

**Disturbance by optimal discrimination**

Ryûitirô Kawakubo\* and Tatsuhiko Koike†

*Department of Physics, Keio University, Yokohama 223-8522, Japan*

(Received 29 September 2017; published 7 March 2018)

We discuss the disturbance by measurements which unambiguously discriminate between given candidate states. We prove that such an optimal measurement necessarily changes distinguishable states indistinguishable when the inconclusive outcome is obtained. The result was previously shown by Chefles [*Phys. Lett. A* **239**, 339 (1998)] under restrictions on the class of quantum measurements and on the definition of optimality. Our theorems remove these restrictions and are also applicable to infinitely many candidate states. Combining with our previous results, one can obtain concrete mathematical conditions for the resulting states. The method may have a wide variety of applications in contexts other than state discrimination.

DOI: [10.1103/PhysRevA.97.032102](https://doi.org/10.1103/PhysRevA.97.032102)**I. INTRODUCTION**

Optimal quantum measurements play fundamental roles in many subjects in quantum foundations and quantum information. The subjects include error-disturbance relations [1], quantum coding theorems [2], entanglement distillation [3] state estimation [4], state discrimination [5], state protection [6], etc. Although it is sometimes difficult to obtain concrete forms of optimal measurements, the characterization of those measurements that maximize the precision, maximize the transmission rate, etc., gives fundamental bounds in quantum mechanics and on efficiency of various information processing. In this paper, we discuss a natural and intuitive principle in optimality: the measurement which does “something” the best leaves no room for doing something afterwards. Intuitively, if the room remained, one would be able to improve the original measurement by composing that with a measurement which does an amount of something later. This simple reasoning seems to have a wide scope. However, it is not necessarily trivial to implement the idea as a rigorous statement in general or in each subject. One must carefully choose the definition of optimality, assumptions and conclusions. We will demonstrate, as a prototype, that the principle works well and makes the original problem more transparent in the context of unambiguous state discrimination, where we also notice subtleties in the application thereof. The principle generalizes the previous results and simplifies the proof, which may not be achieved by other approaches such as extremity (in the mathematical sense) of the optimal measurements, even if we restrict ourselves to convex evaluation functions.

Unambiguous discrimination is one of possible frameworks for state discrimination. There, one must answer the correct

input state among the candidates after performing a quantum measurement. One must not take a state for another, though one can answer “inconclusive” or “?”. Unambiguous discrimination between two states was introduced by Ivanovic [7] and developed in Refs. [8–10]. Chefles [11] showed that finitely many pure states are distinguishable if and only if they are linearly independent. Feng *et al.* [12] extended the result to mixed states. There are also interesting examples where infinitely many candidate states are involved. An example is the set of coherent states corresponding to a lattice in the classical phase space, which was considered by von Neumann [13] in the context of simultaneous measurements of position and momentum. The present authors [14] generalized the results above on unambiguous discrimination to infinitely many candidate states. The application to von Neumann’s lattice led to a natural characterization of Planck’s constant from the viewpoint of state discrimination.

The studies of unambiguous discrimination above mainly discuss its possibility and accuracy. Not much attention was paid to the disturbance caused by unambiguous discrimination measurements. One exception is a part of Chefles’s work [11]. He showed, under some restrictions explained below, that optimal discrimination measurements change the input states to linearly dependent (and thus indistinguishable) ones if the inconclusive outcome ? is obtained. The claim is interesting because it concerns the disturbance of an optimal measurement. However, besides the finiteness of the candidate states, the results obtained there were restrictive in the following sense. First, the states to be discriminated were assumed to be pure. Second, not all measurements allowed by quantum mechanics were considered. The measurements were restricted to those which change pure states to pure states when the outcome is inconclusive. It is quite common, however, that the output state is mixed even if the input is pure. Third, a particular evaluation function was considered to define the optimality. Namely, existence of a prior probability distribution is assumed and the average success probability was chosen. Even in the unambiguous discrimination between a finite number of states, it may be as natural, for example, to define the optimality by maximization of the minimum success probability taken over the candidate states.

\*rkawakub@rk.phys.keio.ac.jp

†koike@phys.keio.ac.jp

*Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article’s title, journal citation, and DOI.*

In this paper, we show that optimal unambiguous discrimination measurements make distinguishable states indistinguishable provided that the outcome is the inconclusive one. We apply the simple principle described at the beginning of this paper and derive the result directly from the definition of the optimality, not resorting to the detailed mathematical properties of the states. The results are free from all restrictions mentioned above, and are valid for mixed states, for general measurements, for a wide class of evaluation functions (which are not necessarily convex), and for infinitely many candidate states. A careful treatment is necessary for infinitely many states since distinguishability naturally splits into slightly different levels, distinguishability and uniform distinguishability [14]. One can obtain detailed mathematical characterization of the states resulting from optimal measurements by combining the results in Ref. [14] and those in the present work.

The paper is organized as follows. In Sec. II, we briefly review quantum measurement theory. We introduce the concept of unambiguous discrimination in Sec. III. We present our main result on distinguishability in Sec. IV and that on uniform distinguishability in Sec. V. The excluded case in the main results, which is itself of interest, is addressed in Sec. VI. Section VII is for conclusion and discussions.

## II. BRIEF REVIEW OF QUANTUM MEASUREMENT THEORY

In this section, we quickly review quantum measurement theory [15] to the extent that is necessary for this paper. We consider the measurement with countably many outcomes. Let  $\Omega$  denote the set of possible outcomes and  $\mathcal{H}$  denote a system (a separable Hilbert space) to be measured. A state is expressed by a density operator  $\rho \in B^1(\mathcal{H})$ ,  $\rho \geq 0$ ,  $\text{tr} \rho = 1$ , where  $B^1(\mathcal{H})$  denotes the Banach space of trace class operators on  $\mathcal{H}$ . A measurement on  $\mathcal{H}$  with the outcome set  $\Omega$  is mathematically described by an *instrument*  $(\mathcal{I}_\omega)_{\omega \in \Omega}$ . Each element  $\mathcal{I}_\omega$  of the instrument is a linear map  $\mathcal{I}_\omega : B^1(\mathcal{H}) \rightarrow B^1(\mathcal{H})$  [that is bounded with respect to the trace norm on  $B^1(\mathcal{H})$ ], which describes a weighted state change caused by the measurement. Each  $\mathcal{I}_\omega$  sends a state  $\rho$  to an “unnormalized” state, namely,

$$\mathcal{I}_\omega \rho = \text{Pr}(\omega|\rho) \rho_\omega, \quad (1)$$

where  $\text{Pr}(\omega|\rho)$  is the probability of obtaining an outcome  $\omega \in \Omega$  and  $\rho_\omega$  is the resulting state when the outcome  $\omega$  is obtained. Equation (1) defines  $\text{Pr}(\omega|\rho)$  and  $\rho_\omega$  uniquely (unless  $\text{tr}[\mathcal{I}_\omega \rho] = 0$ ):

$$\text{Pr}(\omega|\rho) = \text{tr}[\mathcal{I}_\omega \rho], \quad \rho_\omega = \frac{\mathcal{I}_\omega \rho}{\text{tr}[\mathcal{I}_\omega \rho]}. \quad (2)$$

In order to interpret these quantities as probabilities and quantum state, respectively, an instrument is assumed to satisfy the following two conditions.

(CP) Completely positivity:  $\mathcal{I}_\omega$  is completely positive for all  $\omega \in \Omega$ , i.e., its trivial extensions  $\mathcal{I}_\omega \otimes \text{id}_{\mathbb{C}^{n \times n}} : B^1(\mathcal{H} \otimes \mathbb{C}^n) \rightarrow B^1(\mathcal{H} \otimes \mathbb{C}^n)$  are positive maps for all  $n \in \mathbb{N}$ .

(TP) Trace preserving property: the sum  $\sum \mathcal{I}_\omega$  preserves trace of operators, i.e.,  $\text{tr}[(\sum \mathcal{I}_\omega) \rho] = \text{tr} \rho$  for all  $\rho \in B^1(\mathcal{H})$ .

The property CP, especially the positivity, guarantees positivity of  $\rho_\omega$  and  $\text{Pr}(\omega|\rho)$ . The property TP guarantees the conservation of probability. It is known that the instruments

with the above two properties correspond exactly to the realizable measurements (e.g. [16]).

## III. UNAMBIGUOUS DISCRIMINATION

We begin with the precise definition of unambiguous discrimination.

*Definition 1.* An *unambiguous discrimination measurement* between  $(\rho_j)_{j \in J}$  is an instrument  $(\mathcal{I}_\omega)_{\omega \in J \cup \{?\}}$  that satisfies, for  $j, k \in J$  with  $j \neq k$ ,

$$\text{tr}[\mathcal{I}_j \rho_k] = 0, \quad \text{tr}[\mathcal{I}_j \rho_j] > 0. \quad (3)$$

We call  $\text{tr}[\mathcal{I}_j \rho_j]$  *success probabilities*. The states  $(\rho_j)_{j \in J}$  are said to be *distinguishable* when they admit at least one unambiguous discrimination measurement between them.

Let  $(\rho_j)_{j \in J}$  be distinguishable candidate states. Then  $(\rho_j)_{j \in J}$  admit an unambiguous discrimination measurement  $(\mathcal{I}_\omega)_{\omega \in J \cup \{?\}}$ . When the true state is  $\rho_j$ , one obtains the outcome  $j$  or  $?$  with probabilities  $\text{tr}[\mathcal{I}_j \rho_j]$  or  $\text{tr}[\mathcal{I}_? \rho_j]$ , respectively, and does not obtain any other outcome, namely,  $\text{tr}[\mathcal{I}_j \rho_j] + \text{tr}[\mathcal{I}_? \rho_j] = 1$ . When the outcome is  $j$ , we can decide the true state is  $\rho_j$  *with certainty*.

In the rest of this section, we would like to discuss the ways to quantify how good an unambiguous discrimination measurement is. In the presence of a prior probability density  $(p_j)_{j \in J}$ , which was assumed by Dieks [8] and Chefles [11], it is reasonable to evaluate an unambiguous discrimination measurement  $(\mathcal{I}_\omega)_{\omega \in J \cup \{?\}}$  by the average success probability

$$f_{\text{av}} := \sum_{j \in J} p_j q_j, \quad (4)$$

where  $q_j = \text{tr}[\mathcal{I}_j \rho_j]$  are success probabilities.

However, this is not the only way. Even in the case of finite  $J$ , there are important examples in which  $f$  is not of the form (4). The minimum success probability

$$f_{\text{min}} := \min\{q_j \mid j \in J\} \quad (5)$$

is such an example. Roughly speaking,  $1/f_{\text{min}}$  trials are enough to determine the true state. This is the operational meaning of the minimum success probability  $f_{\text{min}}$ . Generalizing these two examples, we define the class of evaluation functions for unambiguous discrimination.

*Definition 2.* We call a function  $f : (0, 1]^J \rightarrow \mathbb{R}$  an *evaluation function* if

$$x_j > y_j \quad \text{for all } j \in J \Rightarrow f(x_j) > f(y_j) \quad (6)$$

holds for all  $(x_j)_{j \in J}, (y_j)_{j \in J} \in (0, 1]^J$ . We say an unambiguous discrimination measurement  $(\mathcal{I}_\omega)_{\omega \in J \cup \{?\}}$  between the states  $(\rho_j)_{j \in J}$  is *better* if the value  $f(\text{tr}[\mathcal{I}_j \rho_j])$  is larger and optimal if no other measurement exceeds the value.

The class of “evaluation functions” defined here is so large that, when  $J$  is finite, one could hardly imagine any functions that suit the term and do not belong to the defined class. For example, the class contains  $f_{\text{av}}$  and  $f_{\text{min}}$ . An evaluation function need not be convex nor linear. When  $J$  is infinite, however,  $f_{\text{inf}}$  [see (16)], which is a natural generalization of  $f_{\text{min}}$ , is excluded from the class. Such functions are more suitably discussed in the context of uniform distinguishability (see Sec. V).

#### IV. RESULT ON DISTINGUISHABILITY

Now, we can state the first main result.

*Theorem.* Optimal discriminations make distinguishable states indistinguishable when the discrimination fails (gives ?). Namely, assume that the measurement  $(\mathcal{I}_\omega)_{\omega \in J \cup \{?\}}$  achieves an optimal unambiguous discrimination between the states  $(\rho_j)_{j \in J}$  and that the condition

$$\text{tr}[\mathcal{I}_j \rho_j] < 1 \quad \text{for all } j \in J \quad (7)$$

holds. Then the resulting states under the condition that the outcome is ?, defined by

$$\left( \frac{\mathcal{I}_? \rho_j}{\text{tr}[\mathcal{I}_? \rho_j]} \right)_{j \in J}, \quad (8)$$

are not distinguishable. Here, the optimality is defined by an arbitrary evaluation function in Definition 2.

Note that the condition  $\text{tr}[\mathcal{I}_j \rho_j] < 1$  is equivalent to  $\text{tr}[\mathcal{I}_? \rho_j] > 0$  and this ensures the well definedness of the resulting states (8). We will discuss the case that this condition fails in Sec. VI.

We explain the idea of the proof first and then give the formal one. The simple but important idea is to prove the contrapositive, namely, if the states (8) are distinguishable then the discrimination measurement  $(\mathcal{I}_\omega)_{\omega \in J \cup \{?\}}$  is not optimal. We thus construct a discrimination measurement that is better than the original one,  $(\mathcal{I}_\omega)_{\omega \in J \cup \{?\}}$ , assuming that (8) are distinguishable. The measurement consisting of the following two steps does the task.

(1) Perform the original discrimination  $(\mathcal{I}_\omega)_{\omega \in J \cup \{?\}}$ . If the outcome of this measurement is  $j \in J$ , then decide the true state is  $\rho_j$  regardless of the second step. If the outcome is ?, then defer the decision and proceed to the next step.

(2) Perform the discrimination of states (8), whose existence is guaranteed by the assumption. If the outcome of this measurement is  $j \in J$ , decide the true state is  $\rho_j$ . Otherwise, give up on the decision and answer ?.

We show below that this combined discrimination measurement truly improves the original one  $(\mathcal{I}_\omega)_{\omega \in J \cup \{?\}}$ .

*Proof of the Theorem.* Note that

$$\text{tr}[\mathcal{I}_? \rho_j] = 1 - \text{tr}[\mathcal{I}_j \rho_j] > 0 \quad (9)$$

by the assumption of the Theorem.

We will prove the contrapositive. Let us assume (8) admits an unambiguous discrimination measurement  $(\mathcal{I}'_\omega)_{\omega \in J \cup \{?\}}$ . Because  $(\mathcal{I}_\omega)_{\omega \in J \cup \{?\}}$  and  $(\mathcal{I}'_\omega)_{\omega \in J \cup \{?\}}$  discriminate between  $(\rho_j)_{j \in J}$  and between (8), respectively, one obtains, by Definition 1,

$$\text{tr}[\mathcal{I}_j \rho_k] = 0, \quad \text{tr}[\mathcal{I}_j \rho_j] > 0, \quad (10)$$

$$\text{tr} \left[ \mathcal{I}'_j \frac{\mathcal{I}_? \rho_k}{\text{tr}[\mathcal{I}_? \rho_k]} \right] = 0, \quad \text{tr} \left[ \mathcal{I}'_j \frac{\mathcal{I}_? \rho_j}{\text{tr}[\mathcal{I}_? \rho_j]} \right] > 0 \quad (11)$$

for  $j, k \in J$  with  $j \neq k$ .

Let us define an instrument  $(\mathcal{J}_\omega)_{\omega \in J \cup \{?\}}$  by

$$\mathcal{J}_j := \left( \sum_{\omega' \in J \cup \{?\}} \mathcal{I}'_{\omega'} \right) \mathcal{I}_j + \mathcal{I}'_j \mathcal{I}_?, \quad j \in J, \quad (12)$$

$$\mathcal{J}_? := \mathcal{I}'_? \mathcal{I}_?. \quad (13)$$

We will prove that the instrument  $(\mathcal{J}_\omega)_{\omega \in J \cup \{?\}}$  unambiguously discriminates states  $(\rho_j)_{j \in J}$  better than  $(\mathcal{I}_\omega)_{\omega \in J \cup \{?\}}$  in the rest of this proof.

First, it is readily seen that  $(\mathcal{J}_\omega)_{\omega \in J \cup \{?\}}$  is an instrument since  $\sum_{\omega \in J \cup \{?\}} \mathcal{J}_\omega = \sum_{\omega' \in J \cup \{?\}} \mathcal{I}'_{\omega'} \mathcal{I}_\omega$ .

Second, we calculate the probabilities  $\text{tr}[\mathcal{J}_j \rho_k]$  for all  $j, k \in J$ :

$$\begin{aligned} \text{tr}[\mathcal{J}_j \rho_k] &= \text{tr} \left[ \left( \sum_{\omega'} \mathcal{I}'_{\omega'} \right) \mathcal{I}_j \rho_k \right] + \text{tr}[\mathcal{I}'_j \mathcal{I}_? \rho_k] \\ &= \text{tr}[\mathcal{I}_j \rho_k] + (\text{tr}[\mathcal{I}_? \rho_k]) \text{tr} \left[ \mathcal{I}'_j \frac{\mathcal{I}_? \rho_k}{\text{tr}[\mathcal{I}_? \rho_k]} \right] \\ &= \delta_{j,k} \left( \text{tr}[\mathcal{I}_k \rho_k] + (\text{tr}[\mathcal{I}_? \rho_k]) \text{tr} \left[ \mathcal{I}'_k \frac{\mathcal{I}_? \rho_k}{\text{tr}[\mathcal{I}_? \rho_k]} \right] \right), \end{aligned} \quad (14)$$

where the first equality is by the definition of  $(\mathcal{J}_\omega)_{\omega \in J \cup \{?\}}$ , the second follows from the TP property of  $\sum \mathcal{I}'_{\omega'}$ , and the third is by (10) and (11).

Finally, we show that  $(\mathcal{J}_\omega)_{\omega \in J \cup \{?\}}$  unambiguously discriminates between  $(\rho_j)_{j \in J}$  better than  $(\mathcal{I}_\omega)_{\omega \in J \cup \{?\}}$ . We see that each  $\text{tr}[\mathcal{J}_j \rho_j]$  is strictly larger than  $\text{tr}[\mathcal{I}_j \rho_j]$ :

$$\text{tr}[\mathcal{J}_j \rho_j] - \text{tr}[\mathcal{I}_j \rho_j] = \text{tr}[\mathcal{I}_? \rho_j] \text{tr} \left[ \mathcal{I}'_j \frac{\mathcal{I}_? \rho_j}{\text{tr}[\mathcal{I}_? \rho_j]} \right] > 0, \quad (15)$$

where the equality is by (14) and the inequality is by (9) and (11). Equations (14) and (15) prove, in particular, that  $(\mathcal{J}_\omega)_{\omega \in J \cup \{?\}}$  unambiguously discriminates  $(\rho_j)_{j \in J}$ . Let  $f$  be any evaluation function described in the Definition 2. Then, by (15), we have  $f(\mathcal{J}_j \rho_j) > f(\mathcal{I}_j \rho_j)$ . In other words, the instrument  $(\mathcal{J}_\omega)_{\omega \in J \cup \{?\}}$  discriminates between  $(\rho_j)_{j \in J}$  better than  $(\mathcal{I}_\omega)_{\omega \in J \cup \{?\}}$ . This completes the proof. ■

We will discuss the assumption (7) in the Theorem in Sec. VI.

#### V. RESULT ON UNIFORM DISTINGUISHABILITY

As the number of states becomes infinite, the concept of distinguishability is naturally divided into two: “distinguishability” and “uniform distinguishability” [14]. We discussed the former in the preceding sections. We consider the latter in this section.

We recall that the class of evaluation functions in the Definition in Sec. III becomes slightly restricted when the index set  $J$  is not finite. Although  $f_{\text{av}}$  is included in the class, an important evaluation function

$$f_{\text{inf}}(x_j) := \inf\{x_j \mid j \in J\}, \quad (16)$$

which generalizes  $f_{\text{min}}$ , is excluded. The function  $f_{\text{inf}}$  has a definite operational meaning similar to  $f_{\text{min}}$ . Hence it is a natural demand to include such an evaluation function. It can be done by introducing the uniform distinguishability. We provide the uniform version of Definitions 1 and 2 and the Theorem.

*Definition 1'.* A uniform unambiguous discrimination measurement between  $(\rho_j)_{j \in J}$  is an instrument  $(\mathcal{I}_\omega)_{\omega \in J \cup \{?\}}$  that satisfies, for  $j, k \in J$  with  $j \neq k$ ,

$$\text{tr}[\mathcal{I}_j \rho_k] = 0, \quad \inf\{\text{tr}[\mathcal{I}_j \rho_j] \mid j \in J\} > 0. \quad (17)$$

The states  $(\rho_j)_{j \in J}$  are said to be uniformly distinguishable when they admit at least one uniform unambiguous discrimination measurement between them.

Distinguishability is a weaker condition than uniform distinguishability. Consider, for example, the states  $(\rho_j)_{j \in \mathbb{N}}$  that are merely distinguishable with success probabilities  $q_j = \text{tr}[\mathcal{I}_j \rho_j] = 1/j$ . In this case, one cannot predict how many trials suffices to determine the true state before performing the discrimination since  $1/f_{\text{inf}}(q_j) = 1/0 = \infty$ . Uniform distinguishability cures this problem and provides a natural framework for infinitely many states.

*Definition 2'.* We call a function  $f : (0, 1]^J \rightarrow \mathbb{R}$  an evaluation function for uniform discrimination if

$$\inf\{x_j > y_j \mid j \in J\} > 0 \Rightarrow f(x_j) - f(y_j) > 0 \quad (18)$$

holds, where  $(x_j)_{j \in J}, (y_j)_{j \in J} \in (0, 1]^J$ .

Note that the function  $f_{\text{inf}}$  is an evaluation function for uniform discrimination as well as  $f_{\text{av}}$ . Evaluation functions for uniform discrimination comprise a larger class than mere evaluation functions do.

*Theorem'.* Optimal uniform discrimination measurements make uniformly distinguishable states not uniformly distinguishable when the discrimination fails (gives ?). Namely, assume that the measurement  $(\mathcal{I}_\omega)_{\omega \in J \cup \{?\}}$  achieves an optimal uniform unambiguous discrimination between the states  $(\rho_j)_{j \in J}$  and that the condition

$$\sup\{\text{tr}[\mathcal{I}_j \rho_j]\} < 1 \quad (19)$$

holds. Then the resulting states under the condition that the outcome is ?, defined by

$$\left( \frac{\mathcal{I}_? \rho_j}{\text{tr}[\mathcal{I}_? \rho_j]} \right)_{j \in J}, \quad (20)$$

are not uniformly distinguishable. Here, the optimality is defined by an arbitrary evaluation function for uniform discrimination in Definition 2'.

The definitions and theorem with prime symbols are equivalent to those without prime in Sec. IV when  $J$  is finite. When  $J$  becomes countably infinite, “uniform distinguishability” becomes a stronger condition than mere “distinguishability” and evaluation functions for uniform discrimination form a larger class than that of mere evaluation functions. The proof of the Theorem' can be given in a way similar to that of the Theorem and is omitted [the essential point is to replace the inequalities “ $\dots > 0$ ” with “ $\inf\{\dots \mid j \in J\} > 0$ ” in Eqs. (9), (10), (11), and (15)].

We note that an assumption (19), which is stronger than (7) in the previous Theorem, is necessary in the Theorem'. In fact, when  $\sup q_j = 1$  and  $q_j < 1$ , the original uniform discrimination (with success probabilities  $q_j$ ) is not necessarily *uniformly* improved by a subsequent uniform discrimination measurement (with  $q'_j$ ). The reason is because the improvements in success probabilities are given by  $(1 - q_j)q'_j$  [see (15)].

## VI. SEPARATION OF PERFECTLY DISTINGUISHABLE STATES

In the Theorem, we assumed that optimal discrimination measurement satisfies  $\text{tr}[\mathcal{I}_j \rho_j] < 1$ . The excluded case was  $\text{tr}[\mathcal{I}_? \rho_j] = 0$ , or equivalently,  $\text{tr}[\mathcal{I}_j \rho_j] = 1$ . We discuss such cases in this section.

*Definition 3.* Let  $(\rho_j)_{j \in J}$  be states and  $K$  be a subset of  $J$ . The states  $(\rho_j)_{j \in K}$  are said to be perfectly distinguishable if there exists an unambiguous discrimination measurement  $(\mathcal{I}_\omega)_{\omega \in J \cup \{?\}}$  between  $(\rho_j)_{j \in J}$  such that

$$\text{tr}[\mathcal{I}_k \rho_k] = 1 \quad (21)$$

holds for all  $k \in K$ .

*Proposition.* Assume that the states  $(\rho_j)_{j \in J}$  are distinguishable and that  $(\rho_j)_{j \in K}$ ,  $K \subset J$ , are perfectly distinguishable. Then there exists a (two-outcome projection) measurement  $(\mathcal{L}, \mathcal{L}')$  such that

$$\mathcal{L} \rho_k = \rho_k, \quad \mathcal{L}' \rho_k = 0, \quad k \in K, \quad (22)$$

and

$$\mathcal{L} \rho_\ell = 0, \quad \mathcal{L}' \rho_\ell = \rho_\ell, \quad \ell \in J \setminus K \quad (23)$$

holds.

*Proof.* Assume  $(\rho_k)_{k \in K}$  are perfectly distinguishable by an unambiguous discrimination measurement  $(\mathcal{I}_\omega)_{\omega \in J \cup \{?\}}$ . Let  $L \in \mathcal{B}(\mathcal{H})$  be an operator such that, for all  $\rho \in B^1(\mathcal{H})$ ,

$$\sum_{k \in K} \text{tr}[\mathcal{I}_k \rho] = \text{tr}[LL^* \rho], \quad (24)$$

i.e.,  $LL^*$  is the sum of so-called positive-operator valued measure (POVM) elements for outcomes in  $K$ . Then  $LL^* \leq 1$ . Let  $P$  be the projection onto the (norm) closure of  $L\mathcal{H} = \{L\xi \mid \xi \in \mathcal{H}\}$ . Then  $PL = L$ . Define the instrument  $(\mathcal{L}, \mathcal{L}')$  by

$$\mathcal{L} \rho = P \rho P, \quad \mathcal{L}' \rho = (1 - P) \rho (1 - P), \quad (25)$$

where  $\rho \in B^1(\mathcal{H})$ . We will prove this instrument satisfies the conditions (22) and (23) in the Proposition.

First, we prove (22). Fix  $k \in K$ . Because  $(\mathcal{I}_\omega)_{\omega \in J \cup \{?\}}$  is perfect on  $K$ , we have

$$\begin{aligned} 1 &= \text{tr}[\mathcal{I}_k \rho_k] \\ &\leq \sum_{k' \in K} \text{tr}[\mathcal{I}_{k'} \rho_k] = \text{tr}[LL^* \rho_k] = \text{tr}[LL^*(P \rho_k P)] \\ &\leq \text{tr}[1(P \rho_k P)]. \end{aligned} \quad (26)$$

Therefore,  $\text{tr}[P \rho_k P] = 1$ . Thus, by the cyclic property of trace,  $\text{tr}[(1 - P) \rho_k (1 - P)] = 0$ . Then one has  $\rho_k^{1/2} (1 - P) = 0$ , which proves (22).

Second, we prove (23). Fix  $\ell \in J \setminus K$ . By the assumption that  $(\mathcal{I}_\omega)_{\omega \in J \cup \{?\}}$  unambiguously discriminates between the states, we have  $\text{tr}[\mathcal{I}_k \rho_\ell] = 0$  for all  $k \in K$ . Then  $0 = \sum_{k \in K} \text{tr}[\mathcal{I}_k \rho_\ell] = \text{tr}[L^* \rho_\ell L]$  and  $\rho_\ell^{1/2} L = 0$ . Since  $P$  is the projection onto the closure of  $L\mathcal{H}$ , we have  $\rho_\ell^{1/2} P = 0$ . Thus  $(1 - P) \rho_\ell (1 - P) = \rho_\ell$ , which proves (23). ■

By the measurement described in the Proposition, we can see whether the true state  $\rho_j$  is in  $(\rho_k)_{k \in K}$  or  $(\rho_\ell)_{\ell \in J \setminus K}$ , without disturbing the set that contains  $\rho_j$ .

The Theorem in Sec. IV is not applicable when an optimal instrument perfectly discriminates between some of the states. In such a case, one can remove all perfectly distinguishable states beforehand by the Proposition above and then apply the Theorem. Therefore, we can assume (7) in the Theorem without loss of generality. However, we cannot always assume (19) in the Theorem' physically. When  $q_j := \text{tr}[\mathcal{I}_j \rho_j] < 1$  and  $\sup q_j = 1$ , the states after an optimal uniform discrimination measurement with the inconclusive outcome may be uniformly distinguishable.



## VII. CONCLUSION AND DISCUSSIONS

We have shown that optimal unambiguous discrimination makes distinguishable states indistinguishable under the condition that the inconclusive outcome  $?$  is obtained. The results extend the previously known ones to infinitely many candidates and output states, which can be pure or mixed, to all quantum mechanically possible measurements, and to virtually all evaluation functions that define optimality. Our proof was based on a simple principle. The best measurements leave no room to carry out the task further. If the resulting states are distinguishable, we can achieve more accurate discrimination by discriminating the resulting states. This made the proof almost obvious and, at the same time, removed restrictions in the previous work [11]. We would like to emphasize that our proofs of the Theorems do not depend on the criteria of the distinguishability such as linear independence. The Theorems are direct consequences of the definition of optimality.

We have also discussed the uniform discrimination, which becomes slightly stronger than the mere distinguishability when the number of candidate state becomes infinite. We showed the Theorem' for the uniform distinguishability. The conclusion of the Theorem' becomes slightly weaker, while the class of evaluation functions is enlarged so that it includes a natural measure  $f_{\text{inf}}$ . The difference between the Theorem and Theorem' exhibits the subtlety of the handling of infinite many states.

Besides the main results, we have discussed the case that some candidate states are perfectly distinguishable. We showed that such candidate state can be separated by a two-outcome measurement without disturbing the true state. This reflects the nature of unambiguous discrimination.

If one is interested in more detailed descriptions of the property of the resulting states, he or she can make use of the results in our previous work. It was proved that countably many pure states are distinguishable if and only if they are minimal and that they are uniformly distinguishable if and only if they are Riesz-Fisher (both of the mathematical properties are generalizations of the linear independence). We also derived the condition for the countably many general (possibly mixed) states to be distinguishable (see also [12] for the case of finitely

many states). On the other hand, the condition for countably many general states to be *uniformly* distinguishable can be stated based on our previous work in principle, though the derived condition seems to be complicated so that it is not practical enough. It may be an interesting problem to simplify the condition, which helps understand the resulting states and the disturbance of the optimal discrimination.

The main results in this paper show that most accurate unambiguous discrimination must destroy the distinguishability of given states, which is an extreme case of error-disturbance trade-off relation. We expect that there exist other trade-off relations that can be understood clearly within the framework of unambiguous discrimination. An example is wave-particle duality, which is formulated as various trade-off relations between coherence and distinguishability of quantum states [17,18]. Among them, Bera *et al.* [18] used the average success probability of unambiguous discrimination as a measure of distinguishability. Our argument may be useful in the context of wave-particle duality when one wants to generalize the results in some directions or simplify the problem.

Finally, we would like to make a general comment on the principle which we have used to prove the theorems (that the optimal measurements leave no room to carry out the task further). The idea is itself very simple and obvious so that everyone understands it readily. However, it is not trivial in what context this idea really works well and how to apply the idea to each context. Note that the fact that the idea works in the context of unambiguous discrimination under an appropriate setting, as we have demonstrated in this paper, is itself not trivial. For example, without the inconclusive outcome, it might be impossible to make use of the idea in a similar manner. The idea seems to be applicable to a wide variety of subjects and to allow a unified discussion as well. It would be an interesting work to find other subjects in which the idea can draw useful conclusions.

## ACKNOWLEDGMENTS

R.K. acknowledges the Research Grant of Keio Leading-edge Laboratory of Science and Technology from Keio University.

- 
- [1] For modern treatments, see M. Ozawa, *Phys. Rev. A* **67**, 042105 (2003); Y. Watanabe, T. Sagawa, and M. Ueda, *ibid.* **84**, 042121 (2011).
  - [2] See A. S. Holevo, *Russ. Math. Surv.* **53**, 1295 (1998), and reference therein.
  - [3] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin, and W. K. Wootters, *Phys. Rev. Lett.* **76**, 722 (1996); C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, *Phys. Rev. A* **53**, 2046 (1996); C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, *ibid.* **54**, 3824 (1996).
  - [4] For example, C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976).
  - [5] For an overview, see A. Chefles, *Contemp. Phys.* **41**, 401 (2000).
  - [6] For example, H. Wakamura, R. Kawakubo, and T. Koike, *Phys. Rev. A* **95**, 022321 (2017); **96**, 022325 (2017).
  - [7] I. Ivanovic, *Phys. Lett. A* **123**, 257 (1987).
  - [8] D. Dieks, *Phys. Lett. A* **126**, 303 (1988).
  - [9] A. Peres, *Phys. Lett. A* **128**, 19 (1988).
  - [10] G. Jaeger and A. Shimony, *Phys. Lett. A* **197**, 83 (1995).
  - [11] A. Chefles, *Phys. Lett. A* **239**, 339 (1998).
  - [12] Y. Feng, R. Duan, and M. Ying, *Phys. Rev. A* **70**, 012308 (2004).
  - [13] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Springer, Berlin, 1932).
  - [14] R. Kawakubo and T. Koike, *J. Phys. A* **49**, 265201 (2016).
  - [15] E. B. Davies and J. T. Lewis, *Commun. Math. Phys.* **17**, 239 (1970).
  - [16] M. Ozawa, *J. Math. Phys.* **25**, 79 (1984).
  - [17] D. M. Greenberger and A. Yasin, *Phys. Lett. A* **128**, 391 (1988); B.-G. Englert, *Phys. Rev. Lett.* **77**, 2154 (1996); E. Bagan, J. A. Bergou, S. S. Cottrell, and M. Hillery, *ibid.* **116**, 160406 (2016).
  - [18] M. N. Bera, T. Qureshi, M. A. Siddiqui, and A. K. Pati, *Phys. Rev. A* **92**, 012118 (2015).