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Relational time in anyonic systems

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In a seminal paper [Phys. Rev. D **27**, 2885 (1983)], Page and Wootters suggest that time evolution could be described solely in terms of correlations between systems and clocks, as a means of dealing with the "problem of time" stemming from vanishing Hamiltonian dynamics in many theories of quantum gravity. Their approach seeks to identify relational dynamics given a Hamiltonian constraint on the physical states. Here we present a "state-centric" reformulation of the Page and Wootters model better suited to cases where the Hamiltonian constraint is satisfied, such as anyons emerging in Chern–Simons theories. We describe relational time by encoding logical "clock" qubits into topologically protected anyonic degrees of freedom. The minimum temporal increment of such anyonic clocks is determined by the universality of the anyonic braid group, with nonuniversal models naturally exhibiting discrete time. We exemplify this approach by using $SU(2)_2$ anyons and discuss generalizations to other states and models.

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In general relativity, the Hamiltonian is constrained to vanish on physical states [1]. Canonical quantization preserves this constraint, resulting in the Wheeler–DeWitt equation [2]. This equation embodies the "problem of time" in canonical quantum gravity: the vanishing of the Hamiltonian on physical states means that all quantum-mechanical operators, including the density matrix describing the state of any system, must be time independent, in contrast to everyday experience. This apparent paradox has many facets and various approaches attempt to solve some of them (see Refs. [3–5] for in-depth reviews). One possible solution is that time is relational [6,7]: that is, it emerges from correlations between subsystems of the Universe, some of which we call "clocks."

One relational approach is the model for a conditional probability interpretation of Page and Wootters [6,8,9] (PaW), which was experimentally demonstrated recently [10]. The PaW Universe is formulated in terms of qubits represented as spins, which implicitly carry internal Hamiltonian dynamics. To conform to the Hamiltonian constraint, the state of the Universe is an energy eigenstate, which factors into "system" and "clock" subspaces. Then, the "system" dynamics emerge with respect to correlations with the "clock" subsystem.

In this paper we show how the PaW relational time emerges in a Universe lacking Hamiltonian dynamics. To do so we reformulate PaW in state-centric terms, which makes explicit the need for additional requirements on the global state beyond the Hamiltonian constraint—requirements implicit in PaW. A departure from the PaW model's kinematical clocks is inevitable—qubits emerge from topologically protected anyonic degrees of freedom, which arise in the charge sectors of $(2+1)$-dimensional Chern–Simons theories, and in quantum double models [11]. We show that these topologically nontrivial degrees of freedom are useful clocks despite being gauge invariant. A consequence of our model is that, whereas in PaW any ordered set of clock states is accessible, the set of clock measurement outcomes in computationally nonuniversal anyonic groups is finite, leading to a natural discretization of relational "time."

First we give a brief overview of the PaW model. Page and Wootters divide the Hilbert space into a "clock" part and a "system" part, with total Hamiltonian $\mathrm{H} = \mathrm{H}_\mathrm{c} + \mathrm{H}_\mathrm{s}$, where $\mathrm{H}_{\mathrm{c,s}}$ are Hamiltonians for the clock and system parts, respectively. In Page and Wootters [6,8] time emerges kinematically. This is embodied in their assumption that the "Universe" is in a pure, maximally entangled state, $|\Psi_0\rangle_{\mathrm{cs}}$, stationary under unitary evolution $U(t) = \exp(-i\mathrm{H}t)$, where t is the unobservable coordinate time. Apart from the initial entanglement in $|\Psi_0\rangle_{\mathrm{cs}}$, there is no dynamical interaction between the system and the clock. A reference state, $|\tau_0\rangle_\mathrm{c}$, which is not an eigenstate of H_c, is defined to be the "zero" tick of the clock (i.e., "midnight") [12,13]. Subsequent clock states $|\tau\rangle_\mathrm{c}$, are then generated by H_c,

$$|\tau\rangle_\mathrm{c} := e^{-i\mathrm{H}_\mathrm{c}(\tau-\tau_0)}|\tau_0\rangle_\mathrm{c}, \tag{1}$$

where τ signifies the "clock time." We note that the clock time τ is not associated with any particular value of the coordinate time t; instead it is a possible outcome for a measurement on the clock.

The state of the system at clock time τ is defined by conditioning $|\Psi_0\rangle_{\mathrm{cs}}$ on the measured clock state $|\tau\rangle_\mathrm{c}$. PaW showed that this conditional state of the system is consistent with Schrödinger evolution of the system under H_s for a time

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$\tau - \tau_0$, i.e.,

$$|\psi(\tau)\rangle_s := \frac{{}_c\langle\tau|\Psi_0\rangle_{cs}}{\mathrm{tr}[{}_c\langle\tau|\Psi_0\rangle_{cs}\langle\Psi_0|\tau\rangle_c]^{1/2}} \tag{2}$$

$$= e^{-i\mathrm{H}_s(\tau-\tau_0)}|\psi(\tau_0)\rangle_s. \tag{3}$$

This remarkable result rests on the fact that the Universe is in a stationary, maximally entangled state, and that there are no interactions between the clock and system [6]. This process is a form of gate teleportation from the clock to the system [14,15]. We note that global stationarity leads to problems if a clock is conditioned upon more than once [4, Ch. 13], a point we discuss at the end.

The PaW approach outlined above is *Hamiltonian centric* in that it starts by defining Hamiltonians for the clock and for the system. Page and Wootters then require the joint state to be an eigenstate of the total Hamiltonian H. From there, unitary evolution of the system in clock time follows, as in Eq. (3).

The Hamiltonian-centric approach is conceptually unsatisfying for systems which already satisfy any Hamiltonian constraints. These include Chern–Simons theories in which the Hamiltonian vanishes on physical states—a consequence of the Chern–Simons Lagrangian being linear in time derivatives [11,16].

In the context of anyons, it is more natural to adopt a *state-centric* approach in which we start by defining the maximally entangled clock-system state $|\Psi_0\rangle_{cs}$ as well as a canonically ordered set of generalized measurement (POVMs) outcomes on the clock.

Entanglement is also a requirement in the PaW model and is not guaranteed by the Hamiltonian constraint alone. In fact, one must choose a specific type of entangled state in order to recover unitary dynamics [8]. Furthermore, whereas in the PaW model the ordering is implicit in the choice of a clock Hamiltonian, in our state-centric approach we impose it explicitly. One convenient way to describe the ordering is to introduce an effective clock Hamiltonian H_c that rotates sequentially between the POVM outcomes. One of the POVM outcomes, $|\tau_0\rangle_c$, on the clock subspace is nominated as the clock reference state (i.e., midnight), from which the clock "evolves" in the manner of Eq. (1). We then find an effective Hamiltonian H_s for the system partition such that the resulting state after the measurement, Eq. (2), can be obtained from the initial state $|\psi(\tau_0)\rangle_s$ by evolving it in clock time, as in Eq. (3).

To exemplify this construction we consider a clock and a system each consisting of a single qubit, prepared in a maximally entangled Bell state,

$$|\Psi_0\rangle_{cs} := \frac{1}{\sqrt{2}}(|-+\rangle - |+-\rangle)_{cs}, \tag{4}$$

where $|\pm\rangle$ are the eigenstates of Pauli X.

In line with the spin-j example in the original PaW paper [6] we restrict ourselves to clock states on the Bloch sphere's equator (x-y plane), and choose $|\tau_0\rangle_c = |+\rangle_c$. Subsequent clock states are defined by rotations around the z axis,

$$|\tau\rangle_c := \mathcal{R}_z^{(c)}[2\pi(\tau-\tau_0)]|+\rangle_c, \tag{5}$$

where $\tau - \tau_0 = \mathrm{T}/\mathcal{N}_c$ with $\mathrm{T} \in \{0,\dots,\mathcal{N}_c - 1\}$ enumerates the $\mathcal{N}_c$ "ticks" of the clock, and $\mathcal{R}_z(\phi) \equiv \exp(i\phi \mathrm{Z}/2)$. To connect with PaW, we observe that this clock time is generated by $\mathrm{H}_c = -\pi \mathrm{Z}_c$, so that the $\mathcal{N}_c$ clock states are equally spaced around the Bloch equator, separated by the time interval $\Delta\tau = 1/\mathcal{N}_c$. We note that if $\mathcal{N}_c$ is larger than the clock Hilbert space, then the POVM outcomes are not mutually orthogonal.

Conditioning the global state, Eq. (4), on the clock state $|\tau\rangle_c$ gives the state of the system at clock time τ:

$$|\psi(\tau)\rangle_s = \mathcal{R}_z^{(s)}[2\pi(\tau-\tau_0)]|-\rangle_s. \tag{6}$$

By noting that $|\psi(\tau_0)\rangle_s = |-\rangle_s$, Eq. (6) exactly corresponds to unitary evolution in clock time, Eq. (3), generated by an effective system Hamiltonian $\mathrm{H}_s = -\pi \mathrm{Z}_s$. The state $|\Psi_0\rangle_{cs}$ is an eigenstate of the total Hamiltonian, $\mathrm{H} = \mathrm{H}_c + \mathrm{H}_s$. In PaW this was a requirement; here it is merely a consequence of our choice of initial state and clock measurements. We emphasize that H_c and H_s are not fundamental but emerge from the structure of $|\Psi_0\rangle_{cs}$ and the ordered clock POVM outcomes.

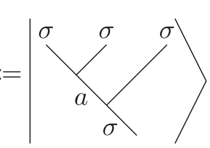
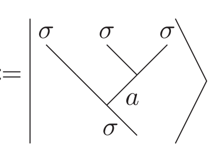
The state-centric approach is applicable to the anyonic Hilbert space in Chern–Simons theories. Physical states can be prepared by anyon pair-production from the vacuum, braiding, and fusion [11].

We describe anyonic relational time explicitly in the context of the $\mathrm{SU}(2)_2$ theory. It deals with three particle species, labeled $\mathbb{1}$, σ, ψ, where $\mathbb{1}$ is the vacuum (spin-0 irreducible representation, or irrep), ψ is a neutral fermion (spin-1 irrep), and σ is the only non-Abelian anyon (spin-$\frac{1}{2}$ irrep). Measurement of the total topological charge of two σ may have more than one possible outcome, as given by the *fusion rules*:

$$\sigma \times \sigma \to \mathbb{1} + \psi, \quad \sigma \times \psi \to \sigma, \quad \psi \times \psi \to \mathbb{1}. \tag{7}$$

The nondeterministic $\sigma \times \sigma$ fusion rule is what allows a collection of three or more non-Abelian anyons to display nontrivial topological degrees of freedom, even when the underlying manifold is contractible [17,18]. These topological degrees of freedom can be used to define qubits, thus enabling clocks in the anyonic PaW universe.

Consider three σ anyons and the associated *fusion Hilbert space*.[1] The order in which we choose to fuse them consecutively defines a basis for this Hilbert space. A given state specifies all intermediate outcomes for that fusion order and is commonly represented as a labeled tree. We define two possible bases, the "z" and "x" bases, for fusing three σ as $\{|\mathbb{1}_z\rangle, |\psi_z\rangle\}$ and $\{|\mathbb{1}_x\rangle, |\psi_x\rangle\}$, where

$|a_z\rangle :=$  , $|a_x\rangle :=$  , (8)

with $a \in \{\mathbb{1}, \psi\}$. We can encode a single qubit in this collective degree of freedom by identifying the z basis with the

[1]Strictly speaking, due to superselection rules, the Hilbert space is defined by the anyons and their total charge, whatever it may be. The $\mathrm{SU}(2)_2$ fusion rules constrain the total charge of three σ to be σ, so we speak of three, rather than four σ as comprising our qubit. In general, for n anyons in any $\mathrm{SU}(2)_k$ model, each local subsystem needs to be postselected on a particular outcome of measurement on the total charge of its anyons such that there are still degrees of freedom associated with intermediate fusion outcomes.

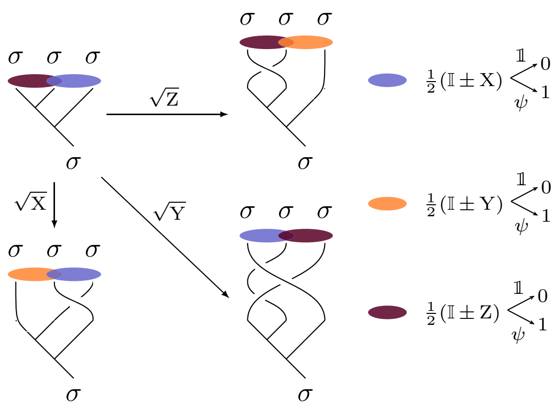


FIG. 1. Three σ SU(2)$_2$ anyons with total charge σ encode a logical qubit. Pauli measurements, X, Y, or Z, on the qubit are implemented by fusing a pair of anyons (i.e., measuring their total charge), indicated by the colored ellipses. Fusion of two σ yields $\mathbb{1}$ or ψ, corresponding to a projective measurement in one of the Pauli bases {X,Y,Z}, depending on which pair is fused. Braiding among the three anyons effects $\frac{\pi}{2}$ rotations, $\sqrt{X}$, $\sqrt{Y}$, $\sqrt{Z}$, around the three axes. $\sqrt{X}$, $\sqrt{Y}$, $\sqrt{Z}$ are equivalent to swapping anyons (2,3), (1,3), and (1,2), respectively.

computational basis, $|0\rangle = |\mathbb{1}_z\rangle$, $|1\rangle = |\psi_z\rangle$. We also define $|+\rangle = |\mathbb{1}_x\rangle$, $|-\rangle = |\psi_x\rangle$, so that $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$.

The transformation between the x and z bases is given by F, whose elements are determined by the fusion rules

$$\mathrm{F} = \frac{1}{\sqrt{2}} \begin{array}{c} \\ \mathbb{1} \\ \psi \end{array} \begin{array}{c} \begin{array}{cc} \mathbb{1} & \psi \end{array} \\ \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \end{array}. \tag{9}$$

Exchanging two σ is trivial if their total charge is $\mathbb{1}$ and introduces a $\frac{\pi}{2}$ phase if their charge is ψ. This is encoded the exchange matrix

$$\mathrm{R}_{i,j} = \begin{array}{c} \\ \mathbb{1} \\ \psi \end{array} \begin{array}{c} \begin{array}{cc} \mathbb{1} & \psi \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \end{array}, \tag{10}$$

given in a basis where the ith and jth σ share a fusion channel.[2] The "y" basis, $\{|\pm i\rangle\}$, can be defined in terms of the z basis and braids on anyons 2 and 3 as

$$|+i\rangle := e^{+i\pi/4}\mathrm{B}_{2,3}|1\rangle, \quad |-i\rangle := e^{-i\pi/4}\mathrm{B}_{2,3}|0\rangle, \tag{11}$$

where $\mathrm{B}_{2,3}$ is given by $\mathrm{B}_{2,3} = \mathrm{F}^\dagger\, \mathrm{R}_{2,3}\mathrm{F}$.

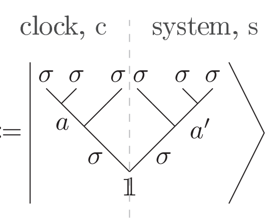
Qubit measurement is effected by pair-wise anyon fusion (i.e., detecting the total charge of a pair, yielding $\mathbb{1}$ or ψ), indicated by colored ellipses in Fig. 1 (top left). The three possible ways to fuse pairs of the σ anyons correspond to measurements in the three Pauli bases X, Y, or Z.

The braid group of three σ anyons is generated by $\mathrm{R}_{1,2}$ and $\mathrm{B}_{2,3}$, so it follows that the braid group of σ in the SU(2)$_2$ model is isomorphic to the one-qubit Clifford group [19]. Because the Clifford group (braiding), normalizes Pauli measurements (fusion), braiding in this model does not give access to any additional measurement bases. Thus, projective measurement outcomes on a single, anyonic SU(2)$_2$ qubit are restricted to one of the six states, $|0\rangle$, $|1\rangle$, $|\pm\rangle$, or $|\pm i\rangle$, of which only four are on the Bloch equator. We identify these $\mathcal{N}_c = 4$ possible times as follows: $|+\rangle_c \leftrightarrow |\mathrm{T}=0\rangle_c$, $|+i\rangle_c \leftrightarrow |\mathrm{T}=1\rangle_c$, $|-\rangle_c \leftrightarrow |\mathrm{T}=2\rangle_c$, and $|-i\rangle_c \leftrightarrow |\mathrm{T}=3\rangle_c$, in cyclic order around the Bloch equator. Below, we discuss how this further in the context of POVM measurements.

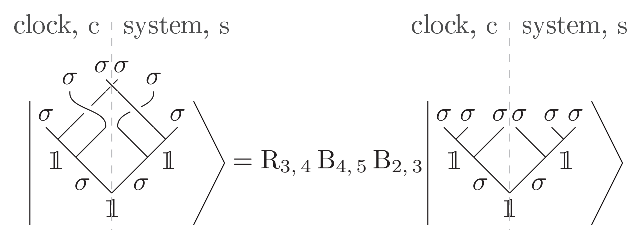
To define relational time in this anyonic Universe, we require (i) at least two subsystems in the Hilbert space, (ii) entanglement between the subsystems, and (iii) a POVM on the clock subsystem.

A minimal anyonic model with two two-dimensional subspaces consists of six σ particles with total charge $\mathbb{1}$ [20]. We define the computational basis as

$$|a_z, a'_z\rangle_{cs} := \left| \; \text{[tree diagram]} \; \right\rangle, \quad a, a' \in \{\mathbb{1}, \psi\} \tag{12}$$



Entanglement requires braiding between the two subsystems. A maximally entangled state is produced when pairs of anyons created from the vacuum are shared between the two subsystems [21] as represented by the following tree:



$$\equiv \frac{1}{\sqrt{2}}(|+,0\rangle + |-,1\rangle)_{cs} := |\Psi_0\rangle_{cs}\,. \tag{13}$$

To illustrate a possible clock POVM outcome: the braid $\mathrm{B}_{2,3}$ effects a Hadamard gate on the clock [as in Eq. (11)], and a measurement outcome $|\mathbb{1}\rangle_{1,2}$ would project the clock onto state the $|+i\rangle_c \leftrightarrow |\mathrm{T}=1\rangle_c$, in accordance with the bottom left panel of Fig. 1.

A POVM on the clock can be modelled by coupling the clock to k ancilla, followed by projective measurements on the clock and ancilla, as shown in Fig. 2. We initialize the system (s) and clock (c) qubits in a Bell state, Eq. (13), and introduce k ancillary qubits. A unitary U, together with projective measurements on the clock and ancilla, yields a POVM on the clock qubit with $\mathcal{N}_c \leqslant 2^{k+1}$ outcomes. In a computationally universal model, for which any unitary U is physically accessible, the inequality can be saturated, so that the temporal increment of the clock, $\Delta\tau = 2^{-(k+1)}$, can be made arbitrarily fine by increasing k.

The SU(2)$_2$ braid group, however, is isomorphic to the Clifford group, which is *not* universal. In this case, the set of unitary gates generated by the braid group is finite. The maximum number of POVM outcomes, $\mathcal{N}_c$, on the clock is

[2] Note that the SU(2)$_2$ model with the same F matrix but with $\mathrm{R} \to i\mathrm{R}^\dagger$ gives the Ising anyon model.

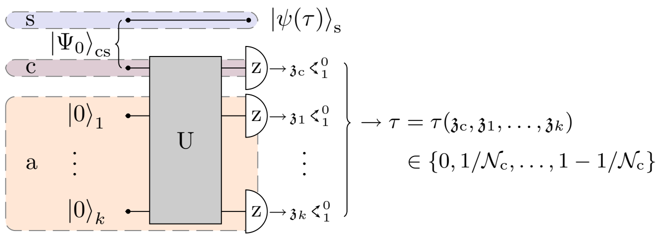


FIG. 2. System qubit s and clock qubit c are prepared in a Bell state $|\Psi_0\rangle_{\mathrm{cs}}$. To implement a POVM on the clock, the clock is coupled to a collection of k ancilla via a unitary gate U. Depending on the universality class of the model, U yields a POVM on the clock qubit with $\mathcal{N}_{\mathrm{c}} \leqslant 2^{k+1}$ possible outcomes, which are in direct correspondence with the set of Z-measurement outcomes, $\{\mathfrak{z}_{\mathrm{c}}, \mathfrak{z}_1, \ldots, \mathfrak{z}_k\}$, on the clock and ancilla qubits. An ordering of those outcomes gives the clock time $\tau \in \{0, 1, \ldots, \mathcal{N}_{\mathrm{c}} - 1\}$.

thus bounded: $\mathcal{N}_{\mathrm{c}} \leqslant \mathcal{M}_{\mathrm{c}}$ for some $\mathcal{M}_{\mathrm{c}}$ which depends on the braid group. Time in such a Universe is a discrete quantity, indivisible into intervals smaller than $1/\mathcal{M}_{\mathrm{c}}$, regardless of the number of the ancilla used to effect the clock POVM.

We note that $\mathcal{M}_{\mathrm{c}}$ is independent of the number of ancillae used. If we consider the joint clock-plus-ancillae system as a generalized clock, the input state in Fig. 2 is generic: "c" simply labels the degree of freedom that is maximally entangled with the system. Because the $SU(2)_2$ braid group is not universal, and braiding on the clock qubit is sufficient to generate all projective measurements in the Pauli basis, the inclusion of ancillae cannot increase the number of accessible POVM outcomes and therefore does not change the number of possible times measurable on the clock.

The construction here extends to other non-Abelian anyonic models. The Universe is modelled as a collection of N anyons with trivial total charge. We isolate a subset of the fusion Hilbert space $\mathcal{H}$ having $n < N$ degrees of freedom. These degrees of freedom are to be interpreted as qudits, with d depending on the number of possible fusion outcomes. $\mathcal{H}$ is split into two noninteracting subsystems—the "clock" and the "system"—such that n_{c} "qudits" go to the clock while the remaining n_{s} go to the system. We do this in a way that results in an entangled state of the two subsystems. Clock time is given by an ordered set of POVM outcomes, where the POVM is implemented by using k ancillary qudits. In nonuniversal models, the minimum temporal interval, $\Delta\tau_{\min} \geqslant 1/\mathcal{M}_{\mathrm{c}}$, is determined by the braid group and the number of clock qudits, independent of the number of ancilla.

The connection between the computational universality class of the clock system and the discreteness of relational time is a key result of this paper. For example, the braid group of $SU(2)_4$ is not computationally universal, so an $SU(2)_4$ Universe would also exhibit discrete relational time [although we note that $SU(2)_4$ anyonic models are capable of universal computation under postselection and feedforward [22], which is not suitable for defining relational time].

In light of this observation, we briefly speculate on several possible research directions. First, we conjecture that such discrete relational time is generically present in nonuniversal theories. An example is the permutation quantum computation (PQC) model of Marzuoli and Rasetti [23,24], in which braiding of anyons is replaced by swap gates on spin-$\frac{1}{2}$ particles with total spin $S = 0$, leaving the total spin invariant.

Second, it is known that nonlocal, multipartite states may admit a local hidden variable theory when the set of allowed measurements is constrained [25]. For instance, CHSH inequalities cannot be violated with only Pauli measurements [as in the $SU(2)_2$ anyon model we have considered] [18,21,25–27]. In an $SU(2)_2$ Universe at least five σ pairs (i.e., four qubits) shared between two parties are needed to show some nonlocality [21]. Thus, nonlocality in nonuniversal measurement models is somewhat limited and we speculate that this might be a generic feature of discrete time theories.

Third, in the PaW model one cannot condition more than once on a clock [4, Ch. 13]. In particular, correlations between clock and system are lost. To allow for multitime measurements on the same clock in the context of PaW, Gambini *et al.* [28] (GPPT) suggest constructing a stationary "quantum clock" which is conditioned on a dynamical classical variable, similarly to the way a single system is conditioned on a dynamical clock in the PaW model. In our context, one could distribute a second Bell pair $|\Psi_0\rangle_{\mathrm{cs}}$ between clock and system and simply repeat the protocol but with the updated basis of the clock rotated so that $|\tau_0\rangle_{\mathrm{c}} \to |\tau_m\rangle_{\mathrm{c}}$.

We have presented a conditional probability approach to relational time where qudits are defined in an anyonic fusion space, and where POVMs are generated by braiding and fusion. Our state-centric reformulation of the Page and Wootters approach is directly applicable to anyonic models which arise in Chern–Simons theories, for which physical states are in the nullspace of the Hamiltonian and thus embody the problem of time. We have shown that $SU(2)_k$ theories which are nonuniversal for computation (i.e., $k = 2$ or $k = 4$) are only capable of supporting discrete relational time, which may have implications for other models that have discrete, emergent time.

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[1] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman, New York, USA, 1973).
[2] B. S. DeWitt, Phys. Rev. **162**, 1195 (1967).
[3] C. J. Isham, in *Integrable Systems, Quantum Groups, and Quantum Field Theories*, NATO Advanced Science Institutes Series, Series C, Vol. 409 (Springer, Netherlands, 1993), pp. 157–287.
[4] K. V. Kuchař, Int. J. Mod. Phys. D **20**, 3 (2011).
[5] E. Anderson, Ann. Phys. (Berlin, Ger.) **524**, 757 (2012).
[6] D. N. Page and W. K. Wootters, Phys. Rev. D **27**, 2885 (1983).

[7] R. Gambini and J. Pullin, Phys. Rev. Lett. **94**, 101302 (2005).
[8] W. K. Wootters, Int. J. Theor. Phys. **23**, 701 (1984).
[9] D. N. Page, in *Physical Origins of Time Asymmetry*, edited by J. J. Halliwell, J. Mercader-Pérez, and W. H. Zurek (Cambridge University Press, Cambridge, England, 1996).
[10] E. Moreva, G. Brida, M. Gramegna, V. Giovannetti, L. Maccone, and M. Genovese, Phys. Rev. A **89**, 052122 (2014).
[11] J. K. Pachos, *Introduction to Topological Quantum Computation* (Cambridge University Press, Cambridge, England, 2012).
[12] G. J. Milburn and D. Poulin, Int. J. Quantum Inform. **4**, 151 (2005).
[13] A. Boette, R. Rossignoli, N. Gigena, and M. Cerezo, Phys. Rev. A **93**, 062127 (2016).
[14] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
[15] I. L. Chuang and D. Gottesman, Nature (London) **402**, 390 (1999).
[16] G. V. Dunne, in *Topological Aspects of Low Dimensional Systems* (Springer, New York, USA, 1998), pp. 177–263.
[17] A. Y. Kitaev, Ann. Phys. (NY) **321**, 2 (2006).
[18] G. K. Brennen, S. Lblisdir, J. K. Pachos, and J. K. Slingerland, New J. Phys. **11**, 103023 (2009).
[19] A. Ahlbrecht, L. S. Georgiev, and R. F. Werner, Phys. Rev. A **79**, 032311 (2009).
[20] C. Nayak and F. Wilczek, Nucl. Phys. B **479**, 529 (1996).
[21] E. T. Campbell, M. J. Hoban, and J. Eisert, Quant. Inf. Comp. **14**, 0981 (2014).
[22] S. X. Cui and Z. Wang, J. Math. Phys. **56**, 032202 (2015).
[23] A. Marzuoli and M. Rasetti, Ann. Phys. (NY) **318**, 345 (2005).
[24] S. P. Jordan, Quantum Inf. Comput. **10**, 470 (2010).
[25] N. Ratanje and S. Virmani, Phys. Rev. A **83**, 032309 (2011).
[26] M. Howard and J. Vala, Phys. Rev. A **85**, 022304 (2012).
[27] D. J. Clarke, J. D. Sau, and S. Das Sarma, Phys. Rev. X **6**, 021005 (2016).
[28] R. Gambini, R. A. Porto, J. Pullin, and S. Torterolo, Phys. Rev. D **79**, 041501 (2009).