

Global Dirac bispinor entanglement under Lorentz boosts

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The effects of Lorentz boosts on the quantum entanglement encoded by a pair of massive spin- $\frac{1}{2}$ particles are described according to the Lorentz covariant structure described by Dirac bispinors. The quantum system considered incorporates four degrees of freedom: two of them related to the bispinor intrinsic parity and the other two related to the bispinor spin projection, i.e., the Dirac particle helicity. Because of the natural multipartite structure involved, the Meyer-Wallach global measure of entanglement is preliminarily used for computing global quantum correlations, while the entanglement separately encoded by spin degrees of freedom is measured through the negativity of the reduced two-particle spin-spin state. A general framework to compute the changes on quantum entanglement induced by a boost is developed and then specialized to describe three particular antisymmetric two-particle states. According to the results obtained, two-particle spin-spin entanglement cannot be created by the action of a Lorentz boost in a spin-spin separable antisymmetric state. On the other hand, the maximal spin-spin entanglement encoded by antisymmetric superpositions is degraded by Lorentz boosts driven by high-speed frame transformations. Finally, the effects of boosts on chiral states are shown to exhibit interesting invariance properties, which can only be obtained through such a Lorentz covariant formulation of the problem.

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I. INTRODUCTION

Relativistic quantum information is a fast developing field that merges quantum information with relativistic quantum mechanics to devise communication protocols in relativistic frameworks involving, for instance, clock synchronization [1], position verification [2], and teleportation protocols [3]. The effects of a relativistic frame transformation on quantum correlations have been investigated [4–12] and, considering spin as the natural tool for quantum information engineering, the effects of frame transformations (Lorentz boosts) on the quantum entanglement encoded by a pair of spin- $\frac{1}{2}$ particles have been investigated as well.

From a kinematic point of view, the action of the linear transformation given by a Lorentz boost Λ describes the change of space-time coordinates from an inertial frame \mathcal{S} to another one \mathcal{S}' , which moves with respect to \mathcal{S} , to set, for instance, the quadrimomentum transformation relation $p'' = \Lambda_{\mu}^{\nu} p^{\mu}$, summarized by $p' = \Lambda p$ in a matricial representative notation, where p' and p are described by coordinates at \mathcal{S}' and \mathcal{S} , respectively. As seminally stated by Wigner [13], under such a transformation between inertial frames, an observable spin (projector operator) described by the SU(2) adjoint representation realized by $\sigma = (\sigma_x, \sigma_y, \sigma_z)$, where $\sigma_{x,y,z}$ are the Pauli matrices, has its spin projection onto the particle momentum direction, $\hat{e}_p \cdot \hat{\sigma}$, with $e_p = \mathbf{p}/|\mathbf{p}|$, changed to return $\hat{e}_{p'} \cdot \hat{\sigma} \neq \hat{e}_p \cdot \hat{\sigma}$, where boldface variables $\mathbf{v} = (v_x, v_y, v_z)$ denote spatial

vectors with modulus $v = \sqrt{\mathbf{v} \cdot \mathbf{v}}$ and circumflexes denote quantum operators.

The rigorous treatment of the above kinematic properties and of their imprints on quantum states of spin- $\frac{1}{2}$ particles involves a description of their observable related properties in terms of the irreducible representations of the Poincaré group [14,15]. For instance, for a particle with momentum \mathbf{p} , in an inertial frame \mathcal{S} , described by the quantum state $|\phi_s(\mathbf{p})\rangle$, where $s = 1, 2$ denotes accessible spin states, the action of a Lorentz boost Λ that describes the change from \mathcal{S} to \mathcal{S}' is given by the unitary transformation [13–15]

$$|\phi_s(\mathbf{p})\rangle \rightarrow \hat{D}[\Lambda]|\phi_s(\mathbf{p})\rangle = \sum_r c_{sr}(\Lambda, \mathbf{p})|\phi_r(\mathbf{p}')\rangle, \quad (1)$$

where the unitary operator $\hat{D}[\Lambda]$ and its explicit component dependence c_{sr} , according to the Poincaré group representations [14,15] (cf. Sec. II), are given in agreement with the irreducible representation of the quantum state $|\phi_s(\mathbf{p})\rangle$, which can describe, for instance, a spinor (in a doublet representation, like electrons and positrons described either as Weyl or as Majorana fermions), a vector (in a triplet representation, like 3S_1 positronium or even photons), bispinors (in a doublet representation, like electrons and positrons described by Dirac fermions), or even scalar (in a singlet representation, like 1S_0 positronium) and higher-order (maybe nonphysical) tensor states. The point in this paper is that when quantum states depend on the momentum, namely, those described by Dirac equation solutions for spin- $\frac{1}{2}$ states, different inertial observers will see different superpositions and if somehow the momentum degrees of freedom are traced out, the quantum entanglement between spin states might change [4–12]. Of course, for two-particle states, the question related to the influence of the reference frame in the computation of quantum

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correlations is raised much more in the framework of relativistic quantum mechanics supported by the Dirac formalism.

Despite the effectiveness of the irreducible representations of the Poincaré group, in the Lorentz covariant Hamiltonian formulation of quantum mechanics, one has to pay attention to inclusion of mass in the relativistic formalism described by the Dirac equation. As will be shown in Sec. II, it requires the inclusion of the parity symmetry and the equalization of its role with the helicity (spin- $\frac{1}{2}$ projection, $\hat{e}_p \cdot \hat{\sigma} \sim \hat{\sigma}_z$) symmetry, as an accomplished $SU(2)$ symmetry. It supports, for instance, the description of electrons as Dirac Hamiltonian eigenstates in the double doublet irreducible representation of $SU(2) \otimes SU(2)$. Spatial parity couples positive and negative parity states with positive and negative helicity states as they were described by irreducible representations of the Poincaré group [14] and, in order to have complete invariance under parity, one needs to consider the extended Poincaré group [15,16]. In this case, spin $\frac{1}{2}$ is carried by Dirac four-component spinors, the bispinors satisfying the Dirac equation, in a representation supported by a subgroup of $SL(2, \mathbb{C}) \otimes SL(2, \mathbb{C})$, i.e., $SU(2) \otimes SU(2)$. In fact, the description of massive charged fermions (such as electrons, muons, quarks, etc.) requires the irreducible representations of the complete Lorentz group,¹ namely, the Dirac (bi)spinors [16].

The intrinsic spin-parity (or helicity-parity) entanglement exhibited by a single Dirac bispinor has already been investigated in the context of quantum correlations driven by interactions with external fields [17], which has been used for simulating Dirac-like systems as, for instance, four-level ion traps [18] and lattice and layer schemes in bilayer graphene [19]. Generically, such an intrinsic entanglement encoded by Dirac-like $SU(2) \otimes SU(2)$ structures can also be generated, for example, by quantum electrodynamics (QED) scattering processes [20].

Not in the same scope, but also emphasizing the bispinor structure of fermionic quantum states, states constructed with the solutions of the Dirac equation have been considered in the scrutiny of Bell inequalities [21] and to obtain proper covariant spin density matrices and definitions of the position operator in the context of relativistic quantum mechanics [22,23]. Likewise, the effects of Lorentz boosts in quantum entanglement encoded in bispinors were described in connection with Wigner rotations for a specific class of states [24] and in the context of the Foldy-Wouthuysen (FW) spin operator [25], with a focus on the properties of transformation of spin-spin entanglement encoded in FW eigenstates [26]. However, considering the focus on the most phenomenologically appealing measurement of two-particle spin-spin entanglement, the intrinsic $SU(2) \otimes SU(2)$ covariant structure of Dirac bispinors, which is associated with intrinsic parity and spin [27] for each particle, has not yet been completely incorporated into such an overall relativistic framework.

The aim of this work is therefore to estimate the influence of Lorentz boosts on the quantum entanglement encoded in the

intrinsic $SU(2) \otimes SU(2)$ structure of two spin- $\frac{1}{2}$ Dirac particles which are also spin-spin entangled. As each particle described by Dirac bispinors carries two qubits, the whole state is a four-qubit one, and since multipartite entanglement is generally present in such states, the Meyer-Wallach global measure of entanglement [28] will be considered as a measure of the entanglement encoded in the four qubits of the system. Alternatively, the net result for the spin-spin entanglement, encoded only in a two-qubit mixed state, will be computed through the negativity [29,30]. The effects of a Lorentz boost on the entanglement content of generic two-particle Dirac bispinor states will be obtained for the case where superpositions of helicity plane waves are considered. The results obtained will be specialized to antisymmetric states showing, for example, that a Lorentz boost cannot create spin-spin entanglement in an initial separable antisymmetric state.

The paper is structured as follows. In Sec. II, the complete Lorentz covariant structure of the Dirac equation solutions, namely, that associated with the properties of $SU(2)$ spinor doublet representations and with the composition of higher-order multiplet representations, is reported and the foundation for establishing and discussing the spin-parity intrinsic entanglement is introduced. In Sec. III, by using the intrinsic $SU(2) \otimes SU(2)$ structure of the Dirac equation, the entanglement profile of a generic superposition of Dirac bispinors is described. In Sec. IV, by using the transformation laws of bispinors under Lorentz boosts, the effects of such transformations on the quantum correlations encoded by two-particle states are computed, with particular emphasis on antisymmetric states. In addition, the investigation of the effects of boosts on the superposition of chiral bispinors shows that some subtle invariance properties can be obtained. A summary is given in Sec. V, where conclusions concerning the importance of accounting for the Lorentz covariant structures in the computation of quantum correlations are drawn.

II. LORENTZ COVARIANT STRUCTURE OF THE DIRAC EQUATION AND SPIN-PARITY INTRINSIC ENTANGLEMENT

In quantum mechanics, the free-particle Dirac Hamiltonian in the coordinate space reads

$$\begin{aligned} \hat{H}\psi(x) &= i \frac{\partial \psi(x)}{\partial t} = (-i\nabla \cdot \hat{\alpha} + m\hat{\beta})\psi(x) \\ &= (-i\hat{\alpha}_i \partial^i + m\hat{\beta})\psi(x) = \pm E_p \psi(x), \end{aligned} \quad (2)$$

where $E_p = \sqrt{p^2 + m^2}$, the space-time dependence has been resumed by $x \sim (t, \mathbf{x})$, the Dirac matrix operators $\hat{\alpha} = (\hat{\alpha}_x, \hat{\alpha}_y, \hat{\alpha}_z)$ and $\hat{\beta}$ satisfy the anticommuting relations $\hat{\alpha}_i \hat{\alpha}_j + \hat{\alpha}_j \hat{\alpha}_i = 2\delta_{ij} \hat{I}_4$ and $\hat{\alpha}_i \hat{\beta} + \hat{\beta} \hat{\alpha}_i = 0$ for $i, j = x, y, z$, with $\hat{\beta}^2 = \hat{I}_4$, \hat{I}_N the N -dimensional identity matrix, and \hat{H} is expressed in natural units, i.e., with $c = \hbar = 1$. The above Dirac Hamiltonian dynamics exhibits some symmetries that are supported by a group representation described by a direct product between two algebras which compose a subset of the group $SL(2, \mathbb{C}) \otimes SL(2, \mathbb{C})$, the group $SU(2) \otimes SU(2)$. To clarify this point, before discussing the above statement in the enhanced language of Lie algebra and Lie groups, one simply notices that left-handed spinors are described by a doublet

¹The complete Lorentz group is composed of the proper Lorentz group together with space inversion. The extended Poincaré group is given by the complete Lorentz group with the addition of space-time translation [16].

(two-dimensional) representation of SU(2) (left) and a singlet (one-dimensional) representation of SU(2) (right), $(\mathbf{2}, \mathbf{1}) \equiv \psi_L^\dagger(x) = (\psi_{L1}(x), \psi_{L2}(x))$, and, analogously, right-handed spinors are described by a doublet (two-dimensional) representation of SU(2) (right) and a singlet (one-dimensional) representation of SU(2) (left), $(\mathbf{1}, \mathbf{2}) \equiv \psi_R^\dagger(x) = (\psi_{R1}(x), \psi_{R2}(x))$, in order to support the following decomposition for the Dirac state vectors, $\psi^\dagger(x) = (\psi_{L1}(x), \psi_{L2}(x), \psi_{R1}(x), \psi_{R2}(x)) \equiv (\mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2})$, in a not unique double doublet representation of the SU(2) \otimes SU(2) group. Therefore, the free-particle Dirac equation is thus mapped into coupled differential equations for left- and right-handed components, respectively,

$$\begin{aligned} i\bar{\sigma}^\mu \partial_\mu \psi_L(x) - m\psi_R(x) &= 0, \\ i\sigma^\mu \partial_\mu \psi_R(x) - m\psi_L(x) &= 0, \end{aligned}$$

in the so-called chiral representation $(\hat{I}_2, \hat{\sigma}) \equiv \sigma^\mu$ and $(\hat{I}_2, -\hat{\sigma}) \equiv \bar{\sigma}^\mu$, for which the Lagrangian density reads

$$\mathcal{L} = i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L + i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R - m(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L), \quad (3)$$

from which a correspondence to the spinor chirality is identified.

As it has been mentioned, the above choice is not unique. Another particular representation of the Dirac matrices is the Pauli-Dirac representation in which the Dirac matrices are decomposed into tensor products of Pauli matrices [27], as $\hat{\alpha}_i = \hat{\sigma}_x^{(P)} \otimes \hat{\sigma}_i^{(S)}$, for $i = x, y, z$ and $\hat{\beta} = \hat{\sigma}_z^{(P)} \otimes \hat{I}^{(S)}$, which has another subadjacent $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ algebra from the SU(2) \otimes SU(2) group which, in this case, does not correspond to left- and right-handed chiral projection representations but instead is associated with intrinsic parity P and spin (or helicity) S . In this case, the Dirac Hamiltonian is thus rewritten in terms of Kronecker products between Pauli matrices as

$$\hat{H} = \mathbf{p} \cdot (\hat{\sigma}_x^{(P)} \otimes \hat{\sigma}^{(S)}) + m(\hat{\sigma}_z^{(P)} \otimes \hat{I}^{(S)}), \quad (4)$$

from which, according to the interpretation of quantum mechanics as an information theory for particles, where the superscripts P and S refer to the qubits of parity and spin, one can identify the Dirac equation solutions as they were described by two-qubit states encoded in a massive particle whose dynamics is constrained by continuous variables.

Within this framework, from the Hamiltonian (4), the normalized stationary eigenstates in the momentum coordinate are written in terms of a sum of direct products describing spin-parity entangled states

$$\begin{aligned} |u_s(\mathbf{p})\rangle &= \frac{1}{\sqrt{2E_p(E_p + m)}} \{ (E_p + m)|+\rangle \otimes |\chi_s(\mathbf{p})\rangle \\ &+ |-\rangle \otimes [\mathbf{p} \cdot \boldsymbol{\sigma} |\chi_s(\mathbf{p})\rangle] \}, \end{aligned} \quad (5)$$

$$\begin{aligned} |v_s(\mathbf{p})\rangle &= \frac{1}{\sqrt{2E_p(E_p + m)}} \{ (E_p + m)|-\rangle \otimes |\chi_s(\mathbf{p})\rangle \\ &+ |+\rangle \otimes [\mathbf{p} \cdot \boldsymbol{\sigma} |\chi_s(\mathbf{p})\rangle] \} \end{aligned} \quad (6)$$

for positive and negative eigenvalues (associated frequencies) $\pm E_p = \pm \sqrt{p^2 + m^2}$, respectively, where $|\chi_s(\mathbf{p})\rangle \in \mathcal{H}_s$, with $s = 1, 2$, are the spinors related to the spatial motion of the particle, i.e., the particle's helicity, which describes the

dynamics of a fermion (in momentum representation) coupled to its spin, and $|\pm\rangle \in \mathcal{H}_p$ are intrinsic parity states. In the bispinorial form, one has

$$\begin{aligned} u_s(\mathbf{p}) &= \frac{1}{\sqrt{2E_p(E_p + m)}} \begin{bmatrix} (E_p + m) & \chi_s(\mathbf{p}) \\ \mathbf{p} \cdot \boldsymbol{\sigma} & \chi_s(\mathbf{p}) \end{bmatrix}, \\ v_s(\mathbf{p}) &= \frac{1}{\sqrt{2E_p(E_p + m)}} \begin{bmatrix} \mathbf{p} \cdot \boldsymbol{\sigma} & \chi_s(\mathbf{p}) \\ (E_p + m) & \chi_s(\mathbf{p}) \end{bmatrix}, \end{aligned} \quad (7)$$

with the orthogonality relations identified by $u_s^\dagger(\mathbf{p})u_r(\mathbf{p}) = v_s^\dagger(\mathbf{p})v_r(\mathbf{p}) = \delta_{sr}$ and $u_s^\dagger(\mathbf{p})v_r(-\mathbf{p}) = v_s^\dagger(\mathbf{p})u_r(-\mathbf{p}) = 0$ and the completeness relation given by

$$\sum_{s=1}^2 [u_s(\mathbf{p})u_s^\dagger(\mathbf{p}) + v_s(\mathbf{p})v_s^\dagger(\mathbf{p})] = \hat{I}_4.$$

States as described by Eqs. (7) are composite quantum systems in a total Hilbert space $\mathcal{H} = \mathcal{H}_p \otimes \mathcal{H}_s$ and, in the general form of Eq. (5), they are spin-parity entangled [27]. Of course, they are a superposition of orthonormal parity eigenstates $|\pm\rangle$ and therefore they do not have a defined intrinsic parity quantum number.²

To summarize, the spin degree of freedom (DOF) identified by the index s is associated with irreducible representations of the proper Poincaré group and the positive or negative associated energy eigenstates of the spin- $\frac{1}{2}$ particles can be reindexed through the notation

$$\begin{aligned} |u_{\pm,s}(\mathbf{p})\rangle &= \frac{1}{\sqrt{2E_p(E_p + m)}} [(E_p + m)|\pm\rangle \otimes |\chi_s(\mathbf{p})\rangle \\ &+ |\mp\rangle \otimes [\mathbf{p} \cdot \boldsymbol{\sigma} |\chi_s(\mathbf{p})\rangle] \end{aligned} \quad (8)$$

for vectors belonging to the irreducible representation labeled by $(\pm, \frac{1}{2})$, associated with the SU(2) \otimes SU(2) group [13–15]. Therefore, the invariance under spatial parity symmetry requires an analysis with the complete Lorentz group in order to include irreducible representations of SU(2) \otimes SU(2) which merge spin with the additional DOF of intrinsic parity [14,16].

In the context of a group theory, the above assertion can be better understood when the representations of $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$, which corresponds to the Lie algebra of the SL(2, \mathbb{C}) \otimes SL(2, \mathbb{C}) Lie group, are irreducible, i.e., they correspond to tensor products between linear complex representations of $\mathfrak{sl}(2, \mathbb{C})$, as it has been observed by considering the subgroup SU(2) \otimes SU(2) \subset SL(2, \mathbb{C}) \otimes SL(2, \mathbb{C}). Unitary irreducible representations of SU(2) \otimes SU(2) are built through tensor products between unitary representations of SU(2), in a one-to-one correspondence with the group SL(2, \mathbb{C}) \otimes SL(2, \mathbb{C}). Since it is a simply connected group, there is also a unique correspondence to the $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$ algebra.

²A defined total parity operator \hat{P} acts on the direct product $|\pm\rangle \otimes |\chi_s(\mathbf{p})\rangle$ in the form $\hat{P}[|\pm\rangle \otimes |\chi_s(\mathbf{p})\rangle] = \pm[|\pm\rangle \otimes |\chi_s(-\mathbf{p})\rangle]$ and, for instance, corresponds to the Kronecker product of two operators $\hat{P}^{(P)} \otimes \hat{P}^{(S)}$, where $\hat{P}^{(P)}$ is the intrinsic parity (with two eigenvalues $\hat{P}^{(P)}|\pm\rangle = \pm|\pm\rangle$) and $\hat{P}^{(S)}$ is the spatial parity [with $\hat{P}^{(S)}\chi_s(\mathbf{p}) = \chi_s(-\mathbf{p})$].

As it has been identified above for the chiral basis and for the spin-parity basis, the existence of inequivalent representations of $SU(2) \otimes SU(2)$ follows from the above-mentioned one-to-one correspondences. Inequivalent representations may not correspond to the complete set of representations of $SL(2, \mathbb{C}) \otimes SL(2, \mathbb{C})$ and therefore of the proper Lorentz transformations that compose the $SO(1,3)$ group, i.e., the Lorentz group.³

Returning to our point, as the transformations of $SU(2) \otimes SU(2)$ can be described by a subset of $SL(2, \mathbb{C}) \otimes SL(2, \mathbb{C})$, one may choose at least two inequivalent subsets of $SU(2)$ generators such that $SU(2) \otimes SU(2) \subset SL(2, \mathbb{C}) \otimes SL(2, \mathbb{C})$, with each group transformation generator having its own irreducible representation. Therefore, a fundamental object of $SU_\xi(2)$, a spinorlike object ξ described by $(\pm, 0)$, transforms as a doublet (the fundamental representation) of $SU_\xi(2)$ and as a singlet [a transparent object under any $SU_\chi(2)$ transformations]. Reciprocally, the fundamental object of $SU_\chi(2)$, a typical spinor χ described by $(0, \frac{1}{2})$, transforms as a doublet of $SU_\chi(2)$ and as a singlet of $SU_\xi(2)$. Under improved notation generalized to higher-dimensional representations ($\dim[SU_\xi(2)], \dim[SU_\chi(2)]$), the spinor ξ is an object given by $(\mathbf{2}, \mathbf{1})$. Following the generalized idea for an arbitrary $SU(2) \otimes SU(2)$ composition, one has the representations as given by $(\mathbf{1}, \mathbf{1})$ for the scalar or singlet, with angular momentum projection $j = 0$; $(\mathbf{2}, \mathbf{1})$ for the spinor $(\frac{1}{2}, 0)$, with angular momentum projection $j = 1/2$, which corresponds to $(\pm, 0)$ in the case of $SU_\xi(2) \otimes SU_\chi(2)$ and also applies for designating left-handed spinors in the case of an inequivalent representation of the $SU_{\text{left}}(2) \otimes SU_{\text{right}}(2)$ group; $(\mathbf{1}, \mathbf{2})$ for the spinor $(0, \frac{1}{2})$, with angular momentum projection $j = 1/2$, which also applies for designating right-handed spinors in the case of an inequivalent representation of the $SU_{\text{left}}(2) \otimes SU_{\text{right}}(2)$ group; $(\mathbf{2}, \mathbf{2})$ for a vector, with angular momentum projection $j = 0$ and $j = 1$; etc. With respect to the fundamental representations of $SL(2, \mathbb{C})$, one may construct more complex objects like $(\mathbf{1}, \mathbf{2}) \otimes (\mathbf{1}, \mathbf{2}) \equiv (\mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3})$, a representation that composes Lorentz tensors like

$$C_{\alpha\beta}(x) = \epsilon_{\alpha\beta} D(x) + G_{\alpha\beta}(x), \quad (9)$$

where $D(x)$ is a scalar and $G_{\alpha\beta} = G_{\beta\alpha}$ is totally symmetric, or even $(\mathbf{2}, \mathbf{1}) \otimes (\mathbf{1}, \mathbf{2}) \equiv (\mathbf{2}, \mathbf{2})$, such that $(\mathbf{2}, \mathbf{2}) \otimes (\mathbf{2}, \mathbf{2}) \equiv (\mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{3}, \mathbf{1}) \oplus (\mathbf{3}, \mathbf{3})$, which composes Lorentz tensors like

$$\varphi^{\mu\nu}(x) = A^{\mu\nu}(x) + S^{\mu\nu}(x) + \frac{1}{4}g^{\mu\nu}\Omega(x), \quad (10)$$

which correspond to a decomposition into smaller irreducible representations related to the Poincaré classes quoted at [17], with $A^{\mu\nu} \equiv (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{3}, \mathbf{1})$ totally antisymmetric under $\mu \leftrightarrow \nu$, $S^{\mu\nu} \equiv (\mathbf{3}, \mathbf{3})$ totally symmetric under $\mu \leftrightarrow \nu$, and $\Omega \equiv (\mathbf{1}, \mathbf{1})$ transforming as a Lorentz scalar, multiplied by the metric tensor $g^{\mu\nu}$.

As a matter of completeness, the above properties, as discussed in Ref. [17], support the inclusion of interacting fields which also transform according to Poincaré symmetries described by the extended Poincaré group [16], as they appear

in a full Dirac Hamiltonian like [18,19]

$$\begin{aligned} \hat{H} = & A^0(\mathbf{x})\hat{I}_4 + \hat{\beta}[m + \phi_S(\mathbf{x})] + \hat{\alpha} \cdot [\hat{\mathbf{p}} - \mathbf{A}(\mathbf{x})] + i\hat{\beta}\hat{\gamma}_5\mu(\mathbf{x}) \\ & - \hat{\gamma}_5 q(\mathbf{x}) + \hat{\gamma}_5 \hat{\alpha} \cdot \mathbf{W}(\mathbf{x}) + i\hat{\boldsymbol{\gamma}} \cdot [\zeta_a \mathbf{B}(\mathbf{x}) + \kappa_a \mathbf{E}(\mathbf{x})] \\ & + \hat{\gamma}_5 \hat{\boldsymbol{\gamma}} \cdot [\kappa_a \mathbf{B}(\mathbf{x}) - \zeta_a \mathbf{E}(\mathbf{x})], \end{aligned} \quad (11)$$

where a fermion with mass m and momentum \mathbf{p} interacts with an external vector field with time component $A^0(\mathbf{x})$ and spatial components $\mathbf{A}(\mathbf{x})$ and is nonminimally coupled to magnetic and electric fields $\mathbf{B}(\mathbf{x})$ and $\mathbf{E}(\mathbf{x})$ (via κ_a and ζ_a). The above Hamiltonian also admits the inclusion of pseudovector field interactions with time component $q(\mathbf{x})$ and spatial components $\mathbf{W}(\mathbf{x})$ besides both scalar and pseudoscalar field interactions through $\phi_S(\mathbf{x})$ and $\mu(\mathbf{x})$, respectively. Algebraic strategies [17] for obtaining the analytical expression for the matrix density of the associated eigenstates of the above Hamiltonian problem have been developed; however, they are beyond the central scope of this paper.

III. BISPINOR ENTANGLEMENT UNDER LORENTZ BOOSTS

With the normalized bispinors from Eq. (8), one can construct a general quantum state of two particles A and B , with momenta (energy) \mathbf{p} (E_p) and \mathbf{q} (E_q), respectively, as a generic M -term normalized superposition

$$|\Psi^{AB}(\mathbf{p}, \mathbf{q})\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^M c_i |u_{s_i}(\mathbf{p})\rangle^A \otimes |u_{r_i}(\mathbf{q})\rangle^B, \quad (12)$$

with the normalization given by $\sum_{i=1}^M |c_i|^2 = N$, and where the subindex \pm has been omitted. Such two-particle states can be generated, for instance, in a QED elastic scattering process [20].⁴

As a matter of convenience, $u_{s_i}(\mathbf{p})$ [as well as $u_{r_i}(\mathbf{q})$] can be described by helicity eigenstates such that $\mathbf{e}_p \cdot \hat{\boldsymbol{\sigma}}^{(S)} |\chi_{s_i}(\mathbf{p})\rangle = (-1)^{s_i} |\chi_{s_i}(\mathbf{p})\rangle$ (where $\mathbf{e}_p = \mathbf{p}/|\mathbf{p}|$) can be factorized from Eq. (8) to set $u_{s_i}(\mathbf{p})$ as a spin-parity separable state. In terms of projected states $|z_\pm\rangle$, eigenstates of $\hat{\sigma}_z^{(S)}$, one can write

$$|\chi_{s_i}(\mathbf{p})\rangle = \frac{\hat{I}_2^{(S)} + (-1)^{s_i} \mathbf{e}_p \cdot \hat{\boldsymbol{\sigma}}^{(S)}}{\sqrt{1 + |\mathbf{e}_p \cdot \mathbf{e}_z|}} |z_\pm\rangle, \quad (13)$$

and if \mathbf{e}_p is in the z direction, $\mathbf{e}_p \equiv \mathbf{e}_z$, one has $|\chi_1(\mathbf{p})\rangle = |z_+\rangle$ and $|\chi_2(\mathbf{p})\rangle = |z_-\rangle$ such that, from now on, the label s_i (and also r_i), when set equal to 1 or 2, denotes positive or negative helicity, respectively.

Under the above assumptions, the density matrix of the generic superposition from Eq. (12) is written as

$$\rho(\mathbf{p}, \mathbf{q}) = \frac{1}{N} \sum_{i,j} c_i c_j^* \rho_{s_i s_j}^A(\mathbf{p}) \otimes \rho_{r_i r_j}^B(\mathbf{q}), \quad (14)$$

³Instead, they describe a subset of transformations of the $SO(4) \equiv SO(3) \otimes SO(3)$ group, for instance, those which include the double covering rotations.

⁴The choice of different momenta $\{\mathbf{p}_i\} \neq \mathbf{p}$ for each particle state of the same vector subspace (either A or B) introduces additional quantum correlations between spin and momentum variables, turning the problem into a more complex and nonrealistic one.

where

$$\begin{aligned} \rho_{s_i s_j}^A(\mathbf{p}) &= [|u_{s_i}(\mathbf{p})\rangle \langle u_{s_j}(\mathbf{p})|]^A \\ &= \frac{1}{2E_p} \{ (E_p \delta_{s_i s_j} + m \delta_{s_i s_j + 1}) \hat{I}_2^{(P)A} \\ &\quad + (E_p \delta_{s_i s_j + 1} + m \delta_{s_i s_j}) \hat{\sigma}_z^{(P)A} \\ &\quad + \sqrt{E_p^2 - m^2} [(-1)^{s_j} \hat{\sigma}_+^{(P)A} \\ &\quad + (-1)^{s_i} \hat{\sigma}_-^{(P)A}] \} \otimes \Xi_{s_i s_j}^{(S)A}(\mathbf{p}), \end{aligned} \quad (15)$$

where $\hat{\sigma}_\pm = \hat{\sigma}_x \pm i \hat{\sigma}_y$ and the factorized dependence on the momentum direction is implicitly given by

$$\Xi_{s_i s_j}^{(S)A}(\mathbf{p}) = [|\chi_{s_i}(\mathbf{p})\rangle \langle \chi_{s_j}(\mathbf{p})|]^A, \quad (16)$$

with a similar expression for $\rho_{r_i r_j}^B(\mathbf{q})$ by replacing $\{\mathbf{p}; s_{i(j)}\}$ by $\{\mathbf{q}; r_{i(j)}\}$ and A by B . As each of the components of the state (14) is a two-qubit state, the joint state $\rho(\mathbf{p}, \mathbf{q})$ is a four-qubit state. Differently from the case where a system composed of two subsystems has the quantum entanglement supported by the Schmidt decomposition theorem, the classification and quantification of entanglement in the above-constructed multipartite states is an open problem. Subsystems in a multipartite state can share entanglement in different nonequivalent ways and the corresponding multipartite entanglement can be approached from different points of view. As the joint state (14) is a pure state, its corresponding multipartite entanglement can be computed through the Meyer-Wallach global measure of entanglement $E_G[\rho]$, expressed in terms of the linear entropy $E_L[\rho]$ as [28]

$$\begin{aligned} E_G[\rho] &= \bar{E}[\rho^{\{\alpha_k\}}] = \frac{1}{4} (E_L[\rho^{(S)A}] + E_L[\rho^{(P)A}] \\ &\quad + E_L[\rho^{(S)B}] + E_L[\rho^{(P)B}]), \end{aligned} \quad (17)$$

with

$$E_L[\rho] = \frac{d}{d-1} (1 - \text{Tr}[\rho^2]),$$

where d is the dimension of the underlying Hilbert space in which ρ acts, and the reduced density matrix of a given subsystem α_k is obtained by tracing out all the other subsystems $\rho^{\alpha_j} = \text{Tr}_{\{\alpha_k \neq \alpha_j\}}[\rho]$. In the above problem, the subsystems considered correspond to spin S and parity P for particles A and B , i.e., $\{\alpha_k\} \equiv \{(S)A, (S)B, (P)A, (P)B\}$. In particular, the more the subsystems of a given state are mixed, the more entanglement is encoded among them; the global measure $E_G[\rho]$ captures a picture of the quantum correlations distributed among the four DOFs involved here.

The linear entropy of a reduced subsystem ρ^{α_k} of (12), which is a two-qubit state, is evaluated in terms of the components of its Bloch vector $a_n^{\alpha_k} = \text{Tr}[\hat{\sigma}_n^{\alpha_k} \rho^{\alpha_k}]$ as

$$E_L[\rho^{\alpha_k}] = 1 - \sum_{n=\{x,y,z\}} (a_n^{\alpha_k})^2 \quad (18)$$

and the global measure from Eq. (17) can be simplified into

$$E_G[\rho] = 1 - \frac{1}{4} \sum_{\alpha=\{\alpha_k\}} \sum_{n=\{x,y,z\}} (a_n^\alpha)^2, \quad (19)$$

with $\{\alpha_k\} \equiv \{(S)A, (S)B, (P)A, (P)B\}$. The Bloch vectors of the subsystems of A are explicitly given by

$$\begin{aligned} a_n^{(S)A} &= \frac{1}{N} \sum_{i,j} c_i c_j^* \mathcal{M}_{r_i r_j}(\mathbf{q}) \frac{1}{E_p} (E_p \delta_{s_i s_j} + m \delta_{s_i s_j + 1}) \\ &\quad \times \text{Tr}[\hat{\sigma}_n^{(S)A} \Xi_{s_i s_j}^{(S)A}(\mathbf{p})] \end{aligned} \quad (20)$$

for the spin subsystem and

$$\begin{aligned} a_x^{(P)A} &= \frac{1}{N} \sum_i (-1)^{s_i} |c_i|^2 \frac{\sqrt{E_p^2 - m^2}}{E_p}, \\ a_y^{(P)A} &= 0, \\ a_z^{(P)A} &= \frac{1}{N} \sum_{i,j} c_i c_j^* \mathcal{M}_{r_i r_j}(\mathbf{q}) \text{Tr}[\Xi_{s_i s_j}^{(S)A}(\mathbf{p})] \\ &\quad \times \frac{1}{E_p} (E_p \delta_{s_i s_j + 1} + m \delta_{s_i s_j}) \end{aligned} \quad (21)$$

for the parity subsystem, where

$$\begin{aligned} \mathcal{M}_{r_i r_j}(\mathbf{q}) &= \text{Tr}[\rho_{r_i r_j}^B(\mathbf{q})] \\ &= \frac{1}{E_q} (E_q \delta_{r_i r_j} + m \delta_{r_i r_j + 1}) \text{Tr}[\Xi_{r_i r_j}^{(S)B}(\mathbf{q})]. \end{aligned} \quad (22)$$

Analogous expressions for the Bloch vectors of the subsystems of B are given by (20) and (22) with the replacements $\{\mathbf{p}; s_{i(j)}\} \leftrightarrow \{\mathbf{q}; r_{i(j)}\}$ and $A \leftrightarrow B$.

To evaluate the quantum entanglement encoded only by the spin DOFs in (14), one considers the spin-spin reduced density matrix

$$\begin{aligned} \rho^{(S)A, (S)B}(\mathbf{p}, \mathbf{q}) &= \text{Tr}_{(P)A, (P)B}[\rho(\mathbf{p}, \mathbf{q})] \\ &= \frac{1}{N} \sum_{i,j} c_i c_j^* \frac{(E_p \delta_{s_i s_j} + m \delta_{s_i s_j + 1})(E_q \delta_{r_i r_j} + m \delta_{r_i r_j + 1})}{E_p E_q} \\ &\quad \times \Xi_{s_i s_j}^{(S)A}(\mathbf{p}) \otimes \Xi_{r_i r_j}^{(S)B}(\mathbf{q}), \end{aligned} \quad (23)$$

which is, in general, a mixed state. Entanglement in mixed states cannot be evaluated in terms of the linear entropy, as a mixed subsystem does not imply a joint entangled state for mixtures. Instead, the characterization of quantum entanglement, in this case, is given by the Peres separability criterion [29], which asserts that a bipartite state $\rho \in \mathcal{H}_A \otimes \mathcal{H}_B$ is separable if and only if the partial transpose density matrix with respect to the any of its subsystem ρ^{T_A} has only positive eigenvalues. With respect to a fixed basis on the composite Hilbert space $\{|\lambda_i\rangle \otimes |\nu_j\rangle\}$ (with $|\lambda_i\rangle \in \mathcal{H}_A$ and $|\nu_j\rangle \in \mathcal{H}_B$), the matrix elements of the partial transpose with respect to the A subsystem ρ^{T_A} are given by

$$\langle \lambda_i \otimes \langle \nu_j | (\rho)^{T_A} | \lambda_k \otimes |\nu_l \rangle = \langle \lambda_k | \otimes \langle \nu_j | \rho | \lambda_i \otimes |\nu_l \rangle, \quad (24)$$

and in the light of the separability criterion, the negativity $\mathcal{N}[\rho]$ is defined as [30]

$$\mathcal{N}[\rho] = \sum_i |\lambda_i| - 1, \quad (25)$$

where λ_i are the eigenvalues of ρ^{TA} . The spin-spin negativity of (14) $\mathcal{N}[\rho^{(S)A,(S)B}]$ is then evaluated with the eigenvalues of the partial transpose of (23) with respect to $(S)A$ to return

$$\begin{aligned} & (\rho^{(S)A,(S)B})^{TA}(\mathbf{p}, \mathbf{q}) \\ &= \{\text{Tr}_{(P)A,(P)B}[\rho(\mathbf{p}, \mathbf{q})]\}^{TA} \\ &= \frac{1}{N} \sum_{i,j}^M c_i c_j^* \frac{(E_p \delta_{s_i s_j} + m \delta_{s_i s_j + 1})(E_q \delta_{r_i r_j} + m \delta_{r_i r_j + 1})}{E_p E_q} \\ & \quad \times \Xi_{s_j s_i}^{(S)A}(\mathbf{p}) \otimes \Xi_{r_i r_j}^{(S)B}(\mathbf{q}), \end{aligned} \quad (26)$$

where the subtle change with respect to (23) is in the subindex of $\Xi^{(S)A}$.

IV. COVARIANCE OF THE DIRAC EQUATION AND THE EFFECTS OF LORENTZ BOOSTS

Once the global and the spin-spin entanglement of the general superposition (12) are characterized by Eqs. (19) and (20) and the spin-spin negativity is evaluated through the eigenvalues of Eq. (26), one can describe how the Lorentz boosts affect such quantum correlations. First, one notices that the covariant form of the Dirac equation

$$(\hat{\gamma}_\mu p^\mu - m \hat{I}_4) \psi(x) = 0, \quad (27)$$

where $\hat{\gamma}_0 = \hat{\beta}$ and $\gamma_\mu = (\gamma_0, \hat{\boldsymbol{\gamma}})$, with $\hat{\boldsymbol{\gamma}} = \hat{\beta} \hat{\boldsymbol{\alpha}}$, transforms under a Lorentz boost $x^\mu \rightarrow x'^\mu = \Lambda_\nu^\mu x^\nu$ as

$$(\hat{\gamma}^\mu p_\mu - m \hat{I}_4) \psi(x) = 0 \rightarrow [(\hat{\gamma}')^\mu p'_\mu - m \hat{I}_4] \psi'(x') = 0 \quad (28)$$

and its solution $\psi(x)$ transforms as

$$\psi'(x') = \hat{S}[\Lambda] \psi(\Lambda^{-1} x'), \quad (29)$$

where $\hat{S}[\Lambda]$ corresponds to the transformation in the bispinor space representation (cf. $\hat{D}[\Lambda]$ from (1)). Lorentz boosts $\Lambda(\omega)$ can be parametrized in terms of components in the vector representation of $\text{SO}(1,3)$ as $[\Lambda(\omega)]_{ij} = \delta_{ij} + [\cosh(\omega) - 1] n_i n_j$, $[\Lambda(\omega)]_{i0} = [\Lambda(\omega)]_{0i} = \sinh(\omega) n_i$, and $[\Lambda(\omega)]_{00} = \cosh(\omega)$, where $\omega = \text{arccosh}(\sqrt{1 - v^2})$ is the (dimensionless) boost rapidity, v is the reference frame velocity (between \mathcal{S} and \mathcal{S}'), and n_i are the space components of the boost direction \mathbf{n} , with $\mathbf{n} \cdot \mathbf{n} = 1$. In the bispinor space representation, $\hat{S}[\Lambda(\omega)]$ reads

$$\hat{S}[\Lambda(\omega)] = \cosh\left(\frac{\omega}{2}\right) \hat{I}_4 - \sinh\left(\frac{\omega}{2}\right) \mathbf{n} \cdot \hat{\boldsymbol{\alpha}}, \quad (30)$$

which is a nonunitary operator. By following the above-introduced two-qubit prescription, the boost operator (30) can be expressed in the form

$$\hat{S}[\Lambda(\omega)] = \cosh\left(\frac{\omega}{2}\right) \hat{I}_2^{(P)} \otimes \hat{I}_2^{(S)} - \sinh\left(\frac{\omega}{2}\right) \mathbf{n} \cdot (\hat{\sigma}_x^{(P)} \otimes \hat{\sigma}^{(S)}), \quad (31)$$

from which one can evaluate the effects of boosts on parity and spin subsystems. For instance, keeping the covariant notation for the quadrimomentum p , the density matrix of a single helicity bispinor with quantum number s transforms under

boosts as

$$\rho_s(p) \rightarrow \rho'_s(p') = \frac{1}{\cosh(\omega)} \hat{S}[\Lambda(\omega)] \rho_s(\Lambda^{-1} p') \hat{S}^\dagger[\Lambda(\omega)], \quad (32)$$

where the term $[\cosh(\omega)]^{-1}$ was included to keep the normalization of the spinor and (31) can be used to describe the transformation law of the subsystem described by the spin density $\rho_s^{(S)}(p) = \text{Tr}_{(P)}[\rho_s(p)]$ as

$$\begin{aligned} & \rho_s^{(S)}(p) \rightarrow \rho_s^{(S)}(p') \\ &= \frac{1}{\cosh(\omega)} \left[\cosh^2\left(\frac{\omega}{2}\right) \rho_s^{(S)}(p) + \sinh^2\left(\frac{\omega}{2}\right) (\mathbf{n} \cdot \hat{\boldsymbol{\sigma}}) \right. \\ & \quad \times \rho_s^{(S)}(p) (\mathbf{n} \cdot \hat{\boldsymbol{\sigma}}) \\ & \quad \left. - (-1)^s \sinh(\omega) \frac{E_p - m}{E_p} \{ \mathbf{n} \cdot \hat{\boldsymbol{\sigma}}, \rho_s^{(S)}(p) \} \right], \end{aligned} \quad (33)$$

where $\{, \}$ denotes anticommutators and which, in the limit $E_p - m \simeq E_p$, can be subtly simplified to give a transformation law in the form of $\rho_s^{(S)}(p') = \hat{O} \rho_s^{(S)}(p) \hat{O}^\dagger$, where \hat{O} is the unitary operator

$$\hat{O} = \frac{1}{\sqrt{\cosh(\omega)}} \left[\cosh\left(\frac{\omega}{2}\right) \hat{I}_2 - \sinh\left(\frac{\omega}{2}\right) (\mathbf{n} \cdot \hat{\boldsymbol{\sigma}}) (\mathbf{e}_p \cdot \hat{\boldsymbol{\sigma}}) \right]. \quad (34)$$

In fact, such transformation under a Lorentz boost is the same as that obtained for states belonging to the irreducible representation $(+, \frac{1}{2})$ of the Poincaré group, which can be recast in terms of a momentum-dependent rotation and which is the basis of several results in relativistic quantum information.⁵

Considering the generic two-particle state (14) in a reference frame \mathcal{S} , the transformed density matrix describing the state in an inertial frame \mathcal{S}' , related to \mathcal{S} by a Lorentz boost Λ , is given by

$$\begin{aligned} & \rho(\mathbf{p}, \mathbf{q}) \rightarrow \rho'(\mathbf{p}', \mathbf{q}') \\ &= \frac{1}{v} (\hat{S}^A[\Lambda] \otimes \hat{S}^B[\Lambda]) \rho(\mathbf{p}, \mathbf{q}) \{ (\hat{S}^A[\Lambda])^\dagger \otimes (\hat{S}^B[\Lambda])^\dagger \} \\ &= \frac{1}{v} \sum_{i,j}^M c_i c_j^* \varrho_{s_i s_j}^A(\mathbf{p}) \otimes \varrho_{r_i r_j}^B(\mathbf{q}), \end{aligned} \quad (35)$$

where $v = \text{Tr}\{(\hat{S}^A[\Lambda] \otimes \hat{S}^B[\Lambda])^2 \rho(\mathbf{p}, \mathbf{q})\}$ and the transformed term $\varrho_{s_i s_j}^A(\mathbf{p})$ reads

$$\begin{aligned} \varrho_{s_i s_j}^A(\mathbf{p}) &= \cosh^2\left(\frac{\omega}{2}\right) \rho_{s_i s_j}^A(\mathbf{p}) \\ & \quad - \frac{\sinh(\omega)}{2} \{ \rho_{s_i s_j}^A(\mathbf{p}), (\hat{\sigma}_x^{(P)A} \otimes \mathbf{n} \cdot \hat{\boldsymbol{\sigma}}^{(S)A}) \} \end{aligned}$$

⁵The nonunitarity of $\hat{S}[\Lambda(\omega)]$ has also additional implications for the definition of spin operators in the context of relativistic quantum mechanics [22]. Apart from the usual Pauli spin operator proportional to $\hat{\boldsymbol{\Sigma}} = \hat{I}_2^{(P)} \otimes \hat{\boldsymbol{\sigma}}^{(S)}$, other spin operators were also proposed in the literature. For example, the FW spin operator [31] was used in the context of transformation properties of Dirac bispinors to define a covariant spin reduced density matrix [23,26] and states constructed with FW eigenstates were then used in describing transformation properties of spin entropy as well as the spin-spin Bell inequality under Lorentz boosts.

$$\begin{aligned}
 & + \sinh^2\left(\frac{\omega}{2}\right) (\hat{\sigma}_x^{(P)A} \otimes \mathbf{n} \cdot \hat{\sigma}^{(S)A}) \\
 & \times \rho_{s_i s_j}^A(\mathbf{p}) (\hat{\sigma}_x^{(P)A} \otimes \mathbf{n} \cdot \hat{\sigma}^{(S)A}), \quad (36)
 \end{aligned}$$

with an analogous expression for $\varrho_{r_i r_j}^B(\mathbf{q})$. The difference between the global entanglement in \mathcal{S}' and \mathcal{S} ,

$$\Delta E_G = E_G[\rho'(\mathbf{p}', \mathbf{q}')] - E_G[\rho(\mathbf{p}, \mathbf{q})], \quad (37)$$

is evaluated through Eqs. (19) and (20) replaced by transformed Bloch vectors, now renamed by $a \rightarrow \mathcal{A}$, which are given by

$$\begin{aligned}
 \mathcal{A}_k^{(S)A} & = \frac{1}{v} \sum_{i,j}^M c_i c_j^* \mu_{r_i r_j} \frac{1}{E_p} \left[\text{Tr}[\hat{\sigma}_k^{(S)A} \Xi_{s_i s_j}^{(S)A}] \cosh^2\left(\frac{\omega}{2}\right) \right. \\
 & \times (E_p \delta_{s_i s_j} + m \delta_{s_i s_j + 1}) \\
 & - 2(-1)^{s_i} n_k \text{Tr}[\Xi_{s_i s_j}^{(S)A}] \sinh(\omega) \sqrt{E_p^2 - m^2} \delta_{s_i s_j} \\
 & \left. + \text{Tr}[\hat{\sigma}_k^{(S)A} (\mathbf{n} \cdot \hat{\sigma}^{(S)A}) \Xi_{s_i s_j}^{(S)A} (\mathbf{n} \cdot \hat{\sigma}^{(S)A})] \sinh^2\left(\frac{\omega}{2}\right) \right. \\
 & \left. \times (E_p \delta_{s_i s_j} + m \delta_{s_i s_j + 1}) \right] \quad (38)
 \end{aligned}$$

for the spin reduced subsystem of A and

$$\begin{aligned}
 \mathcal{A}_x^{(P)A} & = \frac{1}{v} \sum_{i,j}^M c_i c_j^* \mu_{r_i r_j} \frac{1}{E_p} [(-1)^{s_i} \text{Tr}[\Xi_{s_i s_j}^{(S)A}] \\
 & \times \cosh(\omega) \sqrt{E_p^2 - m^2} \delta_{s_i s_j} \\
 & - \sinh(\omega) \text{Tr}[(\mathbf{n} \cdot \hat{\sigma}^{(S)A}) \Xi_{s_i s_j}^{(S)A} (E_p \delta_{s_i s_j} + m \delta_{s_i s_j + 1})], \\
 \mathcal{A}_x^{(P)A} & = \frac{1}{v} \sum_{i,j}^M c_i c_j^* \mu_{r_i r_j} \frac{E_p \delta_{s_i s_j + 1} + m \delta_{s_i s_j}}{E_p} \text{Tr}[\Xi_{s_i s_j}^{(S)A}] \quad (39)
 \end{aligned}$$

for the parity reduced subsystem, where

$$\begin{aligned}
 \mu_{r_i r_j} & = \text{Tr}[\varrho_{r_i r_j}^B] \\
 & = \frac{1}{E_p} \left\{ \cosh(\omega) (E_q \delta_{r_i r_j} + m \delta_{r_i r_j + 1}) \text{Tr}[\Xi_{r_i r_j}^{(S)B}] \right. \\
 & \left. - (-1)^{r_i} \sinh(\omega) \delta_{r_i r_j} \sqrt{E_q^2 - m^2} \text{Tr}[\mathbf{n} \cdot \hat{\sigma}^{(S)B} \Xi_{r_i r_j}^{(S)B}] \right\} \quad (40)
 \end{aligned}$$

and, in all the above expressions, the explicit dependence on \mathbf{p} and \mathbf{q} has been omitted from the notation. Through the above expressions, again, the Bloch vector for the subsystems of B can be obtained with the replacements $\{\mathbf{p}; s_{i(j)}\} \leftrightarrow \{\mathbf{q}; r_{i(j)}\}$ and $A \leftrightarrow B$ in Eqs. (38)–(40). For any boost one also has $\mathcal{A}_y^{(P)A} = \mathcal{A}_y^{(P)B} = 0$.

The effects of the boost on the spin-spin entanglement, on the other hand, are described by a change of the negativity

$$\Delta \mathcal{N}^{(S)A, (S)B} = \mathcal{N}[\varrho^{(S)A, (S)B}] - \mathcal{N}[\rho^{(S)A, (S)B}], \quad (41)$$

with the transformed spin-spin density matrix given by

$$\varrho^{(S)A, (S)B} = \frac{1}{v} \sum_{i,j}^M c_i c_j^* \varrho_{s_i s_j}^{(S)A} \otimes \varrho_{r_i r_j}^{(S)B}, \quad (42)$$

where

$$\begin{aligned}
 \varrho_{s_i s_j}^{(S)A} & = \cosh^2\left(\frac{\omega}{2}\right) \frac{E_p \delta_{s_i s_j} + m \delta_{s_i s_j + 1}}{E_p} \Xi_{s_i s_j}^{(S)A} \\
 & - (-1)^{s_i} \frac{\sinh(\omega)}{2} \frac{\sqrt{E_p^2 - m^2}}{E_p} \delta_{s_i s_j} \{ \Xi_{s_i s_j}^{(S)A}, \mathbf{n} \cdot \hat{\sigma}^{(S)A} \} \\
 & + \sinh^2\left(\frac{\omega}{2}\right) \frac{E_p \delta_{s_i s_j} + m \delta_{s_i s_j + 1}}{E_p} (\mathbf{n} \cdot \hat{\sigma}^{(S)A}) \\
 & \times \Xi_{s_i s_j}^{(S)A} (\mathbf{n} \cdot \hat{\sigma}^{(S)A}), \quad (43)
 \end{aligned}$$

with a corresponding expression for $\varrho_{r_i r_j}^{(S)B}$. From the above expression one concludes that if the boost is performed in a direction \mathbf{n} such that $\{ \Xi_{s_i s_j}^{(S)A}, \mathbf{n} \cdot \hat{\sigma}^{(S)A} \} = \{ \Xi_{r_i r_j}^{(S)B}, \mathbf{n} \cdot \hat{\sigma}^{(S)B} \} = 0$, then the spin reduced density matrix (42) is invariant.

A. Entanglement for an overall class of antisymmetric states

The above framework describes quantitatively the changes on multipartite quantum correlations, as quantified by E_G , and on spin-spin entanglement induced by Lorentz boosts acting on a generic superposition of two-particle helicity bispinors, as quantified by \mathcal{N} . As the nature of fermionic particles requires antisymmetric wave functions, states that are given by the antisymmetric superpositions have to be considered in the form

$$|\Psi_{sr}^M(\mathbf{p}, \mathbf{q})\rangle = \frac{|u_s(\mathbf{p})\rangle^A \otimes |u_r(\mathbf{q})\rangle^B - |u_r(\mathbf{q})\rangle^A \otimes |u_s(\mathbf{p})\rangle^B}{\sqrt{2}}. \quad (44)$$

Talking about Dirac particles like electrons, quarks, neutrinos, etc., some of the above configurations are very difficult to be produced phenomenologically. Thus, only a few examples will be considered in the following, from the less to the more relevant ones.

At the reference frame \mathcal{S} with $\mathbf{p} = -\mathbf{q}$, the center of momentum frame, positive and negative helicity eigenstates are given by

$$\begin{aligned}
 |\chi_1(\mathbf{p})\rangle & = |\chi_2(\mathbf{q})\rangle = |z_+\rangle, \\
 |\chi_2(\mathbf{p})\rangle & = |\chi_1(\mathbf{q})\rangle = |z_-\rangle
 \end{aligned} \quad (45)$$

and in the unboosted frame \mathcal{S} the states are also eigenstates of the Pauli spin operator σ_z . It is sufficient to consider the boost with direction \mathbf{n} in a plane defined by the unitary vectors \mathbf{e}_z and \mathbf{e}_x , with $\mathbf{n} = \sin(\theta)\mathbf{e}_x + \cos(\theta)\mathbf{e}_z$, as pictorially depicted in Fig. 1.

By adapting the notation to the simplifications from Eq. (45), one has the antisymmetric state given by

$$|\psi_1\rangle = \frac{|u_1(\mathbf{p})\rangle^A \otimes |u_2(\mathbf{q})\rangle^B - |u_2(\mathbf{q})\rangle^A \otimes |u_1(\mathbf{p})\rangle^B}{\sqrt{2}}, \quad (46)$$

in a superposition of helicities which, however, is spin-spin separable. Since $\Xi_{sr}^{(S)A} = \Xi_{sr}^{(S)B} = |z_+\rangle\langle z_+|$ for all s and r , the transformed spin-spin density matrix (42) is invariant under partial transposition with respect to any of its subsystems and thus a Lorentz boost does not create spin-spin entanglement. Nevertheless, the global entanglement E_G is not invariant, as depicted in Fig. 2, which shows ΔE_G as a function of the boost rapidity ω and of the boost angle θ . Of course, this is

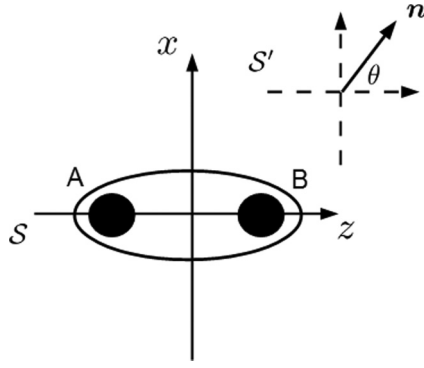


FIG. 1. Schematic representation of the boost scenario. The joint state of A and B is described by the antisymmetric superposition of bispinors (44) with momenta, in \mathcal{S} , $\mathbf{p} = -\mathbf{q} = p\mathbf{e}_z$. A Lorentz boost is performed to describe the joint state in a frame \mathcal{S}' moving with respect to \mathcal{S} in the $\mathbf{n} = \sin(\theta)\mathbf{e}_x + \cos(\theta)\mathbf{e}_z$ direction with rapidity ω .

because $|\psi_1\rangle$ mixes different momentum eigenstates, in a kind of artificial and unrealistic physical composition of particles A and B . Boosts parallel to the momenta in \mathcal{S} do not increase the amount of global entanglement in the state, although for any nonparallel boosts the global entanglement increases due to an increase in both parity and spin reduced entropies, which are essentially constrained by the dependence on the momentum components. It tends to the maximum value (~ 1) for high-speed boosts.

Otherwise, a maximally entangled spin-spin state in \mathcal{S} can be constructed through an antisymmetric superposition between positive helicities

$$|\psi_2\rangle = \frac{|u_1(\mathbf{p})\rangle^A \otimes |u_1(\mathbf{q})\rangle^B - |u_1(\mathbf{q})\rangle^A \otimes |u_1(\mathbf{p})\rangle^B}{\sqrt{2}}, \quad (47)$$

which, according to the correspondence from (45), indeed can be recast as

$$|\psi_2\rangle = \frac{|u_1(\mathbf{p})\rangle^A \otimes |u_1(\mathbf{q})\rangle^B - |u_2(\mathbf{p})\rangle^A \otimes |u_2(\mathbf{q})\rangle^B}{\sqrt{2}}, \quad (48)$$

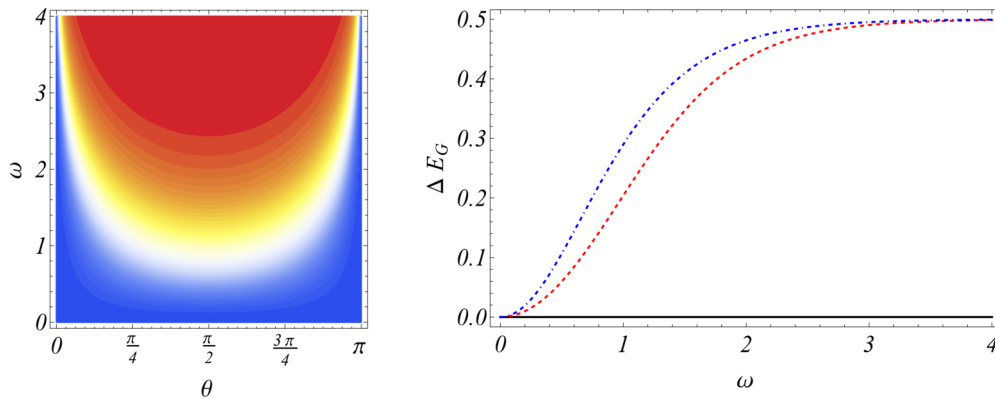


FIG. 2. Variation of the global entanglement for the state (46) as a function of the boost rapidity ω (dimensionless) and of the boost direction angle θ in radians (left plot) and as a function of the boost rapidity for $\theta = 0$ (black solid line), $\pi/4$ (red dashed line), and $\pi/2$ (blue dot–double-dashed line). The initial rapidity of the states in the unboosted frame is $\omega_0 = \text{arccosh}(E_p/m) = 1$. The global entanglement encoded in the DOFs of such a pair of bispinors always increases due to the boost when the transformation is in a direction not parallel to the momenta \mathbf{p} and \mathbf{q} with respect to \mathcal{S} . In the limit of high-speed boosts, E_G has its maximum value 1, as for the unboosted state $E_G[\rho] = 1/2$. Among these correlations, no spin-spin entanglement is present, as in any frame $\mathcal{N}^{(S)A,(S)B}[\psi_1] = 0$.

which corresponds to a much more realistic configuration, for which particles in the subspace A and B have well defined momenta \mathbf{p} and \mathbf{q} , respectively, in agreement with the construction from the preceding section. Figure 3 depicts the variation of the global and the spin-spin entanglement of $|\psi_2\rangle$ as a function of the boost rapidity ω . In this case, the variation of entanglement is independent of the boost angle and, as for the state from Eq. (46), the global entanglement increases under Lorentz boosts. On the other hand, spin-spin entanglement is degraded by the boost transformation and for high-speed boosts the spin-spin state is completely separable.

A third antisymmetric configuration is given by

$$|\psi_3\rangle = \frac{|u_1(\mathbf{p})\rangle^A \otimes |u_2(\mathbf{p})\rangle^B - |u_2(\mathbf{p})\rangle^A \otimes |u_1(\mathbf{p})\rangle^B}{\sqrt{2}}, \quad (49)$$

which describes a two-particle helicity superposition moving in the \mathbf{e}_z direction where both particles have the same momentum. This case is phenomenologically interesting because $\Delta v = 0$ is a kinematic Lorentz invariant. Two electrons in a common rest frame will have $\Delta v = 0$ for any relativistic boost. In this case, the spin-spin entanglement depends on the momentum p even in the unboosted frame. Differently from the preliminary examples, both global entanglement, depicted in Fig. 4, and spin-spin entanglement, depicted in Fig. 5, exhibit a nonmonotonic behavior under Lorentz boosts. In particular, for a boost parallel to the momentum \mathbf{p} with rapidity equal to $\text{arccosh}(E_p/m)$, the global entanglement is minimum, as this frame corresponds to the common rest frame of the particles where there is only spin-spin entanglement. For a high-speed boost, the entanglement shared between the DOFs of the state is enhanced, although the spin-spin entanglement, as in the case of the state (48), is completely degraded.

It is worth mentioning that, although the global measure from Eq. (17) was considered, four-qubit state entanglement can be computed through another global measure of entanglement defined in a fashion similar to (17), but with linear entropies of the reduced subsystems of two qubits. This quantity is calculated with terms of the form $\text{Tr}[\hat{\sigma}_i^{\alpha_k} \hat{\sigma}_j^{\beta_l} \rho^{\{\alpha_k;\beta_l\}}]$ and contains, in addition to the information encoded in E_G

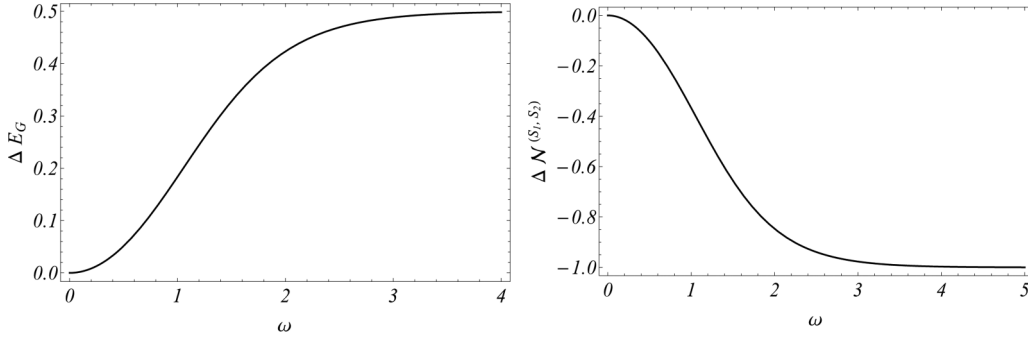


FIG. 3. Variation of the global entanglement (left plot) and spin-spin entanglement (right plot) as a function of the rapidity ω (dimensionless) for the state (48). The transformation of correlations in this state is independent of the boost angle θ and, although the global entanglement increases due to the boost with a behavior similar to the one of Fig. 2, the spin-spin entanglement is degraded. In the limit of high-speed boosts, S_{AB} is separable from all the other DOFs and spin-spin entanglement vanishes: The global entanglement is encoded by the parity DOFs.

(17), also correlations between pairs of the subsystems [32]. Nevertheless, in the case of the antisymmetric states considered here, the behavior of this quantity is qualitatively similar to the behavior depicted in Figs. 3 and 4 and add no information about the variation of quantum entanglement encoded by bispinors under Lorentz boosts. Another point of view of multipartite entanglement is provided by considering the geometry of the composite Hilbert space and by studying distances between a given multipartite state and the set of so-called K -separable states [33]. In this case, the quantification of multipartite entanglement can capture more information about different multipartite components that contribute to the total amount of quantum correlation in a given state, requiring an extremization process. This more complete picture of multipartite entanglement for the two spinors states considered here is postponed to future investigations.

B. Transformation of entanglement in chiral states

Superpositions of eigenstates of the chiral operator $\hat{\gamma}_5 = \hat{\sigma}_x^{(P)} \otimes \hat{I}^{(S)}$ defined in terms of the free bispinors $u_s(\mathbf{p})$ as

$$u_s^f(\mathbf{p}) = \frac{\hat{I} + (-1)^f \hat{\gamma}_5}{2} u_s(\mathbf{p}), \tag{50}$$

with $f = 0, 1$, can also be investigated in the above context. Differently from the helicity, the chirality is a Lorentz invariant given that the chiral and the boost operator commute, i.e., $[\hat{\gamma}_5, \hat{S}[\Lambda(\omega)]] = 0$. However, for massive particles, it is not a dynamical conserved quantity as $[\hat{\gamma}_5, \hat{H}] \neq 0$ [34,35]. This invariance property has implications for the transformation laws of quantum entanglement encoded by superpositions of chiral states

$$\psi^{\text{chiral}}(\mathbf{p}, \mathbf{q}) = \frac{1}{N} \sum_i^M c_i |u_{s_i}^{f_i}(\mathbf{p})\rangle^A \otimes |u_{r_i}^{g_i}(\mathbf{q})\rangle^B, \tag{51}$$

where f_i is the chirality of the bispinor $|u_{s_i}^{f_i}(\mathbf{p})\rangle^A$ and g_i is the chirality of $|u_{r_i}^{g_i}(\mathbf{q})\rangle^A$. Chiral states constructed through projection of helicity states can be written in the simplified form

$$|u_s^f(\mathbf{p})\rangle = |f\rangle \otimes |\chi_s(\mathbf{p})\rangle, \tag{52}$$

where $|f\rangle = (|z_+\rangle + (-1)^f |z_-\rangle)/2$ are the eigenstates of the $\hat{\sigma}_x$ operator, and thus the density matrix of (51) reads

$$\rho^{\text{chiral}} = \frac{1}{N} \sum_{i,j}^M c_i c_j^* (|f_i\rangle \langle f_j|)^A \otimes \Xi_{s_i s_j}^{(S)A} \otimes (|g_i\rangle \langle g_j|)^B \otimes \Xi_{r_i r_j}^{(S)B}, \tag{53}$$

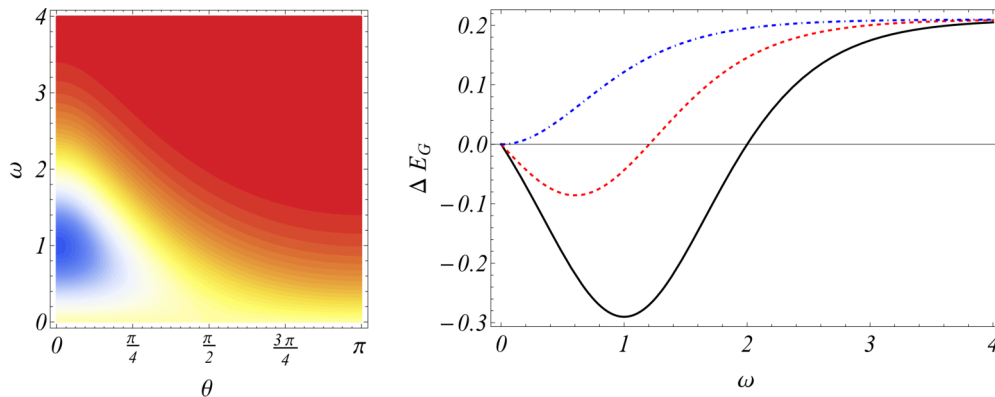


FIG. 4. Variation of the global entanglement under a Lorentz boost for the state (49) as a function of the boost rapidity ω (dimensionless) and of the boost direction angle θ in radians (left plot). The curves of the right plot correspond to those of Fig. 2. Different from Figs. 3 and 4, global entanglement exhibits a nonmonotonic behavior for $\theta < \pi/2$. For a parallel boost $\theta = 0$ (solid curve) global entanglement reaches its minimum for $\omega = 1$, which corresponds to the reference frame in which the bispinors are at rest: All quantum correlations correspond to only spin-spin entanglement.

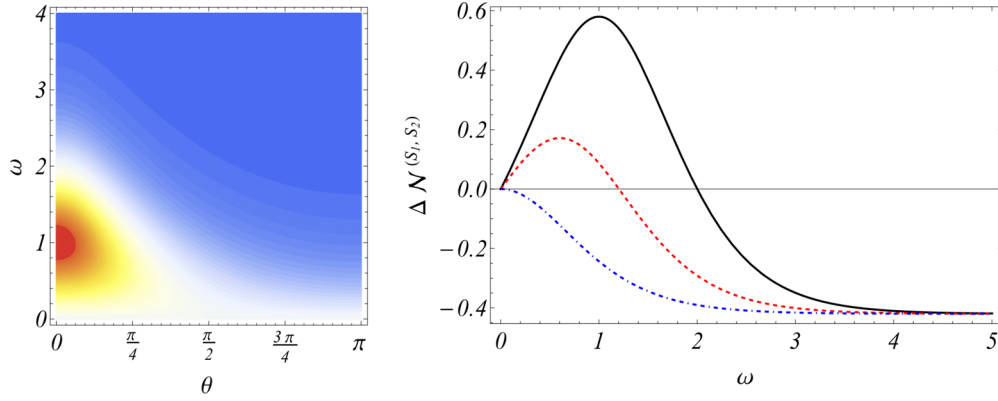


FIG. 5. Variation of spin-spin entanglement under a Lorentz boost for the state (49) as a function of the boost rapidity ω (dimensionless) and of the boost direction angle θ in radians (left plot) and as a function of the boost rapidity for the boost direction corresponding to those of Figs. 2 and 4. The behavior of quantum correlations encoded between the spin degrees of freedom is complementary to that exhibited by the global entanglement of Fig. 4. For $\theta < \pi/2$ the behavior is nonmonotonic with a local maximum corresponding to the rest frame of the bispinors and a complete degradation resulting from a high-speed boost.

where, again, the explicit dependence on momenta has been suppressed. Since the chiral eigenstates are invariant under boosts, the density matrix (53) transforms as

$$\rho'_{\text{chiral}} = \frac{1}{N} \sum_{i,j} c_i c_j^* (|f_i\rangle\langle f_j|)^A \otimes \Xi_{s_i s_j}^{(S)A} \otimes (|g_i\rangle\langle g_j|)^B \otimes \Xi_{r_i r_j}^{(S)B}, \quad (54)$$

where $\Xi_{ij}^{(S)A} = \hat{O}_{f_i} \Xi_{s_i s_j}^{(S)A} \hat{O}_{f_j}$, with

$$\hat{O}_{f_i} = \cosh\left(\frac{\omega}{2}\right) \hat{I} - (-1)^{f_i} \sinh\left(\frac{\omega}{2}\right) \mathbf{n} \cdot \hat{\sigma}, \quad (55)$$

and changes of the global entanglement are exclusively due to changes of the spin terms $\Xi_{s_i s_j}^{(S)A}$. A particular situation is for $f_i = f$ and $g_i = g$, for which $\Xi_{ij}^{(S)A} = \hat{O}_f \Xi_{s_i s_j}^{(S)A} \hat{O}_f$, and

$$\rho'_{\text{chiral}} = \frac{1}{N} \sum_{i,j} c_i c_j^* (|f\rangle\langle f|)^A \otimes \Xi_{ij}^{(S)A} \otimes (|g\rangle\langle g|)^B \otimes \Xi_{ij}^{(S)B},$$

which exhibits an invariant quantum correlation when antisymmetric states as from Eqs. (48) and (49) are considered. In fact, the chiral states

$$|\psi_{2(3)}^{\text{chiral}}\rangle = \left(\frac{\hat{I} + (-1)^f \hat{\gamma}_5}{2} \right)^A \otimes \left(\frac{\hat{I} + (-1)^g \hat{\gamma}_5}{2} \right)^B |\psi_{2(3)}\rangle, \quad (56)$$

with $f, g = 0, 1$, are such that, for a boost direction given by $\mathbf{n} = (\sin(\theta), 0, \cos(\theta))$, one has $\rho_{2(3)}^{\text{chiral}} \rightarrow \rho_{2(3)}^{\text{chiral}} = \rho_{2(3)}^{\text{chiral}}$ and the states are completely Lorentz invariant.

V. CONCLUSION

The relativistic transformation properties of quantum entanglement have been the focus of many recent investigations, with a special interest in describing how the spin-spin entanglement does change under Lorentz boosts. Although the setup usually adopted to describe transformation properties of quantum entanglement has given some interesting insights into the physics of relativistic quantum information, when massive fermions are considered as the physical carriers of spin

$\frac{1}{2}$, a more complete description of the problem is required. The physical particles, such as electrons, muons, etc., are described by QED including, apart from the usual Poincaré symmetry, also invariance under parity transformation. This last symmetry operation exchanges two irreducible representations of the Poincaré group and a proper formulation is given in terms of irreducible representations of the so-called complete Lorentz group. The states of the particles are then described by four-component objects, the Dirac bispinors, which satisfy the Dirac equation.

In this paper we have described how Lorentz boosts affect quantum entanglement shared among the DOFs of a pair of bispinorial particles in a generic framework. As each of the bispinors is supported by an $SU(2) \otimes SU(2)$ structure associated with the spin and intrinsic parity, the corresponding multipartite entanglement was quantified by means of the Meyer-Wallach global measure of entanglement, given in terms of the linear entropies of each subsystem. Additionally, since the reduced spin state is mixed, the spin-spin entanglement was quantified through the appropriate negativity. By means of the $SU(2) \otimes SU(2)$ decomposition of the boost operator $\hat{S}[\Lambda]$, the transformation laws for the Bloch vectors (and for the reduced spin density matrix) of each subsystem were recovered for a generic state, setting a framework in which to describe changes of both global and spin-spin entanglements.

In order to specialize our results we have considered the action of Lorentz boosts in three different antisymmetric states. First we considered a spin-spin separable state in which the particles are moving in opposite directions in the unboosted frame. In such a scenario, Lorentz boosts cannot create spin-spin entanglement and the global entanglement monotonically increases as a function of the boost rapidity. The second antisymmetric state considered here describes particles with opposite momenta and maximal spin-spin entanglement. As in the first case, the global entanglement increases as a consequence of the boost, although a degradation of spin-spin entanglement is induced by the frame transformation. The last specific case consists of a pair of particles with the same momentum and spin-spin entanglement, exhibiting the nonmonotonic behavior of both global and spin-spin

entanglement under a Lorentz boost. Finally, we addressed the effects of Lorentz boosts on chiral states, which exhibit some subtle invariance properties. In particular, the density matrices obtained through projections of the antisymmetric states on definite chiral states are completely invariant under boosts.

The general formalism developed through this paper sets the framework for some future developments, including the computation of quantum entanglement among particles involved in scattering processes [36]. It may also be useful in the aim of a field-theoretic description of relativistic entanglement. Finally, given that some low-energy systems, such as trapped

ions and graphene, emulate the Dirac equation dynamics [19], interactions in such systems can be used to reproduce the effects of Lorentz transformations in feasible manipulable platforms which can work as simulating platforms for high-energy-physics measurements.

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