Universal quantum uncertainty relations between nonergodicity and loss of information

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We establish uncertainty relations between information loss in general open quantum systems and the amount of nonergodicity of the corresponding dynamics. The relations hold for arbitrary quantum systems interacting with an arbitrary quantum environment. The elements of the uncertainty relations are quantified via distance measures on the space of quantum density matrices. The relations hold for arbitrary distance measures satisfying a set of intuitively satisfactory axioms. The relations show that as the nonergodicity of the dynamics increases, the lower bound on information loss decreases, which validates the belief that nonergodicity plays an important role in preserving information of quantum states undergoing lossy evolution. We also consider a model of a central qubit interacting with a fermionic thermal bath and derive its reduced dynamics to subsequently investigate the information loss and nonergodicity in such dynamics. We comment on the "minimal" situations that saturate the uncertainty relations.

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I. INTRODUCTION

In practical situations, it is arguably impossible to completely isolate a quantum system from its surroundings and it is subjected to information loss due to dissipation and decoherence. In modeling open quantum systems, the simpler approach is to consider the environment to be memoryless, i.e., Markovian [1–5]. The system-environment relation is, however, more often than not, non-Markovian, and there are possibilities of information backflow into the system, which can be considered as a resource in information theoretic tasks [6–8]. The systems showing such properties are usually associated with various structured environments without the consideration of weak system-environment coupling and the Born-Markov approximation [9–17]. In a Markovian evolution, this information flow is one way and quickly leads to an unwanted total loss of coherence and other quantum characteristics. Using structured environments, it may be possible to reduce information loss of the associated quantum system.

On the other hand, an important statistical mechanical attribute of a system interacting with an environment, with the later being in a thermal state, is the ergodicity of the system. A physical process is considered to be ergodic if the statistical properties of the process can be realized from a long-time-averaged realization. In the study of the realization of a thermal relaxation process, ergodicity plays a very important role [3,18,19]. It also has important applications in quantum control [20–23], quantum communication [24], and beyond [25,26]. Here we intend to capture the notion of "nonergodicity" from the perspective of quantum channels, i.e., considering only the reduced dynamics of a quantum system interacting with an environment. In the framework of open quantum systems, a rigorous study on ergodic quantum channels can be found in [27]. Ergodic quantum channels are channels having a unique fixed point in the space of density matrices [28]. Nonergodicity of a dynamical process can then be quantified as the amount of deviation from a ergodic process in open system dynamics.

In this work, we find a connection between information loss of a general open quantum system and the nonergodicity therein. We propose a measure of information loss in a quantum system based on distinguishability of quantum states, which in turn is based on distance measures on the space of density operators [29–34]. We quantify the nonergodicity of the dynamics based on the distance between the time-averaged state after sufficiently long processing time and the corresponding thermal equilibrium state. Within this paradigm, we derive an uncertainty relation between information loss and the amount of nonergodicity for an arbitrary quantum system interacting according to an arbitrary quantum Hamiltonian with an arbitrary environment. The derivation is not for a particular distance measure, but for all such which satisfies a set of intuitively satisfactory axioms. In the illustrations, we mainly focus on the trace distance, and to a certain extent, also on the relative entropy. We find that our relations are compatible with Markovian ergodic dynamics, where the system loses all the information.

Finally, we have considered a particular structured environment model where a central qubit interacts with a collection of mutually noninteracting spins in thermal states at an arbitrary temperature. A spin-bath model of this type, which has been considered previously in the literature [10,11,16,17,35], shows a highly non-Markovian nature. Here we have derived the reduced dynamics of a particular spin-bath model without the weak coupling and Born-Markov approximations. Subsequently, we investigate the information loss and nonergodicity and find the status of the uncertainty for this system.

The organization of the paper is as follows. In Sec. II, we present the definitions for loss of information and nonergodicity. We derive the uncertainty relations between information loss and nonergodicity in Sec. III. In Sec. IV, we consider the central spin model, derive the reduced dynamics of the central qubit, and analyze the corresponding information loss and nonergodicity. We conclude in Sec. V.

II. DEFINITIONS: MEASURES FOR LOSS OF INFORMATION AND NONERGODICITY

Before proceeding to the main results, let us define the two primary quantities under the present investigation, i.e., loss of information and a measure of nonergodicity, based on distance measures.

A. Loss of information

We quantify the loss of information in quantum systems due to environmental interaction in terms of distinguishability measures for quantum states. The loss of information, denoted by $I_\Delta(t)$, at any instant of time, can be quantified by the maximal difference between the initial distinguishability between a pair of states, $\rho_1(0),\rho_2(0)$, and that for the corresponding time-evolved states $\rho_1(t)=\Phi(\rho_1(0)),\rho_2(t)=\Phi(\rho_2(0))$ at time t, where Φ denotes the open quantum evolution of the initial states. Mathematically, it is given by

$$I_\Delta(t)=\max_{\rho_1(0),\rho_2(0)}[D(\rho_1(0),\rho_2(0))-D(\rho_1(t),\rho_2(t))], \quad (1)$$

where the distance measure $D(\rho,\sigma)$ must satisfy the following conditions:

P1. $D(\rho,\sigma)\geqslant 0\,\forall$ density matrices ρ,σ.

P2. $D(\rho,\rho)=0\,\forall\,\rho$ and $D(\rho,\sigma)=0 \iff \rho=\sigma, \forall\,\rho,\sigma$.

P3. $D(\Phi(\rho),\Phi(\sigma))\leqslant D(\rho,\sigma)\ \forall\ \rho,\sigma$ and $\forall$ completely positive trace-preserving maps $\Phi(\cdot)$, on the space of density operators $\mathcal{B}(\mathcal{H})$ on the Hilbert space $\mathcal{H}$.

The class of distance measures satisfying these conditions include trace distance, Bures distance, and Hellinger distance [36–38]. Though the von Neumann relative entropy and Jensen-Shannon divergence also satisfy the aforementioned conditions, they are not generally considered as geometric distances, since they certain other metric properties. But also note here that the square root of Jensen-Shannon divergence does satisfy metric properties [39–41] and can be considered as a valid distance measure. It is also important to mention that all the aforementioned valid distance measures are bounded.

Loss of information for time-averaged states. To draw the connection with nonergodicity, discussed below, we now define the long-time-averaged state as

$$\bar{\rho}=\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau \rho(t)dt. \quad (2)$$

The information loss for the time-averaged state, which we call "average loss of information," can then be defined as

$$\bar{I}_\Delta=\max_{\rho_1(0),\rho_2(0)}[D(\rho_1(0),\rho_2(0))-D(\bar{\rho}_1,\bar{\rho}_2)], \quad (3)$$

which is lower and upper bounded by 0 and 1, respectively. From Eq. (3) we can infer that when the open system dynamics has a unique steady state or fixed point, independent of initial state, the entire information $D(\rho_1(0),\rho_2(0))$ is lost for arbitrary inputs $\rho_1(0),\rho_2(0)$. Later in the paper, we draw a connection between average loss of information and nonergodicity of the underlying dynamics, with the latter being defined in the succeeding section.

B. Nonergodicity

Ergodicity plays an important role in statistical mechanics to describe the realization of relaxation of a system to thermal equilibrium. The ergodic hypothesis states that if a system evolves over a long period of time, the long-time-averaged state of the system is equal to its thermal state corresponding to the temperature of the environment with which the system is interacting. Ergodicity can also be defined in terms of observables. For any observable f, if its long-time average $\langle f\rangle_T$ is equal to its ensemble average $\langle f\rangle_{en}$, the dynamics is considered to be ergodic for the observables. Here the time and ensemble averages of the observable are respectively defined as

$$\langle\bar{f}\rangle=\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau \mathrm{Tr}[f\rho(t)]=\mathrm{Tr}[f\bar{\rho}]\ ;\ \langle f\rangle_{en}=\mathrm{Tr}[f\rho_{th}].$$

Ergodicity further assumes the equality of $\langle\bar{f}\rangle$ and $\langle f\rangle_{en}$, independent of the initial state of the evolution. Therefore, nonergodicity of the dynamics for the observable can be quantified by the difference between the time average and ensemble average, i.e., by $|\langle\bar{f}\rangle-\langle f\rangle_{en}|=|\mathrm{Tr}[f(\bar{\rho}-\rho_{th})]|$. Based on these understandings of ergodicity of a dynamics, we define a measure of nonergodicity as the distance between the long-time-averaged state ($\bar{\rho}$) of the system and its corresponding thermal state (ρ_{th}), and so is given by

$$\mathcal{N}_\epsilon(\bar{\rho})=D(\bar{\rho},\rho_{th}). \quad (4)$$

Here we impose two further conditions on the allowed distance measures:

P4. The measure must be symmetric, i.e., $D(\rho,\sigma)=D(\sigma,\rho), \forall\rho,\sigma$.

P5. The measure must satisfy the triangle inequality, given by $D(\rho,\sigma)\leqslant D(\rho,\kappa)+D(\kappa,\sigma)$, $\forall$ density matrices ρ,σ,κ.

The conditions P1–P5 are satisfied by the geometric distance measures like trace distance, Bures distance, and Hellinger distance. Note that the von Neumann relative entropy neither satisfies the symmetry property nor the triangle inequality and hence we cannot use it directly for our investigation. However, we will later show the possibility of overcoming such "shortcomings" of the relative entropy distance. Interestingly, it has been shown [40] that Jensen-Shannon divergence satisfies the symmetry property and for its square root, the triangle inequality holds. Therefore, the square root of Jensen-Shannon divergence can also be taken as a proper distance measure for our investigation. Note that the measure of nonergodicity, given in (4), depends on the initial state. Hence, to obtain a measure of nonergodicity which is state independent, we introduce

$$\mathcal{N}_\epsilon^M=\max_{\rho(0)}\mathcal{N}_\epsilon(\bar{\rho}), \quad (5)$$

where maximization is performed over all initial states ($\rho(0)$).

III. CONNECTING INFORMATION LOSS WITH NONERGODICITY

With the definitions given in the preceding section, we now establish a connection between loss of information and nonergodicity. For the distance measures, which satisfy P1-P5, we obtain

$$D(\bar{\rho}_1,\bar{\rho}_2) \leqslant \mathcal{N}_\epsilon(\bar{\rho}_1) + \mathcal{N}_\epsilon(\bar{\rho}_2).$$

Using Eq. (3), we therefore have the inequality

$$\bar{I}_\Delta \geqslant \max_{\rho_1(0),\rho_2(0)} [D(\rho_1(0),\rho_2(0)) - (\mathcal{N}_\epsilon(\bar{\rho}_1) + \mathcal{N}_\epsilon(\bar{\rho}_2))]. \quad (6)$$

It draws a direct connection between nonergodicity and loss of information in open system dynamics. Using the state-independent measure of nonergodicity [Eq. (5)], we can arrive at an uncertainty relation between information loss and a measure of nonergodicity, given by

$$\bar{I}_\Delta + 2\mathcal{N}_\epsilon^M \geqslant \max_{\rho_1(0),\rho_2(0)} [D(\rho_1(0),\rho_2(0))]. \quad (7)$$

The above relation is valid for any distance measure which satisfies the conditions P1–P5, and for any quantum system interacting with an arbitrary environment.

In this paper, we mainly work on the uncertainty relation based on the distance measure given by $D^T(\rho,\sigma) = \frac{1}{2}\mathrm{Tr}|\rho - \sigma|$ for pairs of states ρ and σ. The importance of quantum relative entropy [42–44] as a "distance-type" measure, notwithstanding its inability in satisfying symmetry and other relations, from the perspective of quantum thermodynamics is unquestionable, and hence obtaining the uncertainty relation in terms of quantum relative entropy can be interesting. Towards this aim, we use a relation between relative entropy and trace distance [45], given by

$$S(\rho||\sigma) \equiv \mathrm{Tr}[\rho(\log\rho - \log\sigma)] \geqslant 2(D^T(\rho,\sigma))^2. \quad (8)$$

The above inequality helps us to overcome the drawbacks of relative entropy for not satisfying P4 and P5. Let us first rewrite (6) in terms of trace distance as

$$\bar{I}_\Delta^T \geqslant \max_{\rho_1(0),\rho_2(0)} \left[D^T(\rho_1(0),\rho_2(0)) - \left(\mathcal{N}_\epsilon^T(\bar{\rho}_1) + \mathcal{N}_\epsilon^T(\bar{\rho}_2)\right)\right]. \quad (9)$$

Using inequalities (8) and (9), we arrive at

$$\bar{I}_\Delta^T \geqslant \max_{\rho_1(0),\rho_2(0)} \left[D^T(\rho_1(0),\rho_2(0)) - \left(\sqrt{\frac{\mathcal{N}_\epsilon^{\mathrm{Rel}}(\bar{\rho}_1)}{2}} + \sqrt{\frac{\mathcal{N}_\epsilon^{\mathrm{Rel}}(\bar{\rho}_2)}{2}}\right)\right], \quad (10)$$

where $\mathcal{N}_\epsilon^{\mathrm{Rel}}(\bar{\rho}_i) = S(\bar{\rho}_i||\rho_{th})$ denotes the measure of nonergodicity for the time-averaged state $\bar{\rho}_i$ in terms of relative entropy. As before, we can define a state-independent measure of nonergodicity as

$$\mathcal{N}_\epsilon^{M(\mathrm{Rel})} = \max_{\rho(0)} S(\bar{\rho}||\rho_{th}). \quad (11)$$

The above definition and the inequality (10) leads to another uncertainty relation,

$$\bar{I}_\Delta^T + \sqrt{2\mathcal{N}_\epsilon^{M(\mathrm{Rel})}} \geqslant \max_{\rho_1(0),\rho_2(0)} [D^T(\rho_1(0),\rho_2(0))], \quad (12)$$

in terms of trace and relative entropy distances. But it is to be noted that there is a certain limitation in this relation because of the fact that the relative entropy is not a bounded function. When $\mathrm{supp}\rho \nsubseteq \mathrm{supp}\rho_{th}$, the relative entropy diverges. One such example is obtained for the zero-temperature bath, where $\rho_{th} = |0\rangle\langle 0|$ is pure. In that case, the relation (12) becomes trivial. But in that case, we can find state-dependent uncertainty relations by defining state-dependent information loss as

$$\bar{I}_\Delta(\bar{\rho}_1,\bar{\rho}_2) = [D(\rho_1(0),\rho_2(0)) - D(\bar{\rho}_1,\bar{\rho}_2)]. \quad (13)$$

This will lead us to the state-dependent uncertainty relation

$$\bar{I}_\Delta^T(\bar{\rho}_1,\bar{\rho}_2) + \sum_{i=1,2}\sqrt{\frac{\mathcal{N}_\epsilon^{\mathrm{Rel}}(\bar{\rho}_i)}{2}} \geqslant D^T(\rho_1(0),\rho_2(0)). \quad (14)$$

But other than these extreme cases, the relation (12) works perfectly.

Note that the distinguishability measures like trace distance, Bures distance, and Jensen-Shannon divergence, mentioned earlier, not only satisfies all the conditions P1–P5, but they are also bounded. But in the cases of some unbounded distance measure, to avoid the triviality of the uncertainty relation (7), we can use the state-dependent uncertainty relation

$$\bar{I}_\Delta(\bar{\rho}_1,\bar{\rho}_2) + \sum_{i=1,2}\mathcal{N}_\epsilon(\bar{\rho}_i) \geqslant D(\rho_1(0),\rho_2(0)). \quad (15)$$

Qubits

Until now, we have considered an arbitrary density matrix of arbitrary dimension. Let us now restrict our view to the case of a two-level system (TLS) as a simple example to further understand the connection between nonergodicity and information loss. For a TLS, the pair of states maximizing the trace distance is located on the antipodes of the Bloch sphere, i.e., the pair of states consists of pure and mutually orthogonal states [46]. Therefore in the case of trace distance, the uncertainty relation (7), for a qubit, reads as

$$\bar{I}_\Delta^T + 2\mathcal{N}_\epsilon^{M(T)} \geqslant 1. \quad (16)$$

Similarly, the uncertainty relation given in (12) reduces to

$$\bar{I}_\Delta^T + \sqrt{2\mathcal{N}_\epsilon^{M(\mathrm{Rel})}} \geqslant 1. \quad (17)$$

Let us now consider a simple Markovian model, where a qubit is weakly coupled with a thermal bosonic environment. In absence of any external driving Hamiltonian, the qubit eventually thermally equilibrates with the environment. Under the Born-Markov approximation, the master equation for this model is given by

$$\dot{\rho}(\tilde{t}) = \frac{i}{\hbar}[\rho(\tilde{t}),H_0] + \gamma(n+1)\left(\sigma_-\rho(t)\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-,\rho(\tilde{t})\}\right) + \gamma n\left(\sigma_+\rho(\tilde{t})\sigma_- - \frac{1}{2}\{\sigma_-\sigma_+,\rho(\tilde{t})\}\right), \quad (18)$$

where $H_0 = \hbar\Omega_0|1\rangle\langle 1|$ is the Hamiltonian of the system, γ is a constant parameter, and $n = 1/[\exp(\hbar\Omega_0/K\tilde{T}_m) - 1]$ is the Planck number. Here σ_+ and σ_- are respectively the raising and lowering operators of the TLS, with $|1\rangle$ being the excited state of the same. The solution of the Markovian master equation in (18) is given by

$$\rho(\tilde{t}) = \rho_{11}(\tilde{t})|1\rangle\langle 1| + \rho_{22}(\tilde{t})|0\rangle\langle 0| + \rho_{12}(\tilde{t})|1\rangle\langle 0| + \rho_{21}(\tilde{t})|0\rangle\langle 1|,$$

with

$$\rho_{11}(\tilde{t}) = \rho_{11}(0)e^{-\gamma(2n+1)\tilde{t}} + \frac{n}{2n+1}(1 - e^{-\gamma(2n+1)\tilde{t}}),$$
$$\rho_{22}(\tilde{t}) = 1 - \rho_{11}(\tilde{t}),$$
$$\rho_{12}(\tilde{t}) = \rho_{12}(0)\exp\left(-\gamma\frac{(2n+1)\tilde{t}}{2} - 2i\Omega_0\tilde{t}\right).$$

One can find from the solution given above that the long-time-averaged state for this evolution is independent of initial states and equal to the thermal state corresponding to the temperature of the bath $\tilde{T}_m$, which can be expressed as $p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$, with $p = 1/(1+\exp(-\hbar\Omega_0/K\tilde{T}_m))$. Hence the dynamics is ergodic and we find that the information loss $\bar{I}_\Delta^T = 1$, i.e., the system loses all its information. It is also noteworthy that the Markovianity of a quantum evolution does not mean it will be ergodic. An example of such Markovian nonergodic evolution is the dephasing channel expressed by the master equation

$$\dot{\rho} = i\Omega_0[\sigma_z,\rho] + \gamma_d(\sigma_z\rho\sigma_z - \rho). \tag{19}$$

Here the Lindblad operator is in the same basis as the system Hamiltonian σ_z. A system interacting with a bosonic environment can lead to such an evolution [3]. The solution of this equation is given by

$$\rho_{11}(\tilde{t}) = \rho_{11}(0), \quad \rho_{22}(\tilde{t}) = \rho_{22}(0),$$
$$\rho_{12}(\tilde{t}) = \rho_{12}(0)e^{-2(i\Omega+\gamma_d)t}. \tag{20}$$

We realize from Eq. (20) that under this particular evolution, the system will decohere, but the diagonal elements of the density matrix will remain invariant, leading to infinitely many fixed points for the dynamics. So this particular evolution will certainly be nonergodic, since there exists infinitely many fixed points and the time-averaged state will depend on the initial state of the system. This gives a definite example which proves that Markovianity does not imply ergodicity of the dynamics.

IV. NONERGODICITY AND INFORMATION BACKFLOW IN A CENTRAL SPIN MODEL

In this section, we consider a specific non-Markovian model and study the status of uncertainty relation derived in Sec. III. The system here consists of a single qubit interacting with N number of noninteracting spins. The total Hamiltonian of the system governing the dynamics is given by

$$\tilde{H} = \tilde{H}_S + \tilde{H}_B + \tilde{H}_I, \tag{21}$$

where the system Hamiltonian $\tilde{H}_S$, bath Hamiltonian $\tilde{H}_B$, and interaction Hamiltonian $\tilde{H}_I$ are respectively given by

$$\tilde{H}_S = \hbar g\omega_0\sigma_z,$$
$$\tilde{H}_B = \hbar g\frac{\omega}{N}\sum_{i=1}^{N}\sigma_z^i, \tag{22}$$
$$\tilde{H}_I = \hbar g\frac{\alpha}{\sqrt{N}}\sum_{i=1}^{N}\left(\sigma_x\sigma_x^i + \sigma_y\sigma_y^i + \sigma_z\sigma_z^i\right).$$

Here σ_k, $k = x,y,z$ are the Pauli spin matrices, the superscript i represents the ith spin of the bath, g is a constant factor with the dimension of frequency, ω_0 and ω are the dimensionless parameters characterizing the energy-level differences of the system and the bath, respectively, and α denotes the coupling constant of the system-bath interaction. By using the total angular momentum operators $J_k = \sum_{i=1}^{N}\sigma_k^i$ and the Holstein-Primakoff transformation, given by

$$J_+ = \sqrt{N}b^\dagger\left(1 - \frac{b^\dagger b}{2N}\right)^{1/2}, \quad J_- = \sqrt{N}\left(1 - \frac{b^\dagger b}{2N}\right)^{1/2} b,$$

the bath and interaction Hamiltonians can now be rewritten as

$$\tilde{H}_B = -\hbar g\omega\left(1 - \frac{b^\dagger b}{N}\right),$$
$$\tilde{H}_I = 2\hbar g\alpha\left[\sigma_+\left(1 - \frac{b^\dagger b}{2N}\right)^{1/2} b + \sigma_- b^\dagger\left(1 - \frac{b^\dagger b}{2N}\right)^{1/2}\right] - \hbar g\alpha\sqrt{N}\sigma_z\left(1 - \frac{b^\dagger b}{N}\right). \tag{23}$$

We consider the initial (uncorrelated) system-bath state as $\rho_S(0)\otimes\rho_B(0)$. Let us take the initial system qubit as $\rho_S(0) = \rho_{11}(0)|1\rangle\langle 1| + \rho_{22}(0)|0\rangle\langle 0| + \rho_{12}(0)|1\rangle\langle 0| + \rho_{21}(0)|0\rangle\langle 1|$ and the initial bath state to be a thermal state $\rho_B(0) = \exp(-\tilde{H}_B/K\tilde{T})$ in an arbitrary temperature $\tilde{T}$ with K being the Boltzmann constant. The reduced dynamics of the system state can then be calculated by tracing out the bath degrees of freedom and is given by $\rho_S(t) = \mathrm{Tr}_B[\exp(-iHt)\rho_S(0)\otimes\rho_B(0)\exp(iHt)]$, where

$$H = \frac{\tilde{H}}{\hbar g},\ t = g\tilde{t},\ \text{and}\ T = \frac{K\tilde{T}}{\hbar g}$$

are dimensionless, specifying Hamiltonian, time, and temperature, respectively. After solving the global Schrödinger evolution, the reduced dynamics can be exactly obtained [17,47] as

$$\rho_S(t) = \begin{pmatrix}\rho_{11}(t) & \rho_{12}(t)\\ \rho_{21}(t) & \rho_{22}(t)\end{pmatrix}, \tag{24}$$

where

$$\rho_{11}(t) = \rho_{11}(0)(1 - \Theta_1(t)) + \rho_{22}(0)\Theta_2(t),$$
$$\rho_{12}(t) = \rho_{12}(0)\Delta(t), \tag{25}$$

with

$$\Theta_1(t) = \sum_{n=0}^{N}(n+1)\alpha^2(1-n/2N)\left(\frac{\sin(\eta t/2)}{\eta/2}\right)^2\frac{e^{-\frac{\omega}{T}(n/N-1)}}{Z},\quad \Theta_2(t) = \sum_{n=0}^{N}n\alpha^2(1-(n-1)/2N)\left(\frac{\sin(\eta' t/2)}{\eta'/2}\right)^2\frac{e^{-\frac{\omega}{T}(n/N-1)}}{Z},$$

$$\Delta(t) = \sum_{n=0}^{N}e^{-i(\Lambda-\Lambda')t/2}\left(\cos(\eta t/2) - i\frac{\theta}{\eta}\sin(\eta t/2)\right)\left(\cos(\eta' t/2) + i\frac{\theta'}{\eta'}\sin(\eta' t/2)\right)\frac{e^{-\frac{\omega}{T}(n/N-1)}}{Z},$$

$$Z = \sum_{n=0}^{N}e^{-\frac{\omega}{T}(n/N-1)},\quad \eta = 2\sqrt{\left[\omega_0 - \frac{\omega}{2N} - \alpha\sqrt{N}\left(1-\frac{2n+1}{2N}\right)\right]^2 + 4\alpha^2(n+1)\left(1-\frac{n}{2N}\right)},$$

$$\eta' = 2\sqrt{\left[\omega_0 - \frac{\omega}{2N} - \alpha\sqrt{N}\left(1-\frac{2n-1}{2N}\right)\right]^2 + 4\alpha^2 n\left(1-\frac{(n-1)}{2N}\right)},$$

$$\theta = 2\left[\omega_0 - \omega/2N + \alpha\sqrt{N}\left(1-\frac{2n+1}{2N}\right)\right],\quad \theta' = -2\left[\omega_0 - \omega/2N - \alpha\sqrt{N}\left(1-\frac{2n-1}{2N}\right)\right],$$

$$\Lambda = -2\omega\left(1-\frac{2n+1}{2N}\right) - \frac{\alpha}{\sqrt{N}},\quad \Lambda' = -2\omega\left(1-\frac{2n-1}{2N}\right) - \frac{\alpha}{\sqrt{N}}.$$

The time-averaged state for this system can then be calculated as

$$\bar{\rho}_{11} = \rho_{11}(0)(1-\bar{\Theta}_1) + \rho_{22}(0)\bar{\Theta}_2,\quad \bar{\rho}_{12} = \rho_{12}(0)\bar{\Delta}, \tag{26}$$

with

$$\bar{\Theta}_1 = \sum_{n=0}^{N}2(n+1)\alpha^2(1-n/2N)\left(\frac{1}{\eta^2}\right)\frac{e^{-\frac{\omega}{T}(n/N-1)}}{Z},$$

$$\bar{\Theta}_2 = \sum_{n=0}^{N}2n\alpha^2[1-(n-1)/2N]\left(\frac{1}{\eta'^2}\right)\frac{e^{-\frac{\omega}{T}(n/N-1)}}{Z},$$

$$\bar{\Delta} = 0.$$

Note that in general the coherence of the time-averaged state will vanish as $\bar{\Delta} = 0$. But there are specific resonance conditions under which there can be nonzero coherence present in the time-averaged state [17]. But in this work, we will not consider such situations.

Before investigating the uncertainty relation in terms of trace distance given in (7), we explore the behavior of loss of information at instantaneous time with different parameters involved in this dynamics. For such study, let us restrict ourselves to the set of pure initial qubits over which the optimization involved in (7) is performed. In particular, we take the initial pair of orthogonal pure states to be $\cos\frac{\theta}{2}|1\rangle + \sin\frac{\theta}{2}e^{-i\phi}|0\rangle$ and $\sin\frac{\theta}{2}|1\rangle - \cos\frac{\theta}{2}e^{-i\phi}|0\rangle$, with $0 \leqslant \theta \leqslant \pi, 0 \leqslant \phi < 2\pi$. The instantaneous and average information losses in this case are given respectively by

$$I_\Delta^T(t) = \Theta_1(t) + \Theta_2(t),\quad \bar{I}_\Delta^{(T)} = \bar{\Theta}_1 + \bar{\Theta}_2. \tag{27}$$

In Figs. 1, 2, and 3, the instantaneous loss of information is depicted with time for different values of the number of bath spins (N), temperatures ($\tilde{T}$), and system-bath interaction strengths (α), respectively, by keeping the other parameters fixed. From the figures, we deduce the following:

Observation 1. The instantaneous loss of information shows oscillatory behavior whose amplitude decreases with time.

Observation 2. The increase of number of spins of the bath, in temperature as well as in the interaction strength, can be seen as an increase of influence of bath on the system. Hence, expectantly in all cases, the loss of information increases with increase of the above system parameters.

Let us now check the uncertainty relation given in (16) for the qubit case, taking the same initial pair of pure orthogonal states and the thermal state at arbitrary temperature $\rho_{th} = p_1|0\rangle\langle 0| + (1-p_1)|1\rangle\langle 1|$, where $p_1 = \frac{1}{2}[1+\tanh(\frac{\hbar g\omega_0}{K\tilde{T}})]$. After performing the maximization, we find

$$\bar{I}_\Delta^T + 2\mathcal{N}_\epsilon^{M(T)} = \bar{\Theta}_1 + \bar{\Theta}_2 + 2|p_1 - \bar{\Theta}_1|. \tag{28}$$

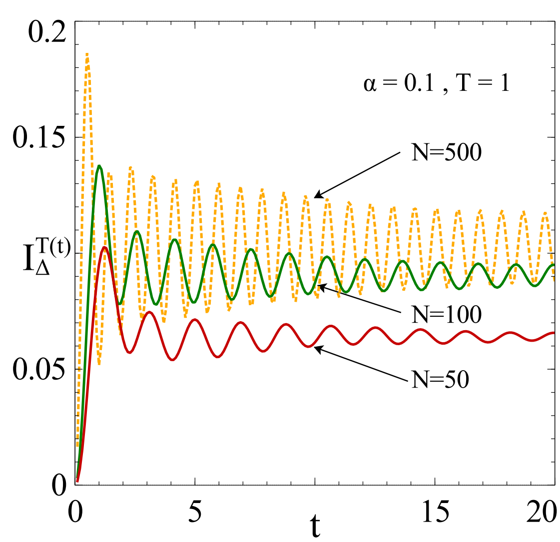

FIG. 1. Time dynamics of instantaneous information loss. We plot $I_\Delta^T(t)$ on the vertical axis against t on the horizontal axis for different values of the total number of bath spins N, where the system-environment duo governed by the Hamiltonian in Eq. (21) is being considered. We set $\alpha = 0.1$ and $T = 1$. All quantities are dimensionless.

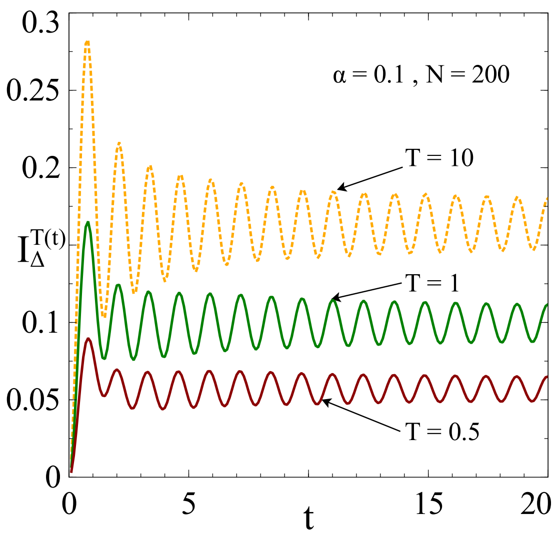

FIG. 2. $I_\Delta^T(t)$ vs t for various temperatures. We set $N = 200$ and $\alpha = 0.1$. The physical system is the same as in Fig. 1. All quantities are dimensionless.

We now examine the conditions for which the uncertainty relation (16) saturates. Note that for the ergodic situations, i.e., if the steady state is unique and is equal to the thermal state, the information loss is equals unity, leading to a trivial equality in (16). Keeping N fixed to 1000 and fixing the temperature to different values, we investigate the values of $\bar{I}_\Delta^T + 2N_\epsilon^{M(T)}$ for increasing interacting strength. We observe that the sum goes close to unity for a strong interaction strength, as depicted in Fig. 4 for high temperature.

In Figs. 5 and 6, we analyze the sum $I_\Delta^T + 2\mathcal{N}_\epsilon^{M(T)}$ as the number of bath spins is ramped up from 100 to 1000. Scrutinizing these figures, we can safely conclude that with the increase in number of spins of the bath, or in the bath temperature, or in the system-bath interaction strength, the sum $I_\Delta^{(T)} + 2\mathcal{N}_\epsilon^{M(T)}$ goes very close to unity in this qubit case, provided the optimization involved is restricted to pure qubits. However, numerical evidence strongly suggests that for this non-Markovian model, given in Eq. (21), there will be no nontrivial situation when the uncertainty relation in Eq. (16) saturates to unity, provided the maximization is carried out over a pure state.

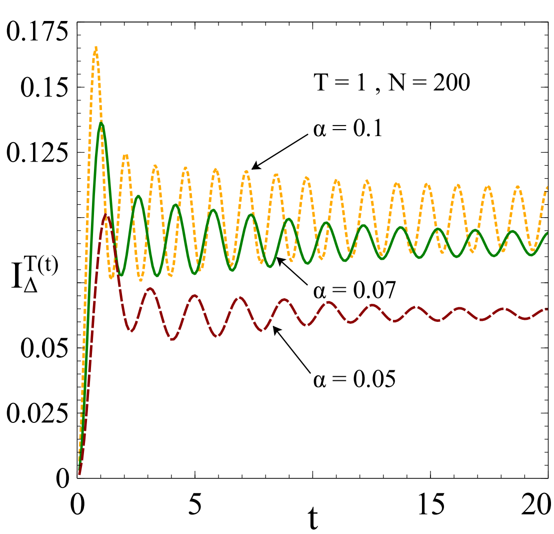

FIG. 3. $I_\Delta^T(t)$ with t for three different values of system-bath coupling (α). We set $T = 1$ and $N = 200$. The physical system is the same as in Fig. 1. All quantities are dimensionless.

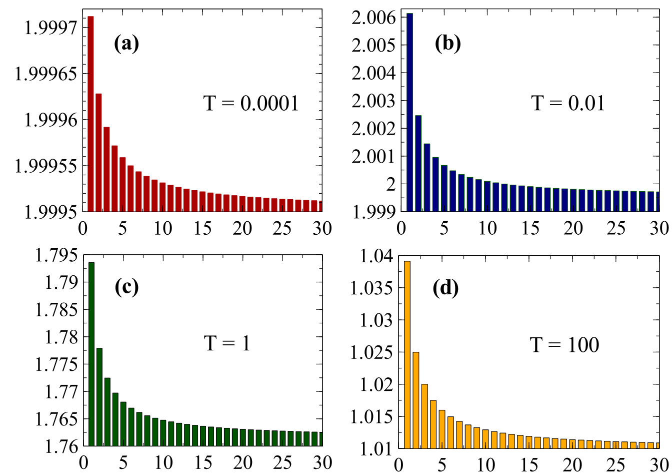

FIG. 4. Behavior of the uncertainty for different system-bath interaction strengths. We denote the values of $I_\Delta^T + 2\mathcal{N}_\epsilon^{M(T)}$ for different α as vertical bars on the horizontal axis that represent α. Here, $N = 1000$, and the different panels are for different T. The system considered is the one given by the Hamiltonian in Eq. (21). At already a moderate interaction strength, the quantity converges to a certain value which is higher than unity. The converged value goes close to unity with the increase of temperature. All quantities are dimensionless.

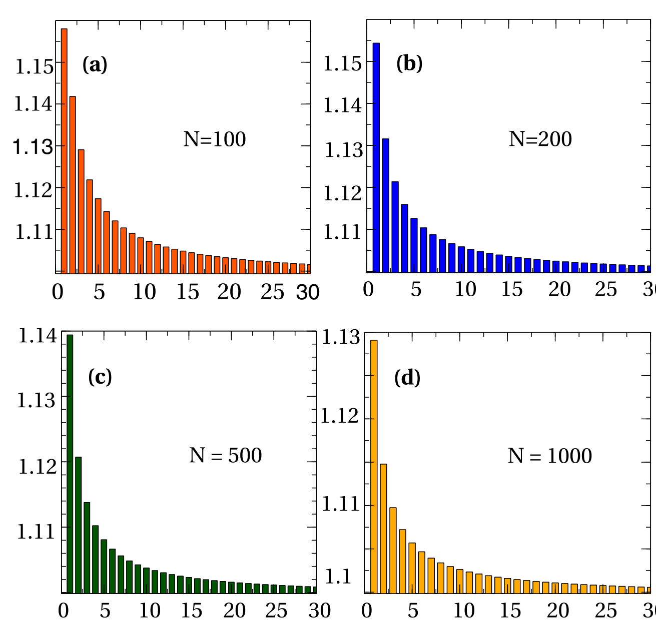

FIG. 5. Bar diagram for $I_\Delta^T + 2\mathcal{N}_\epsilon^{M(T)}$ vs α. The situation here is the same as in Fig. 4, except that the different panels are for different N for fixed T, which is set to be 10. All quantities are dimensionless.

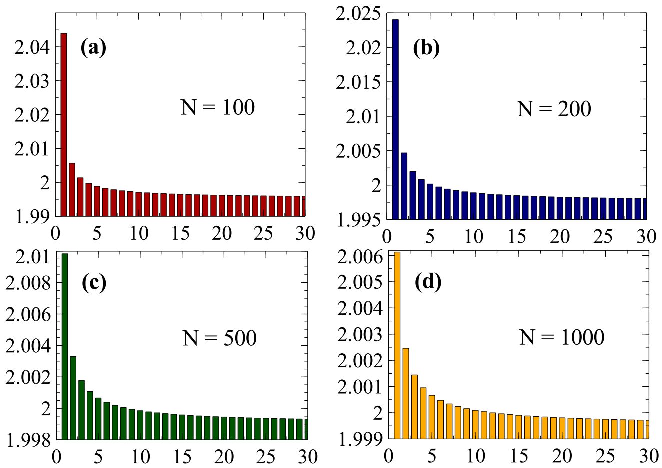

FIG. 6. The panels here are the same as in Fig. 5, except that $T = 0.01$. All quantities are dimensionless.

We find that the saturation values of $\bar{\Theta}_1$ and $\bar{\Theta}_2$, when $\alpha \longrightarrow \infty$, are respectively given by

$$\bar{\Theta}_1^{\mathrm{sat}} = \frac{1}{2}\sum_{n=0}^{n=N} \frac{1}{4 + N\frac{[1-(2n+1)/2N]}{(n+1)(1-n/2N)^2}} \frac{e^{-\frac{\hbar\omega}{KT}(n/N-1)}}{Z},$$

$$\bar{\Theta}_2^{\mathrm{sat}} = \frac{1}{2}\sum_{n=0}^{n=N} \frac{1}{4 + N\frac{[1-(2n-1)/2N]^2}{n[1-(n-1)/2N]}} \frac{e^{-\frac{\hbar\omega}{KT}(n/N-1)}}{Z}. \quad (29)$$

If the interaction Hamiltonian given in Eq. (22) is considered in absence of the $z - z$ interaction, the saturated values of $\bar{\Theta}_1$ and $\bar{\Theta}_2$ in the limit $N \to \infty, \alpha \to \infty$ will be $\bar{\Theta}_1^{\mathrm{sat}} = \bar{\Theta}_2^{\mathrm{sat}} = 1/8$. In the infinite-temperature limit, we have $p_1 = 1/2$, which leads to the equality in (16). It is interesting that in the mentioned limit, the nonergodicity measure is finite and equals 3/8. Therefore, we find a nonergodic situation where the equality of the uncertainty relation holds. When the equality of the mentioned relations hold for a nonergodic evolution, these relations imply that when the nonergodicity of the dynamics increases, the information loss in the system decreases. Nonergodic dynamics are, in general, good for information processing as they have less chance of leakage of information compared to ergodic dynamics. In particular, for nonergodic evolution for which the uncertainty relations discussed in this paper are equalities, the loss of information can be quantified by and attributed to the nonergodicity in the evolution. It is also important to mention that spin-bath models do not always indicate a shows nonergodic dynamics. In a recent work [48], such dynamics have been considered with the Born-Markov approximation region to find the effective reduced dynamics. It is shown in the mentioned work that there are situations where a unique fixed point (stationary state) can exist for the evolution, and hence in those situations, the dynamics is ergodic [27].

V. CONCLUSION

In open quantum dynamics, the information exchange between the system and bath plays an important role, while the time-evolved state's correspondence with the Gibb's ensemble conspire to imply the ergodic nature of the system. In this article, we establish a relation between loss of information and a measure of nonergodicity. Both the definitions are given in terms of distinguishability, which can be measured by a suitably chosen distance measure. We have shown that the information loss and the quantifier of nonergodicity follow an uncertainty relation, valid for a broad class of distinguishability measures, which includes trace distance, Bures distance, Hilbert-Schmidt distance, Hellinger distance, and square root of Jensen-Shannon divergence. We have further considered trace distance between a pair of quantum states as a specific distinguishability measure and connected the corresponding information loss with nonergodicity, which is now defined in terms of relative entropy between the time-averaged state and the thermal state, maximized over all possible initial states. We have shown that in a Markovian model, the uncertainty relation saturates and shows a complete information loss. We also considered a structured environment model of a central quantum spin interacting, according to Heisenberg interaction, with a collection of mutually noninteracting quantum spin-half particles, leading to non-Markovian dynamics. In this case, we observed that with the increase of temperature, number of spins in the bath, and the system-bath interaction strength, there is increase in information loss at instantaneous time. In this scenario, we found that the uncertainty relation shows a nonmonotonic behavior with the increase of temperature for small values of interaction strength, provided the optimization is performed over pure qubits. Moreover, we found that although the uncertainty relation in this model goes close to the saturation value, it fails to saturate exactly. Interestingly, however, we found that in the absence of z-z system-bath interaction and in the limit of large bath size, high bath temperature, and strong system bath interaction, the uncertainty relation between information loss and nonergodicity, based on trace distance measure, is saturated, providing a nonergodic situation that saturates the uncertainty. The uncertainty relations have been obtained by using the usual notion of the ergodicity where it is required that the unique fixed point of the dynamics be thermal. We note that the entire analysis follows through for a more general definition where a single fixed point is sufficient to imply ergodicity.

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