\\&=\mp\sqrt{4\sigma^2(X(0))\sigma^2(P(0))-\hbar^2},\quad \text{if } \mathrm{Im}\lambda=\pm|\mathrm{Im}\lambda|.\end{aligned}\tag{16}$$

The positive and negative signs of $\mathrm{Im}\lambda$ correspond respectively to saturation of the left-hand side and right-hand side of the inequality (12). The right-hand side of inequality (12) sets an upper limit on spreading of the position wave packet and the left-hand side to the amount of contraction possible. The states (14) with positive $\mathrm{Im}\lambda$ derived without any reference to oscillators turn out to be essentially Yuen's contractive twisted coherent states (TCS) [5] of an associated fictitious oscillator. Thus, the above demonstration shows that for given $\sigma(X(0)),\sigma(P(0))$, the TCS are the optimum contractive states.

It is useful to rewrite the left-hand side of the inequality (12) in two alternative forms:

$$\begin{aligned}&\sigma^2(X(t))\\&\geqslant\left(\frac{\hbar}{2\sigma(P(0))}\right)^2+\left(\frac{\sigma(P(0))}{m}\right)^2\left(t-\frac{1}{2}t_M\right)^2\end{aligned}\tag{17}$$

$$\begin{aligned}&=\frac{t}{m}\left(2\sigma(X(0))\sigma(P(0))-\sqrt{4\sigma^2(X(0))\sigma^2(P(0))-\hbar^2}\right)\\&\quad+\left(t\frac{\sigma(P(0))}{m}-\sigma(X(0))\right)^2,\end{aligned}\tag{18}$$

where

$$t_M = \frac{m}{\sigma^2(P(0))}\sqrt{4\sigma^2(X(0))\sigma^2(P(0))-\hbar^2}. \tag{19}$$

Equation (17) shows that the optimal state (14) with positive $\mathrm{Im}\lambda$ remains contractive up to time $t_M/2$, and the variance $\sigma^2(X(t))$ is less than the initial variance $\sigma^2(X(0))$ for time $t < t_M$, i.e.,

$$\sigma^2(X(t)) \leqslant \sigma^2(X(0)),\quad \text{for } t\leqslant t_M, \tag{20}$$

for the optimum contractive state. Equation (18) shows that for a given uncertainty product, by choosing $(t/m)\sigma^2(P(0)) = \sigma(X(0))\sigma(P(0))$, $\sigma^2(X(t))$ can be made as small as $(t/m)(2[\sigma(X(0))\sigma(P(0))] - \sqrt{4\sigma^2(X(0))\sigma^2(P(0))-\hbar^2})$; this is $\approx t\hbar^2/[4m\sigma(X(0))\sigma(P(0))]$ for a large uncertainty product, and can be much smaller than the heuristic standard quantum limit $\hbar t/m$.

IV. RIGOROUS QUANTUM LIMITS ON MONITORING POSITION OR MOMENTUM OF A HARMONIC OSCILLATOR

This problem is specially significant because Hamiltonians for all free bosonic fields, including the electromagnetic field, are sums of harmonic oscillator Hamiltonians. In particular, the limits I derive can be immediately translated into RQLs on time development of quadratures of the electromagnetic field.

The Hamiltonian $H = P^2/(2m) + \frac{1}{2}m\omega^2X^2$ can be rewritten as

$$H = \tfrac{1}{2}\hbar\omega(p^2+x^2) = \hbar\omega(a^\dagger a + 1/2), \tag{21}$$

where

$$\begin{aligned}&p=\frac{P}{\sqrt{m\hbar\omega}},\quad x=\sqrt{\frac{m\omega}{\hbar}}X,\\&a=\frac{x+ip}{\sqrt{2}},\quad a^\dagger=\frac{x-ip}{\sqrt{2}}.\end{aligned}\tag{22}$$

The Heisenberg equations of motion yield

$$\Delta x(t) = \cos(\omega t)\,\Delta x(0) + \sin(\omega t)\,\Delta p(0)$$
$$\Delta p(t) = -\sin(\omega t)\,\Delta x(0) + \cos(\omega t)\,\Delta p(0). \tag{23}$$

Hence,

$$\sigma^2(x(t)) = \cos^2(\omega t)\sigma^2(x(0)) + \sin^2(\omega t)\sigma^2(p(0)) + \tfrac{1}{2}\sin(2\omega t)\langle\psi(0)|\Delta x(0)\Delta p(0) + \Delta p(0)\Delta x(0)|\psi(0)\rangle, \tag{24}$$

$$\sigma^2(p(t)) = \sin^2(\omega t)\sigma^2(x(0)) + \cos^2(\omega t)\sigma^2(p(0)) - \tfrac{1}{2}\sin(2\omega t)\langle\psi(0)|\Delta x(0)\Delta p(0) + \Delta p(0)\Delta x(0)|\psi(0)\rangle. \tag{25}$$

As before, using $[\Delta x(0),\Delta p(0)] = i$ and Schwarz inequality, we obtain

$$(\langle\psi(0)|\Delta x(0)\Delta p(0) + \Delta p(0)\Delta x(0)|\psi(0)\rangle)^2 \leqslant 4\sigma^2(x(0))\sigma^2(p(0)) - 1. \tag{26}$$

Hence, we have the RQL for the oscillator in terms of the dimensionless variables x and p, which can be the quadratures for a mode of frequency ω of the electromagnetic field,

$$\begin{aligned}[&\cos^2(\omega t)\sigma^2(x(0)) + \sin^2(\omega t)\sigma^2(p(0))] - \tfrac{1}{2}|\sin(2\omega t)|\sqrt{4\sigma^2(x(0))\sigma^2(p(0)) - 1}\\ &\leqslant \sigma^2(x(t))\\ &\leqslant [\cos^2(\omega t)\sigma^2(x(0)) + \sin^2(\omega t)\sigma^2(p(0))] + \tfrac{1}{2}|\sin(2\omega t)|\sqrt{4\sigma^2(x(0))\sigma^2(p(0)) - 1},\end{aligned} \tag{27}$$

which corresponds to Eq. (12) for a free mass. We also have RQL for $\sigma^2(p(t))$ for the oscillator,

$$\begin{aligned}[&\sin^2(\omega t)\sigma^2(x(0)) + \cos^2(\omega t)\sigma^2(p(0))] - \tfrac{1}{2}|\sin(2\omega t)|\sqrt{4\sigma^2(x(0))\sigma^2(p(0)) - 1}\\ &\leqslant \sigma^2(p(t))\\ &\leqslant [\sin^2(\omega t)\sigma^2(x(0)) + \cos^2(\omega t)\sigma^2(p(0))] + \tfrac{1}{2}|\sin(2\omega t)|\sqrt{4\sigma^2(x(0))\sigma^2(p(0)) - 1}.\end{aligned} \tag{28}$$

The extremal states saturating these RQL may be written in terms of the dimensionless variables x,p for use in optical quadrature measurements,

$$[\Delta p(0) - i\eta_\pm\Delta x(0)]|\psi(0)_\pm\rangle = 0 \tag{29}$$

$$\langle x'|\psi(0)_\pm\rangle = \left(\frac{\mathrm{Re}\,\eta_\pm}{\pi}\right)^{1/4} \times \exp\left(i\langle p(0)\rangle x') - \left(\frac{\eta_\pm(x' - \langle x(0)\rangle)^2}{2}\right)\right), \tag{30}$$

with

$$\eta_\pm = \frac{1}{2\sigma^2(x(0))}[1 \pm i\sqrt{4\sigma^2(x(0))\sigma^2(p(0)) - 1}]. \tag{31}$$

The values $\eta = \eta_\pm$ yield the values $\sigma^2(x(t))_\pm$ and $\sigma^2(p(t))_\pm$,

$$\sigma^2(x(t))_\pm - \cos^2(\omega t)\sigma^2(x(0)) - \sin^2(\omega t)\sigma^2(p(0)) = \mp\tfrac{1}{2}\sin(2\omega t)\sqrt{4\sigma^2(x(0))\sigma^2(p(0)) - 1}, \tag{32}$$

and

$$\sigma^2(p(t))_\pm - \sin^2(\omega t)\sigma^2(x(0)) - \cos^2(\omega t)\sigma^2(p(0)) = \pm\tfrac{1}{2}\sin(2\omega t)\sqrt{4\sigma^2(x(0))\sigma^2(p(0)) - 1}. \tag{33}$$

We deduce, for example, that for the initial state $|\psi(0)_+\rangle$,

$$\sigma^2(x(t))_+ \leqslant \sigma^2(x(0)), \quad \text{if } 0 \leqslant \omega t \leqslant \omega t'_M, \tag{34}$$

where

$$\omega t'_M \equiv \tan^{-1}\left[\frac{\sqrt{4\sigma^2(x(0))\sigma^2(p(0)) - 1}}{\sigma^2(p(0)) - \sigma^2(x(0))}\right] < \pi\ , \tag{35}$$

which corresponds to Eq. (19) in the free mass case.

Up to time t'_M, the contractive states for the oscillator thus improve on the Schrödinger coherent states which have constant $\sigma^2(x(t))$. Analogous results are easily obtained for $[\sigma^2(p(t))]_-$ for the initial state $|\psi(0)_-\rangle$.

It is easy to rewrite the bounds (27), (28), and extremal states (29) in dimensionless variables in terms of the dimensional X and P for the oscillator. Thus we have the RQL for the oscillator,

$$\begin{aligned}&\cos^2(\omega t)\sigma^2(X(0)) + \frac{\sin^2(\omega t)}{m^2\omega^2}\sigma^2(P(0)) - \frac{|\sin(2\omega t)|}{2m\omega}\sqrt{4\sigma^2(X(0))\sigma^2(P(0)) - \hbar^2}\\ &\leqslant \sigma^2(X(t))\\ &\leqslant \cos^2(\omega t)\sigma^2(X(0)) + \frac{\sin^2(\omega t)}{m^2\omega^2}\sigma^2(P(0)) + \frac{|\sin(2\omega t)|}{2m\omega}\sqrt{4\sigma^2(X(0))\sigma^2(P(0)) - \hbar^2},\end{aligned} \tag{36}$$

which shows that in the limit $\omega \to 0$ the RQL for the oscillator (36) yields the RQL for a free mass, Eq. (12).

V. CONNECTION OF EXTREMAL OSCILLATOR STATES WITH SQUEEZED COHERENT STATES

The extremal oscillator states have a close connection with squeezed coherent states with arbitrary squeezing direction. There are many applications of such optical states in quantum optics [10] and optomechanics. In particular, there has been progress in preparing a mechanical oscillator in non-Gaussian quantum states [11] by transferring such states from optical fields onto the oscillator. Squeezed coherent states have already been utilized in precision measurements needed in gravitational interferometers [12].

Using the definitions,

$$a = \frac{x + ip}{\sqrt{2}}, \quad \alpha = \langle\psi(0)|a|\psi(0)\rangle, \tag{37}$$

the extremal oscillator eigenvalue equation (29) is equivalent to

$$(b-\beta)|\psi(0)\rangle = 0, \quad \text{with } b = \mu a + \nu a^\dagger, \quad \beta = \mu\alpha + \nu\alpha^*,$$
$$\nu/\mu = (\eta-1)/(\eta+1), \quad \eta = (\mu+\nu)/(\mu-\nu), \tag{38}$$

where we have suppressed the subscripts $\pm$ on $|\psi(0)\rangle,\eta,\mu$, and ν for simplicity. Given η, only the ratio ν/μ is fixed, so we can make the convenient choice,

$$|\mu|^2 - |\nu|^2 = 1, \quad \mu > 0, \quad \text{i.e.,} \ \mu = \cosh r, \ \nu = e^{i\theta}\sinh r, \tag{39}$$

with $r > 0,\theta$real, in order to make the transformation from $a,a^\dagger$ to $b,b^\dagger$ canonical, i.e., $[b,b^\dagger] = 1$. Equation (38) is then just a twisted coherent state eigenvalue equation. The unitary displacement operator D and squeeze operator S,

$$D(\beta,b) = D(\alpha,a) = \exp(\alpha a^\dagger - \alpha^* a),$$
$$S(\xi) = \exp\tfrac{1}{2}(\xi^* a^2 - \xi a^{\dagger 2}), \quad \xi \equiv r\exp(i\theta), \tag{40}$$

obey

$$D^\dagger(\beta,b) b D(\beta,b) = b + \beta,$$
$$S^\dagger(\xi) a S(\xi) = a\cosh r - a^\dagger e^{i\theta}\sinh r\ . \tag{41}$$

Defining $a|0\rangle = 0$ we have

$$(b-\beta)|\alpha,\xi\rangle = 0, \quad |\alpha,\xi\rangle \equiv D(\alpha,a)S(\xi)|0\rangle, \tag{42}$$

$$\text{i.e.,}\ |\psi(0)\rangle = |\alpha, r\exp(i\theta)\rangle. \tag{43}$$

Thus the extremal states $|\psi(0)\rangle$ are simply related to the squeezed coherent states. Since

$$\eta = \frac{1 + i\sin\theta\sinh(2r)}{\cosh(2r) - \cos\theta\sinh(2r)}, \tag{44}$$

$\sin\theta > 0$ and $\sin\theta < 0$ correspond respectively to $|\psi(0)\rangle_+$ and $|\psi(0)\rangle_-$. The Heisenberg equations of motion give $a(t) = a\exp(-i\omega t)$, and hence the time-dependent states are

$$\exp(-iHt)|\psi(0)\rangle = e^{-i\omega t/2}|\alpha e^{-i\omega t}, re^{i(\theta - 2\omega t)}\rangle. \tag{45}$$

VI. POSITION MEASUREMENTS ON FREE MASSES AND HARMONIC OSCILLATORS USING CONTRACTIVE STATES

The RQL given above only considers unitary evolution with the system Hamiltonian. Caves [6] noted insightfully that additional considerations involving system-meter interactions during measurement are necessary, and sometimes important. The von Neumann model [13] is a prototype of quantum measurement models which couple the system to a meter and monitor the meter position y to obtain information about the system position x. Caves considered a class of models (which include the von Neumann model) in which, at any time τ, $\sigma^2(y(\tau)) = \sigma^2(x(\tau)) + \sigma_R^2$, where σ_R is the meter resolution. He showed that for measurements at $t=0$ and $t=\tau$ using identical meter states, the assumption $\sigma_R \geqslant \sigma(x(0))$, where $\sigma(x(0))$ is the position uncertainty just after the first measurement, would again imply the heuristic SQL $\sigma^2(X(\tau)) \geqslant \hbar\tau/m$. The SQL also applies to extensions of the Caves [6] model to continuous measurements by Caves and Milburn, and others [14].

In order to exploit the new possibilities allowed by the contractive states which violate the SQL (but obey the RQL), I outline below the use of the Ozawa interaction Hamiltonian [7],

$$H = k[2xp_y - 2p_x y + (xp_x + p_x x - yp_y - p_y y)/2], \tag{46}$$

where x,p_x are position and momentum operators for the system, and y,p_y those for the meter. The important properties of this interaction are that for a carefully chosen interaction time, after the measurement, (i) the meter uncertainty does not contain the additional uncertainty σ_R mentioned above and (ii) the contractive state of the meter is transferred to the system.

Suppose N measurements, each of time duration τ, are made over time intervals

$$t\epsilon\ [0,\tau],[T,T+\tau],[2T,2T+\tau],\ldots[(N-1)T,(N-1)T+\tau]$$

by N meters, each identically prepared at the beginning of the respective measurement in the same contractive state given by Eq. (30),

$$\langle y'|\chi\rangle = \left(\frac{\mathrm{Re}\ \eta_+}{\pi}\right)^{1/4} \exp\left(-\frac{\eta_+ y'^2}{2}\right), \tag{47}$$

where we have chosen $\langle y(0)\rangle = \langle p_y(0)\rangle = 0$ for simplicity, and

$$\eta_+ = \frac{1}{2\sigma^2(y(0))}[1 + i\sqrt{4\sigma^2(y(0))\sigma^2(p_y(0)) - 1}]. \tag{48}$$

The meter may, for example, be an oscillator of frequency Ω with $\Omega \neq \omega$, where ω is the frequency of the system oscillator. The coupling strength k is assumed large enough and the time interval τ small enough for the free Hamiltonians of the system and meter to be negligible during these measurement periods.

During each of the $N-1$ time intervals of duration $T-\tau$ between successive measurements,

$$t\epsilon\ [\tau,T],[T+\tau,2T],\ldots[(N-1)T+\tau,NT],$$

the measurement interaction is switched off and the system (free mass or harmonic oscillator) evolves unitarily according to its free Hamiltonian. At the beginning of each measurement period (e.g., $t = 0,T,2T,..$), i.e., $t = t_i = (i-1)T, i = 1,2,..N$, the joint wave function of the system and meter is

$$\langle x',y'|\Psi(t_i)\rangle = \langle x'|\psi(t_i)\rangle\langle y'|\chi\rangle, \tag{49}$$

where we have suppressed a subscript i referring to the ith meter. Solving the Heisenberg equation of motion using the Ozawa interaction, we get the operators after time τ,

$$x(t_i+\tau) = \frac{2}{\sqrt{3}}\left[\sin\left(k\tau\sqrt{3} + \frac{\pi}{3}\right)x(t_i) - \sin(k\tau\sqrt{3})y(t_i)\right],$$
$$y(t_i+\tau) = \frac{2}{\sqrt{3}}\left[\sin(k\tau\sqrt{3})x(t_i) + \sin\left(\frac{\pi}{3} - k\tau\sqrt{3}\right)y(t_i)\right],$$

and the corresponding wave function

$$\langle x', y'|\Psi(t_i+\tau)\rangle = \left\langle \frac{2}{\sqrt{3}}\left[\sin(k\tau\sqrt{3})y' + \sin\left(\frac{\pi}{3} - k\tau\sqrt{3}\right)x'\right]|\psi(t_i)\right\rangle \times \left\langle \frac{2}{\sqrt{3}}\left[\sin\left(k\tau\sqrt{3} + \frac{\pi}{3}\right)y' - \sin(k\tau\sqrt{3})x'\right]|\chi\right\rangle. \quad (50)$$

If we choose the product of the strength and duration of the interaction such that

$$k\tau = \pi/(3\sqrt{3}), \quad (51)$$

we get the simple operators and wave functions,

$$x(t_i+\tau) = x(t_i) - y(t_i); \quad y(t_i+\tau) = x(t_i), \quad (52)$$

$$\langle x', y'|\Psi(t_i+\tau)\rangle = \langle y'|\psi(t_i)\rangle\langle y' - x'|\chi\rangle. \quad (53)$$

Hence observation of the meter after the measurement will return the correct expectation value for the system before the measurement,

$$\langle\Psi(t_i)|y(t_i+\tau) - x(t_i)|\Psi(t_i)\rangle = 0, \quad (54)$$

and the predicted probability density $P(y')$ for the meter,

$$P(y')(t_i+\tau) = \int dx'|\langle x', y'|\Psi(t_i+\tau)\rangle|^2 = |\langle y'|\psi(t_i)\rangle|^2, \quad (55)$$

which is identical to the system position probability density just before measurement. Hence,

$$\sigma^2[y(t_i+\tau)] = \sigma^2[x(t_i)], \quad (56)$$

without any extra error σ_R corresponding to meter resolution. Further, after a meter reading y', the system is left in the state

$$\langle x'|\psi(t_i+\tau)\rangle = \langle y' - x'|\chi\rangle\left[\frac{\langle y'|\psi(t_i)\rangle}{|\langle y'|\psi(t_i)\rangle|}\right], \quad (57)$$

which, apart from the phase factor in the square bracket on the right-hand side is just the contractive state in which the meter was prepared, but with $\langle x\rangle = y'$. Using this result and our previous results in Eqs. (19) and (35), it follows that the choice

$$T - \tau = t'_M \quad \text{for oscillator,}$$
$$T - \tau = t_M, \quad \text{for free mass,} \quad (58)$$

will ensure that the system state has position uncertainty less than the initial meter uncertainty for $NT > t > \tau$. To justify neglecting the free Hamiltonians during the measurement interval τ we need $\Omega\tau \ll 1$ for the meter and $\omega\tau \ll 1$ if the system is an oscillator, and $\sigma(P(0))/\sigma(X(0))\tau/m \ll 1$ if the system is a free mass; we need the error in the condition $k\tau = \pi/(3\sqrt{3})$ to be negligible, i.e., the error $k\delta\tau \ll 1$. Hence we have the following necessary conditions on the sensitivity of the time setting $\delta\tau$ and the strength k of the measurement interaction:

$$\delta\tau \ll \frac{1}{k} = \tau\frac{3\sqrt{3}}{\pi} \ll \min\left[\frac{1}{\Omega}, \frac{1}{\omega}\right], \quad (59)$$

for measurements on the oscillator; for the case of the free mass $1/\omega \to m\sigma(X(0))/\sigma(P(0))$ on the right-hand side of the above equation.

VII. CONCLUSION

I have obtained rigorous quantum limits on the variance $\sigma^2(X(t))$ in terms of $\sigma^2(X(0))$ and $\sigma^2(P(0))$ for arbitrary quantum states of a free mass and of a harmonic oscillator. I also obtained the states which achieve saturation of the limits and their connection with squeezed coherent states of an oscillator with arbitrary squeezing direction. In order to utilize the contractive states to obtain accuracies beyond the SQL, I have outlined measurement models over finite nonzero time intervals for free mass position and oscillator position using the Ozawa Hamiltonian [7] for system-meter interaction. Between measurements the system evolves according to the free Hamiltonian. In the oscillator case the extremal contractive state improves on the Schrödinger coherent states for a well-defined time interval, and the free evolution period is adjusted to be equal to that interval. I also briefly discuss the experimental sensitivities needed to justify the assumptions on the parameters of the model.

ACKNOWLEDGMENTS

I thank the Indian National Science Academy for support through an INSA honorary scientist position at HBCSE, TIFR. I thank Priyanshi Bhasin and Ujan Chakraborty, who collaborated with me on a related problem [15] investigating generalized twisted coherent states for a free mass using the generalized oscillator coherent states with nonminimum uncertainty [16]. I thank Paul Busch for discussions during the ICQF-17 Conference and for suggesting that I mention that Eq. (11) is a rearrangement of the usual uncertainty relations on position and momentum. I thank the referee for valuable suggestions on connections with squeezed coherent states and on the importance of noninstantaneous measurement models.

[1] B. P. Abbott *et al.* (LIGO Scientific Collaboration and Virgo Collaboration), Phys. Rev. Lett. **116**, 061102 (2016).

[2] E.g., K. S. Thorne, R. W. P. Drever, C. M. Caves, M. Zimmermann, and V. D. Sandberg, Phys. Rev. Lett. **40**, 667 (1978); R. Weiss, in *Sources of Gravitational Radiation*, edited by L. Smarr (Cambridge University Press, Cambridge, 1979); V. B. Braginsky, Y. I. Vorontsov, and K. S. Thorne, Science **209**, 547 (1980); B. Abbott *et al.* (LIGO Scientific Collaboration), New J. Phys. **11**, 073032 (2009); J. Abadi *et al.* (LIGO Scientific Collaboration), Nat. Phys. **7**, 962 (2011); S. L. Danilishin and F. Y. Khalili, Living Rev. Relativ. **15**, 5 (2012); Y. Ma *et al.*, Nat. Phys. **13**, 776 (2017).

[3] V. B. Braginsky and Yu. I. Vorontsov, Ups. Fiz. Nauk **114**, 41 (1974) [Sov. Phys. Usp. **17**, 644 (1975)].

[4] C. M. Caves, K. S. Thorne, R. W. P. Drever, V. D. Sandberg, and M. Zimmermann, Rev. Mod. Phys. **52**, 341 (1980).
[5] H. P. Yuen, Phys. Rev. Lett. **51**, 719 (1983).
[6] C. M. Caves, Phys. Rev. Lett. **54**, 2465 (1985).
[7] M. Ozawa, Phys. Rev. Lett. **60**, 385 (1988); see also, Ann. Phys. (N.Y.) **311**, 350 (2004); and Curr. Sci. **109**, 2006 (2015); J. P. Gordon and W. H. Louisell, in *Physics of Quantum Electronics*, edited by P. L. Kelly, B. Lax, and P. E. Tannenwald (McGraw-Hill, New York, 1966), p. 833.
[8] E. Kennard, Zeitschr. Phys. **44**, 326 (1927); H. Weyl, *Gruppentheorie und quantenmechanik* (Hirzel, Leipzig, 1928); H. Robertson, Phys. Rev. **34**, 163 (1929).
[9] E. Arthurs and J. L. Kelly Jr., Bell System Tech. J. **44**, 725 (1965); P. Busch, T. Heinonen, and P. Lahti, Phys. Rep. **452**, 155 (2007); P. Busch, P. Lahti, and R. F. Werner, Rev. Mod. Phys. **86**, 1261 (2014); S. M. Roy, Curr. Sci. **109**, 2029 (2015).
[10] D. Walls, Squeezed states of light, Nature (London) **306**, 141 (1983); M. O. Scully and M. S. Zubairy, *Quantum Optics*, (Cambridge University Press, Cambridge, UK, 1997); C. Gerry and P. L. Knight, *Introductory Quantum Optics* (Cambridge University Press, Cambridge, UK, 2005); R. Loudon, *The Quantum Theory of Light* (Oxford University Press, Oxford, UK, 2000); D. F. Walls and G. J. Milburn, *Quantum Optics* (Springer, Berlin, 1994); C. W. Gardiner and Peter Zoller, *Quantum Noise*, 3rd ed. (Springer, Berlin, 2004).
[11] F. Khalili, S. Danilishin, H. Miao, H. Muller-Ebhardt, H. Yang, and Y. Chen, Phys. Rev. Lett. **105**, 070403 (2010).
[12] H. Grote, K. Danzmann, K. L. Dooley, R. Schnabel, J. Slutsky, and H. Vahlbruch, Phys. Rev. Lett. **110**, 181101 (2013); J. Aasi *et al.*, Nat. Phontics **7**, 613 (2013); M. Aspelmeyer, P. Meystre, and K. Schwab, Phys. Today **65**, 29 (2012); K. Goda *et al.*, Nat. Phys. **4**, 472 (2008); G. M. Harry *et al.*, Classical Quant. Grav. **27**, 084006 (2010).
[13] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, NJ, 1955), Chap. 6.
[14] C. M. Caves and G. J. Milburn, Phys. Rev. A **36**, 5543 (1987); M. T. Jaekel and S. Reynaud, Europhys. Lett. **13**, 301 (1990); G. J. Milburn, K. Jacobs, and D. F. Walls, Phys. Rev. A **50**, 5256 (1994); H. Mabuchi, *ibid.* **58**, 123 (1998).
[15] Priyanshi Bhasin, Ujan Chakraborty, and S. M. Roy (unpublished).
[16] S. M. Roy and V. Singh, Phys. Rev. D **25**, 3413 (1982).