Coherent scattering from semi-infinite non-Hermitian potentials

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When two identical (coherent) beams are injected at a semi-infinite non-Hermitian medium from left and right, we show that both reflection (r_L,r_R) and transmission (t_L,t_R) amplitudes are nonreciprocal. In a parametric domain, there exists spectral singularity (SS) at a real energy $E = E_* = k_*^2$ and the determinant of the time-reversed two port scattering matrix, i.e., $|\det(S(-k))| = |t_L(-k)t_R(-k) - r_L(-k)r_R(-k)|$, vanishes sharply at $k = k_*$, displaying the phenomenon of coherent perfect absorption (CPA). In the complementary parametric domain, the potential becomes either left or right reflectionless at $E = E_z$. We rule out the existence of invisibility despite $r_R(E_i) = 0$ and $t_R(E_i) = 1$ but $T(E_i) \neq 1$, in this avenue. We present two simple exactly solvable models where expressions for E_*, E_z, E_i, and parametric conditions on the potential have been obtained in explicit and simple forms. Earlier, the phenomena of SS and CPA have been found to occur only in the scattering complex potentials which are spatially localized (vanish asymptotically) and have $t_L = t_R$.

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A non-Hermitian complex potential $V(x) = V_r(x) + iV_i(x)$ which is spatially localized and nonsymmetric displays the nonreciprocity of reflection amplitudes ($r_L \neq r_R$) whereas transmission amplitudes are reciprocal $t_L = t_R$ [1–7]. For non-Hermitian scattering potentials the existence of a special real energy (E_*) has been proposed [8] where all three probabilities ($T = |t|^2, R = |r|^2$) $T(E), R_L(E)$, and $R_R(E)$ become infinity. This special energy is called spectral singularity (SS) [8]. Though SS was first demonstrated to exist in a complex PT-symmetric potential [8], with ample number of examples, it has been found [9] that SS is a property of either a complex non-PT-symmetric potential or the parametric domain of broken PT symmetry of a complex PT-symmetric potential. Very interesting exactly solvable models are available [10] giving the explicit expression of E_* and explicit parametric conditions on the non-Hermitian potential.

The two concepts of nonreciprocity [1–7] of reflection and spectral singularity [8] give rise to an experimentation where coherent (identical) beams are injected into a non-Hermitian optical medium from left and right. In coherent scattering, the determinant of two port scattering matrix $S(k)$ is given as $|\det(S(k))| = |r_L(k)r_R(k) - t^2(k)|$ [11]. It has been further claimed that, if spectral singularity occurs at $E = E_* = k_*^2$, $S(-k)$ becomes zero at $k = k_*$, signifying perfect absorption of coherent beams [11] in the non-Hermitian medium. This novel idea of coherent perfect absorption (CPA) has given rise to time-reversed lasers [11–13]. The complex PT-symmetric potentials have been ruled out [9] for CPA, which is also referred to as coherent perfect absorption without lasing [11–13].

We would like to remark that, unlike the first proposal for the general CPA [11], the authors in [14] have been cautious about choosing the optical non-Hermitian medium as parity symmetric. They set a less general yet simpler and intuitive condition for CPA at a real energy as $t + r_L = 0 = t + r_R$. For parity-symmetric complex potentials the reciprocity ($r_L = r_R$) works. This phenomenon has been called controlled CPA, which is a special case of the more general condition [11]. The existence of SS in this case also supports the conjecture that complex PT symmetry is not necessary for SS. Very interesting exactly solvable models of CPA have been proposed [10]. For the non-Hermitian PT-symmetric potentials which are spatially localized, unidirectional invisibility (UI) [15,16] occurs when either r_L or r_R vanishes at a real energy $E = E_i$ and $t_L = t_R = t = 1$ at this energy. For complex PT-symmetric potentials another novel phenomenon of CPA with lasing has been revealed [17,18]. Very interestingly the aforementioned phenomena occur as a possibility and not as a necessity, so their (non)occurrence in various systems is worth studying. For instance, recently, (non)occurrence of SS has been discussed [19] in terms of various kinds of antilinear symmetry of the spatially localized optical mediums.

In sharp contrast to the aforementioned works [1–19] on scattering where complex potentials are spatially localized, in this paper we study scattering from complex potentials where the real part is semi-infinite and the imaginary part is as usual spatially localized (see Fig. 1). By *semi-infinite*, we mean that $V_r(x \sim -\infty) = 0$ and $V_r(x \sim \infty) = V_1$. So the potentials discussed here are essentially non-PT-symmetric. In these interesting models, we find that both r and t are nonreciprocal [see Eqs. (7) and (8)]. Nevertheless, the transmission probabilities remain reciprocal again: $T_L = T_R$ in a nontrivial way [see Eq. (14)]. The question arising here is as to whether we can observe the novel phenomena of SS, CPA, and UI even in such semi-infinite mediums. In this paper, we derive the two-port S matrix [see Eqs. (7) and (8)] for coherent injection at a semi-infinite potential and investigate the possibility of occurrence of the aforementioned phenomena of SS, CPA, and

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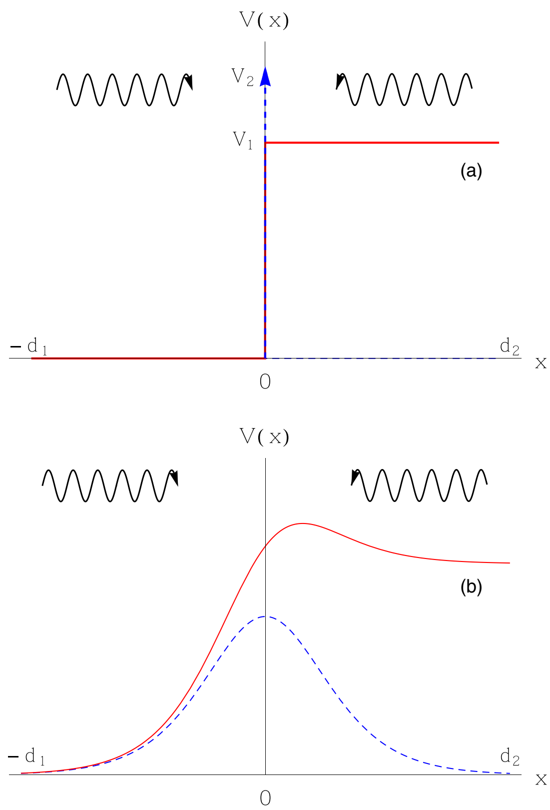

FIG. 1. Two models of non-Hermitian potentials for coherent scattering from left and right. Their real part are semi-infinite. (a) Equation (10). (b) Equation (11). Solid lines denote the real part and dashed lines denote the imaginary part. The vertical arrow in (a) represents the Dirac delta potential.

UI yet again. Earlier, a non-Hermitian semi-infinite potential has been studied, however it being one unit of a periodic array does not remain semi-infinite; nevertheless, this study gives rise to several other interesting issues [20] in scattering from optical potentials.

The Schrödinger equation for a semi-infinite potential $V(x)$ (see Fig. 1),

$$\frac{d^2\psi(x)}{dx^2}+\frac{2\mu}{\hbar^2}[E-V(x)]\psi(x)=0, \tag{1}$$

is solved by defining $k_L{=}\sqrt{2\mu E}/\hbar, k_R{=}\sqrt{2\mu(E-V_1)}/\hbar$. Let d_1 be large asymptotic distance such that $V(-d_1)=0$ and $V(d_2)=V_1$. $u(x)$ and $v(x)$ are two linearly independent solutions of the Schrödinger equation in the interval $[-d_1,d_2]$ such that $u(0)=1, u'(0)=0$; $v(0)=0, v'(0)=1$. These conditions ensure linear independence of the two solutions of the second-order differential equation (1) and the constancy (independence on x) of the Wronskian: $W(x)=[u(x)v'(x)-u'(x)v(x)]$ for all $x \in(-\infty,\infty)$. The solution of (1) for the semi-infinite models is given by

$$\begin{aligned}\psi(x<-d_1)&=A_Le^{ik_Lx}+B_Le^{-ik_Lx},\\ \psi(-d_1<x<d_2)&=Cu(x)+Dv(x),\\ \psi(x>d_2)&=A_Re^{ik_Rx}+B_Re^{-ik_Rx}.\end{aligned} \tag{2}$$

Next the numerical integration on both sides provides us with the end values $u(-d_1), u'(-d_1), v(-d_1)$, and $v'(-d_1)$ at a given energy E on the left of the potential. For short we will denote these values as u_1, u_1', v_1, and v_1', respectively. Similarly, we will have u_2, u_2', v_2, and v_2' evaluated at $x=d_2$. The quantities u_1, v_1, u_2, and v_2 are in general complex.

Further, we use the transfer-matrix method [8,18,21] of scattering in one dimension. Matching these solutions and their first derivative at $x=-d_1$ and d_2, in the matrix notation we can write

$$\begin{pmatrix}A_L\\B_L\end{pmatrix}=\begin{pmatrix}f^{-1}&f\\ik_Lf^{-1}&-ik_Lf\end{pmatrix}^{-1}\begin{pmatrix}u_1&v_1\\u_1'&v_1'\end{pmatrix}\begin{pmatrix}u_2&v_2\\u_2'&v_2'\end{pmatrix}^{-1}\times\begin{pmatrix}g&g^{-1}\\ik_Rg&-ik_Rg^{-1}\end{pmatrix}\begin{pmatrix}A_R\\B_R\end{pmatrix} \tag{3}$$

where $f=e^{ik_Ld_1}$ and $g=e^{ik_Rd_2}$. For short the matrix product can be denoted as $M=M_1^{-1}M_2M_3^{-1}M_4$, which is called the transfer matrix. The Wronskians $u_1v_1'-u_1'v_1=W=u_2v_2'-u_2'v_2$ give us $\det(M)=k_R/k_L$. When $k_L=k_R$, we get $\det(M)=1$. Let us point out that normally $\det(M)=1$ is used as a fundamental property of the transfer matrix wherein the crucial and more basic connection of the Wronskian is often overlooked. We denote the product $M_2M_3^{-1}$ as M_5 to write

$$\begin{aligned}M_5&=\begin{pmatrix}u_1v_2'-v_1u_2'&u_2v_1-u_1v_2\\u_1'v_2'-u_2'v_1'&u_2v_1'-u_1'v_2\end{pmatrix}\\&=\begin{pmatrix}w_{11}&w_{12}\\w_{21}&w_{22}\end{pmatrix}\end{aligned} \tag{4}$$

and $\det(M_5)=1$ holds once again. Now the transfer matrix M can be denoted as

$$M=M_1^{-1}M_5M_4=\begin{pmatrix}m_{11}&m_{12}\\m_{21}&m_{22}\end{pmatrix}. \tag{5}$$

From (3) and (5), we get $A_L=m_{11}A_R+m_{12}B_R$ and $B_L=m_{21}A_R+m_{22}B_R$. So we can write

$$\begin{pmatrix}A_R\\B_L\end{pmatrix}=\begin{pmatrix}-m_{11}&0\\-m_{21}&1\end{pmatrix}^{-1}\begin{pmatrix}-1&m_{12}\\0&m_{22}\end{pmatrix}\begin{pmatrix}A_L\\B_R\end{pmatrix}. \tag{6}$$

Hence the two port S matrix $S(E)$ for coherent injection from left and right gets defined as

$$S(k)=\begin{pmatrix}s_{11}&s_{12}\\s_{21}&s_{22}\end{pmatrix}=\begin{pmatrix}\frac{1}{m_{11}}&-\frac{m_{12}}{m_{11}}\\\frac{m_{21}}{m_{11}}&\frac{\det(M)}{m_{11}}\end{pmatrix}=\begin{pmatrix}t_L&r_R\\r_L&t_R\end{pmatrix}. \tag{7}$$

We also prove the nonreciprocity of transmission amplitude for semi-infinite potentials as

$$k_Lt_R=k_Rt_L. \tag{8}$$

The crucial question arising here is as to whether a semi-infinite potential $V_r(x)$ $(\mathcal{R}(V(x)))$ can also give rise to CPA with a modified two-port S matrix (7), where

$$|\det(S(k_L,k_R))|=|t_Lt_R-r_Lr_R|. \tag{9}$$

For this we propose two semi-infinite models. The first one has a sharp semi-infinite potential step as the real part and a Dirac delta function as the imaginary part:

$$V(x)=V_1\Theta(x)+iV_2\delta(x),\quad \Theta(x<0)=0,\quad \Theta(x>0)=1. \tag{10}$$

The other one has its real part as a defused Fermi step and the imaginary part as $\text{sech}^2 x$:

$$V(x) = \frac{V_1}{2}\left[1 + \tanh\left(\frac{x}{2a}\right)\right] + iV_2\text{sech}^2\left(\frac{x}{2a}\right). \quad (11)$$

In (10) and (11), V_1 is essentially real and V_2 may be nonreal such that $\mathcal{R}(V_2) > 0$. The Hermitian version of the potential (11) is well known as the Eckart [22] or Rosen Morse [23] potential, which are known to be exactly solvable. More recently a complex *PT*-symmetric version of the Eckart or Rosen-Morse potential, $V(x) = A\text{sech}^2 x + iB\tanh x$ $(A,B$ are real), has been studied for scattering [24]. However, here we utilize it as an essentially non-*PT*-symmetric complex non-Hermitian potential.

For the scattering from left for (10) we take $\psi(x<0) = A_L\exp(ik_L x) + B_L\exp(-ik_L x), \psi(x>0) = A_R\exp(ik_R x) + B_R\exp(-ik_R x)$. By matching the solutions at $x = 0$ and mismatching their first derivative at $x = 0$ due to the presence of the Dirac delta potential, we obtain

$$r_L = \frac{B_L}{A_L} = \frac{k_L - k_R + u}{k_L + k_R - u}, \quad t_L = \frac{A_R}{A_L} = \frac{2k_L}{k_L + k_R - u} \quad (12)$$

and

$$r_R = \frac{A_R}{B_R} = \frac{k_R - k_L + u}{k_R + k_L - u}, \quad t_R = \frac{B_L}{B_R} = \frac{2k_R}{k_R + k_L - u}, \quad (13)$$

where $u = 2\mu V_2/\hbar^2$. Reflection probabilities are obtained as $R_L = |r_L|^2$ and $R_R = |r_R|^2$ but transmission probabilities for semi-infinite potentials when $E > V_1$ are found as

$$T_L = \frac{k_R}{k_L}|t_L|^2, \quad T_R = \frac{k_L}{k_R}|t_R|^2 \Rightarrow T_L = T_R. \quad (14)$$

One can readily see that $k_L + k_R = u$ is the condition of spectral singularity at which

$$|\det(S(k_L,k_R))| = \left|\frac{u + (k_L + k_R)}{u - (k_L + k_R)}\right| = \infty,$$
$$|\det(S(-k_L,-k_R))| = 0; \quad (15)$$

the modulus of the determinant of the time-reversed S matrix vanishes; and the real energy turns out to be

$$E_* = \left(\frac{U^2 + V_1}{2U}\right)^2, \quad U = \frac{\sqrt{2\mu}}{\hbar}V_2, \quad U^2 > V_1. \quad (16)$$

But when $U^2 < V_1$ (i.e., $k_L - k_R = u$)

$$E_i = \left(\frac{U^2 + V_1}{2U}\right)^2, \quad r_R(E_i) = 0, \quad t_R(E_i) = 1. \quad (17)$$

Equation (17) does give a scope for right invisibility of (10) yet it is belied by noting that the transmission probability for right incidence is $T(E_i) = k_L/k_R \neq 1$. So the potential (10) becomes only right-reflectionless at $E = E_i$. Notice that parametric conditions (16) and (17) of SS and reflectionlessness are mutually exclusive. CPA can occur even if $V_1 = 0$, which means that the imaginary Dirac delta potential alone (with $V_2 > 0$) is the simplest model of CPA. Interestingly, here it turns out that the presence of the semi-infinite step potential does not hamper CPA. We claim that the model (10) is the second simplest model of CPA so far. It also has pedagogic advantage. However, the semi-infiniteness of the real part adds novelty in the phenomenon of CPA. The analytical demonstration of SS and CPA in (10) has been carried out in Eqs. (12)–(17). For pictorial demonstration, taking $2\mu = 1 = \hbar^2$ and $V_1 = 3, V_2 = 3$, we present $T(E), R_L(E)$, and $R_R(E)$ to show spectral singularity at $E = E_* = 4$ and the determinant of the two port time-reversed S matrix vanishing at $E = E_* = 4$: $|\det(S(-k_L,-k_R))| = 0$ (see Fig. 2).

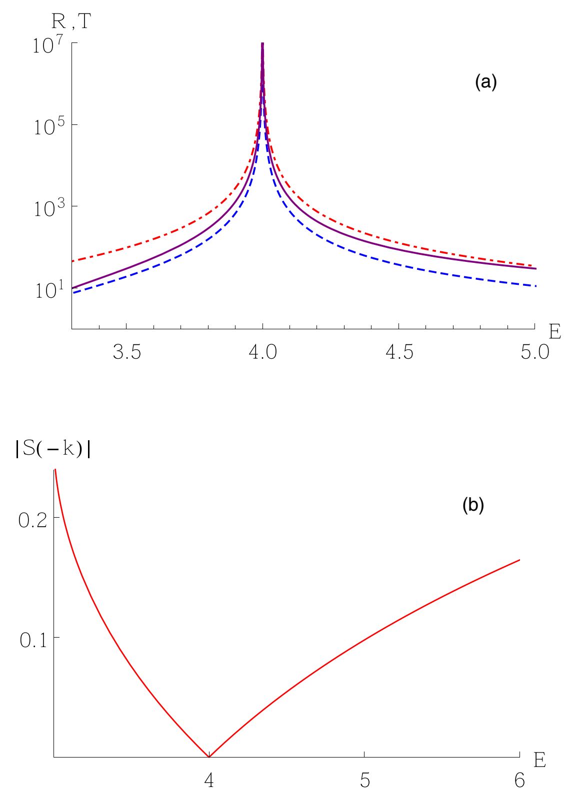

FIG. 2. (a) $T(E), R_L(E)$, and $R_R(E)$ displaying spectral singularity at $E = 4$, for the model (10) of semi-infinite non-Hermitian potential. (b) $|\det(S(-k_L,-k_R))|$ passing sharply through zero at $E = 4$. Here, $2\mu = 1 = \hbar^2, V_1 = 3, V_2 = 3$.

Next, we consider the potential profile (11) in (1), using the standard transformation [22,23]

$$y = \frac{1}{2}\left[1 - \tanh\left(\frac{x}{2a}\right)\right], \quad \psi(x) = y^{-i\beta}(1-y)^{i\alpha}G(y), \quad (18)$$

and we can reduce (1) for (11) in terms of Gauss hypergeometric form

$$y(1-y)\frac{d^2G}{dx^2} + [c - (a+b+1)y]\frac{dG}{dx} - abG = 0, \quad (19)$$

where $a = 1/2 + i(\alpha - \beta + \gamma), b = 1/2 + i(\alpha - \beta - \gamma), c = 1 - 2i\beta$ and $\alpha = \sqrt{\frac{E}{\Delta}} = k_L a, \beta = \sqrt{\frac{E-V_1}{\Delta}} = k_R a, \gamma = \sqrt{\frac{4iV_2}{\Delta} - \frac{1}{4}}, \Delta = \frac{\hbar^2}{2\mu a^2}$. This equation has two linearly independent solutions $G_1 = {}_2F_1(a,b,c,y)$ and $G_2 = y_2^{1-c}F_1(1+a-c,1+b-c,2-c,y)$. When $x \to \infty$, $y \to e^{-x/a}, 1 - y \to 1$. Also when $x \to \infty, {}_2F_1(a,b,c,0) = 1$ and ${}_2F_1(1+a-c,1+b-c,2-c,0) = 1$. The solutions G_1

and G_2 give $\psi \sim e^{ik_R x}$ and $\sim e^{-ik_R x}$. We choose the first one as it represents a transmitted wave traveling from left to right, and finally we seek the solution of (1) for (11) as

$$\psi(x) = N y^{-i\beta}(1-y)^{i\alpha}{}_2F_1(a,b,c,y) \sim N e^{ik_R x}. \tag{20}$$

Using an identity of hypergeometric functions as

$$\begin{aligned}{}_2F_1(a,b,c,y) = &\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}{}_2F_1(a,b,a+b-c+1,1-y)\\ &+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-y)^{c-a-b}{}_2F_1(c-a,c-b,c-a-b+1,1-y),\end{aligned} \tag{21}$$

when $x \to -\infty, y \to 1$, the solution (20) is capable of representing a linear combination of incident (traveling from left to right) and reflected (traveling right to left) waves as $x \to -\infty$:

$$\psi(x) \sim \frac{N'\Gamma(1-2i\beta)\Gamma(-2i\alpha)e^{ik_L x}}{\Gamma(1/2-i(\alpha+\beta+\gamma))\Gamma(1/2-i(\alpha+\beta-\gamma))} + \frac{N'\Gamma(1-2i\beta)\Gamma(2i\alpha)e^{-ik_L x}}{\Gamma(1/2+i(\alpha-\beta-\gamma))\Gamma(1/2+i(\alpha-\beta+\gamma))}. \tag{22}$$

Equations (20) and (22) help us to get the reflection and transmission amplitudes for incidence from the left as

$$\begin{aligned} r(\alpha,\beta) &= \frac{\Gamma(2i\alpha)\Gamma(1/2-i(\alpha+\beta+\gamma))\Gamma(1/2-i(\alpha+\beta-\gamma))}{\Gamma(-2i\alpha)\Gamma(1/2+i(\alpha-\beta-\gamma))\Gamma(1/2+i(\alpha-\beta+\gamma))},\\ t(\alpha,\beta) &= \frac{\Gamma(1/2-i(\alpha+\beta+\gamma))\Gamma(1/2-i(\alpha+\beta-\gamma))}{\Gamma(1-2i\beta)\Gamma(-2i\alpha)}.\end{aligned} \tag{23}$$

Poles or zeros at $\alpha = \pm in$ or $\beta = -i(n+1)/2$ in (23), where $n = 0,1,2,3,\ldots$ are unphysical. For the incidence from the right, similarly we obtain

$$\begin{aligned} r_L(k_L,k_R) &= r(\alpha,\beta) \text{ and } t_L(k_L,k_R) = t(\alpha,\beta),\\ r_R(k_L,k_R) &= r(\beta,\alpha) \text{ and } t_R(k_L,k_R) = t(\beta,\alpha),\end{aligned} \tag{24}$$

leading to nonreciprocity of transmission amplitudes (8):

$$t_L k_R = t_R k_L. \tag{25}$$

Letting $\gamma = q + is, q,s \in \mathcal{R}$, the second term in the numerator of r_L and t_L becomes $\Gamma[1/2 - s - i(\alpha+\beta-q)]$, and by demanding

$$1/2 - s = -n \quad \alpha+\beta = q, \quad n = 0,1,2,3,\ldots \tag{26}$$

we get the real

$$\begin{aligned} E_* &= \left(\frac{W^2+V_1}{2W}\right)^2,\\ V_2 &= \frac{\Delta}{4}\{(2n+1)q + i[n(n+1)-q^2]\}, \quad n \in I^+, \quad W = q\sqrt{\Delta}, W^2 > V_1,\end{aligned} \tag{27}$$

the energy of spectral singularity. Using (23) and (24), we can write $r_L(-k_L,-k_R)r_R(-k_L,-k_R) = X(\alpha,\beta)/Y(\alpha,\beta)$ and $t_L(-k_L,-k_R)t_R(-k_L,-k_R) = X(\alpha,\beta)/Z(\alpha,\beta)$ where

$$X(\alpha,\beta) = \frac{(\Gamma(1/2+i(\alpha+\beta+\gamma))\Gamma(1/2+i(\alpha+\beta-\gamma)))^2}{\Gamma(2i\alpha)\Gamma(2i\beta)}, \tag{28}$$

$$Y(\alpha,\beta) = \frac{\Gamma(1/2-i(\alpha-\beta+\gamma))\Gamma(1/2-i(\alpha-\beta-\gamma))}{\Gamma(-2i\alpha)}\frac{\Gamma(1/2+i(\alpha-\beta+\gamma))\Gamma(1/2+i(\alpha-\beta-\gamma))}{\Gamma(-2i\beta)}, \tag{29}$$

$$Z(\alpha,\beta) = \Gamma(1+2i\alpha)\Gamma(1+2i\beta). \tag{30}$$

Under the condition of spectral singularity (26), $X(\alpha,\beta)$ remains a nonzero and finite multiplicative factor in $\det(S(-k_L,-k_R)) = X(1/Y - 1/Z)$, where

$$Y^{-1} = \frac{\Gamma(-2i\alpha)\Gamma(-2i\beta)}{\Gamma(1+n-2i\alpha)\Gamma(-n+2i\alpha)\Gamma(1+n-2i\beta)\Gamma(-n+2i\beta)}, \tag{31}$$

$$Y^{-1} = -\frac{1}{\pi^2}\sinh 2\pi\alpha \sinh 2\pi\beta\,\Gamma(-2i\alpha)\Gamma(-2i\beta) = Z^{-1}, \tag{32}$$

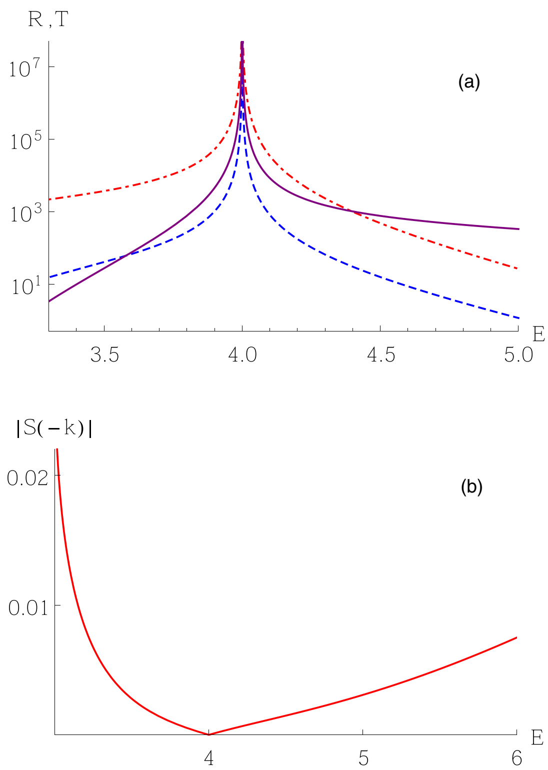

FIG. 3. The same as in Fig. 2 for the non-Hermitian version of Eckart potential profile (11). Here, $2\mu = 1 = \hbar^2$, $V_1 = 3; q = 3, n = 1, E_* = 4$ [see Eq. (27)].

leading to $\det(S(-k_L,-k_R)) = 0$ at $E = E_*$ (26), which is the analytic demonstration of the phenomenon of CPA in (11). For simplifications in (31) to the form (32), we have made multiple use of a property of Gamma functions expressed as $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin \pi z}$ [19].

Next, when $W^2 < V_1$, i.e., $\alpha - \beta = q, s = n + 1/2, V_2$ [as in Eq. (27)],

$$E_z = \left(\frac{W^2 + V_1}{2W}\right)^2, \quad r_R(E_z) = 0, \quad t_{L,R}(E_z) \neq 1. \quad (33)$$

Further, when $W^2 < V_1$ and $n = 0$ [$V_2 = \frac{\Delta}{4}(q - iq^2)$],

$$E_i = \left(\frac{W^2 + V_1}{2W}\right)^2, \quad r_R(E_i) = 0, \quad t_R(E_i) = 1. \quad (34)$$

It can be readily checked that when $s = n + 1/2$ and $\alpha - \beta = q$ the argument of the Gamma function in the denominator of $r_R = r(\beta,\alpha)$ in (23) and (24) becomes a negative integer, as $\Gamma[-n] = \infty, n = 0,1,2,3,\ldots$, and we get $r_R(E_i) = 0$ but $t_R(E_i) = \frac{\Gamma(n+1-2i\alpha)\Gamma(-n-2i\beta)}{\Gamma(1-2i\alpha)\Gamma(-2i\beta)}$, which is 1 only when $n = 0$. However, the result that $T_R(E_i) = k_L/k_R$ (probability of transmission) for incidence from the right would prevent even the right invisibility [14]. So the potential (11) is eventually right-reflectionless where conditions (33) and (34) are met. Apart from analytic demonstration of SS, CPA, and reflectionlessness in the potential (11) through Eqs. (23)–(34), in Fig. 3, we present a pictorial demonstration when $V_1 = 3, q = 3, n = 1$, yielding $E_* = 4$ (26).

The two-port S matrix for coherent scattering derived here [Eq. (7)] is more general than the one discussed previously [8,11,16,18]. Equations (3)–(7) are useful for analytically intractable profiles and our conclusions are also based on other profiles such as $V(x) = V_1[1 + \mathrm{erf}(x/a)] + iV_2 e^{-x^2/a^2}$. We would like to remark that, for energies $E < V_1$, $R_L(E)$ or $R_R(E)$ may alone [not $T(E)$] have singularities which cannot be admitted as spectral singularity. Some of these unphysical poles have been mentioned [Eq. (23)].

We conclude that the semi-infiniteness of the real part and the nonreciprocity of transmission amplitudes do not hamper the interesting critical phenomena of spectral singularity, coherent perfect absorption, and one-sided reflectionlessness: They occur yet again. All of these occur at energies $E > V_1$ and when $\mathcal{R}(V_2) > 0$. The first two phenomena require the strength of the imaginary part of the potential to be larger. We find that one-sided reflectionlessness can occur for lesser values of $\mathcal{R}(V_2) > 0$. Very interestingly, the invisibility gets ruled out despite $r(E_i) = 0$ and $t(E_i) = 1$ [but $T(E_i) \neq 1$] on one side of these semi-infinite models (10) and (11). It may be remarked that the (non)occurrence of invisibility is cumbersome and difficult to detect as experienced in Ref. [15]. However, here this could be done easily. Let us label the situation of $r(E_z) = 0$ and $T(E_z) \neq 1$ on one side as *one-sided reflectionlessness*. We find that existences of one-sided reflectionlessness and spectral singularity are mutually exclusive for a fixed semi-infinite non-Hermitian potential. This is another distinctive feature of the semi-infinite medium. The similarity of the results in Eqs. (16) and (27) for two potentials (10) and (11) is tantalizing.

Investigations of coherent injection at non-Hermitian mediums have been giving interesting surprises and revealing novel phenomena in the recent past. We hope that the two proposed exactly solvable models and their explicit results, which are surprisingly simple, will be found useful in both theory and experiments.

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