Variance uncertainty relations without covariances for three and four observables

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Sum and product uncertainty relations, containing variances of three or four observables, but not containing explicitly their covariances, are derived. Their consequences are, in particular, inequalities, giving nonzero lower bounds for the products of two variances in the case of zero mean value of the commutator between the related operators. Explicit examples show that the bounds can be better than the known Robertson-Schrödinger one.

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I. INTRODUCTION

It is impressive that 90 years after the birth of the concept of uncertainty relations in quantum mechanics [1,2], this subject is still "alive," in the sense that one can observe a burst of publications in this area, devoted to generalizations of the traditional product inequalities for two observables [3,4]

$$\sigma_{AA}\sigma_{BB} \geqslant \tfrac{1}{4}|\langle[\hat{A},\hat{B}]\rangle|^2 \tag{1}$$

or its stronger version [4,5]

$$\sigma_{AA}\sigma_{BB} \geqslant \sigma_{AB}^2 + \tfrac{1}{4}|\langle[\hat{A},\hat{B}]\rangle|^2. \tag{2}$$

Here, $\sigma_{AB} \equiv \frac{1}{2}\langle\{\delta\hat{A},\delta\hat{B}\}\rangle$ and $\delta\hat{A} \equiv \hat{A} - \langle\hat{A}\rangle$. Although the mainstream of the current research is connected to the "entropic uncertainty relations" (see [6–9] for recent reviews and results), several inequalities containing *variances* of observables as measures of "uncertainties" have been discovered recently [10–23]. The goal of this article is to provide families of relatively simple inequalities, containing on an equal footing variances of three and four observables, but not containing explicitly any covariance (the specific choice of numbers 3 and 4 will become clear soon). Remarkable consequences are inequalities for the product of two variances, replacing inequality (1) in the case of zero mean value of the commutator $[\hat{A},\hat{B}]$.

Indeed, what should or can we do if $\langle[\hat{A},\hat{B}]\rangle = 0$? One of possible answers arises, if one considers the triple of angular momentum operators L_x, L_y, L_z. In this case, relation (1) assumes the form

$$L_{xx}L_{yy} \geqslant (\hbar^2/4)L_z^2. \tag{3}$$

The following notation is used hereafter for operators labeled with indices:

$$z_j \equiv \langle\hat{z}_j\rangle, \quad z_{jk} = z_{kj} = \tfrac{1}{2}\langle\hat{z}_j\hat{z}_k + \hat{z}_k\hat{z}_j\rangle - \langle\hat{z}_j\rangle\langle\hat{z}_k\rangle. \tag{4}$$

If $L_z = 0$ (and this can happen for many quantum states), then relation (3) gives no information about the variances L_{xx} and L_{yy}. Looking at relation (3), one can suppose that its insufficient efficiency could be explained, at least partially, by the fact that three equivalent noncommuting operators $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$ enter this relation on an unequal footing. Therefore, a natural way to overcome the difficulty can consist in generalizing inequalities (1) or (2) to systems of *more than two operators*. This problem was considered for the first time by Robertson [24]. [Another possibility, discovered by Trifonov [25,26] and Maccone and Pati [12] (see also more recent papers [14,17,18]), is to use inequalities containing *more than one quantum state*. But, I do not follow this way here.]

Robertson's scheme is as follows. Consider N arbitrary operators $\hat{z}_1$, $\hat{z}_2$, ..., $\hat{z}_N$, and construct the operator $\hat{f} = \sum_{j=1}^N \alpha_j\delta\hat{z}_j$, where α_j are arbitrary complex numbers. The inequalities, which can be interpreted as generalized uncertainty relations, are the consequences of the fundamental inequality $\langle\hat{f}^\dagger\hat{f}\rangle \geqslant 0$, that must be satisfied for any pure or mixed quantum state (the symbol $\hat{f}^\dagger$ means the Hermitian conjugated operator). In the explicit form, this inequality is the condition of positive semidefiniteness of the quadratic form $\alpha_j^* F_{jm}\alpha_m$ (with the summation over repeated indices), whose coefficients $F_{jm} = \langle\delta\hat{z}_j^\dagger\delta\hat{z}_m\rangle$ form the Hermitian matrix $F = \|F_{jm}\|$. One has only to use the known conditions of the positive semidefiniteness of Hermitian matrices to write the explicit inequalities for the elements of matrix F. All such inequalities can be considered as generalizations of inequality (2) to the case of more than two operators. Many of them can be found, e.g., in review [27] or Refs. [20,23,26,28–33]. Applications to the problem of entanglement of continuous variable systems were studied in Refs. [16,20,34–38].

If all operators $\hat{z}_j$ are Hermitian, then it is convenient to split matrix F as $F = X + iY$, where X and Y are real symmetric and antisymmetric matrices, consisting of the elements $X_{mn} = \frac{1}{2}\langle\{\delta\hat{z}_m,\delta\hat{z}_n\}\rangle$ and $Y_{mn} = \frac{1}{2i}\langle[\hat{z}_m,\hat{z}_n]\rangle$. The symbols $\{\ldots,\ldots\}$ and $[\ldots,\ldots]$ mean the anticommutator and commutator, respectively. The fundamental inequality ensuring the positive semidefiniteness of matrix F is

$$\det F = \det\|X + iY\| \geqslant 0. \tag{5}$$

Other inequalities proven by Robertson have the form

$$X_{11}X_{22}\ldots X_{NN} \geqslant \det X \geqslant \det Y. \tag{6}$$

Unfortunately, inequalities (5) and (6) are rather complicated for $N > 2$ observables because they contain, in addition to N variances X_{kk} and $N(N-1)/2$ mean values of commutators

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Y_{jk}, numerous sums and products of various combinations of $N(N-1)/2$ covariances X_{jk} with $j \neq k$.

The presence of covariances in the uncertainty relations is important, both from the point of view of mathematical beauty and completeness [the invariance with respect to arbitrary linear canonical transformations in the case of inequality (2) for the coordinates and momenta] [39], and from the point of view of physical effects, such as tunneling of wave packets through potential barriers [40], quantum optical measurements [41], or low-energy nuclear reactions [42]. An inclusion of covariances into the entropic uncertainty relations was achieved recently in study [9]. Also, the account of covariances between the selected subsystem and the "external world" increases the right-hand side of the Robertson-Schrödinger inequality (2) [43].

However, the analysis of complete inequalities, containing all covariances, is a difficult task for $N > 2$ observables. For example, if $N = 4$, then det X contains 17 different products of covariances [27,44], in addition to 6 different products of mean values of commutators in det Y. The simplest expressions exist for $N = 3$, when (5) can be written in the form (see, e.g., [19,45])

$$X_{11}X_{22}X_{33} \geqslant X_{11}\left(X_{23}^2 + Y_{23}^2\right) + X_{22}\left(X_{13}^2 + Y_{13}^2\right) + X_{33}\left(X_{12}^2 + Y_{12}^2\right) - 2X_{12}X_{23}X_{31} + 2(X_{12}Y_{23}Y_{31} + X_{23}Y_{31}Y_{12} + X_{31}Y_{12}Y_{23}). \tag{7}$$

In contradistinction to the case of the Schrödinger inequality (2), where removing the term σ_{AB} from the right-hand side results in a simplified inequality (1), there is no possibility to simplify (7) by deleting all terms X_{jk} with $j \neq k$ since covariances X_{jk} with $j \neq k$ can be positive or negative. If such a simple trick could be done, then one would obtain the inequality

$$X_{11}X_{22}X_{33} \geqslant X_{11}Y_{23}^2 + X_{22}Y_{13}^2 + X_{33}Y_{12}^2. \tag{8}$$

But, it is not satisfied, e.g., for the triple of dimensionless (scaled) operators (introduced in [13]) x, p, and $\xi = x + p$, in the correlated coherent state [46]

$$\psi_\alpha(x;\sigma,r) = \mathcal{N} \exp\left[-\frac{x^2}{4\sigma}\left(1 - \frac{ir}{\sqrt{1-r^2}}\right) + \frac{\alpha x}{\sqrt{\sigma}}\right] \tag{9}$$

with $\sigma = \hbar/\sqrt{3}$ and the correlation coefficient $r = -\frac{1}{2}$ (here $\mathcal{N}$ is the normalization factor). Indeed, in this case

$$Y_{jk}^2 = \hbar^2/4, \quad \sigma_{xx} = \sigma_{pp} = \sigma_{\xi\xi} = \hbar/\sqrt{3}, \tag{10}$$

so that the left-hand side of (8) equals $L = \hbar^3/(3\sqrt{3})$, while the right-hand side equals $R = \hbar^3\sqrt{3}/4 = 9L/4$. Consequently, inequality (8) is wrong.

The correct simplified form of inequality (7) for the product of three variances is derived in Sec. II together with several other relations for the sums and products of uncertainties. Similar inequalities for four variances are derived in Sec. III.

II. INEQUALITIES WITHOUT EXPLICIT COVARIANCES FOR $N = 3$

The correct simplified inequality without covariances can be obtained using the scheme proposed in [47]. The main idea is to extend the Hilbert space of states $|\psi\rangle$, considering the tensor products $|\Psi\rangle = |\psi\rangle \otimes |\chi\rangle$, where $|\chi\rangle$ is an auxiliary spinor. In this extended space we can introduce the operator $\hat{F} = \sum_{j=1}^{3} \alpha_j \sigma_j \delta\hat{z}_j$, where α_j are arbitrary *real* numbers and σ_j are the standard 2×2 Pauli matrices. Then, using the anticommutativity property of the Pauli matrices and performing averaging over the state $|\psi\rangle$, one can write $\langle\Psi|\hat{F}^\dagger\hat{F}|\Psi\rangle = \langle\chi|\mathcal{A}|\chi\rangle$ with the 2×2 Hermitian matrix (here σ_0 is the 2×2 unit matrix)

$$\mathcal{A} = \left(\alpha_1^2 X_{11} + \alpha_2^2 X_{22} + \alpha_3^2 X_{33}\right)\sigma_0 - 2\sigma_1\alpha_2\alpha_3 Y_{23} - 2\sigma_2\alpha_3\alpha_1 Y_{31} - 2\sigma_3\alpha_1\alpha_2 Y_{12}. \tag{11}$$

We see that matrix (11) does not contain covariances X_{jk} with $j \neq k$. Since $\langle\Psi|\hat{F}^\dagger\hat{F}|\Psi\rangle \geqslant 0$ for any physical state, matrix (11) must be positive semidefinite for arbitrary real parameters $\alpha_1,\alpha_2,\alpha_3$. Unfortunately, the analysis of this condition in Ref. [47] suffered from some drawbacks because the main result of that paper was the incorrect inequality (8). The origin of the mistake is shown in Appendix A.

It is known from the matrix theory that there are three conditions that guarantee the positive semidefiniteness of 2×2 Hermitian matrices: the non-negativity of two diagonal elements and the matrix determinant. The first two conditions result in the Robertson inequality (1) $X_{11}X_{22} \geqslant Y_{12}^2$, whereas the condition det $\mathcal{A} \geqslant 0$ results in the inequality

$$\alpha_1^2 X_{11} + \alpha_2^2 X_{22} + \alpha_3^2 X_{33} \geqslant 2[(\alpha_1\alpha_2 Y_{12})^2 + (\alpha_2\alpha_3 Y_{23})^2 + (\alpha_1\alpha_3 Y_{13})^2]^{1/2}. \tag{12}$$

This inequality must hold for *arbitrary real numbers* α_1, α_2, and α_3. Each set of these parameters yields some kind of uncertainty relations for three observables. In the following subsections, we consider several interesting special cases.

A. Most symmetric inequalities

Looking for the most symmetric relations, let us choose $\alpha_1 = \alpha_2 = \alpha_3$. Then, we arrive at the inequality

$$X_{11} + X_{22} + X_{33} \geqslant 2\left[Y_{12}^2 + Y_{23}^2 + Y_{13}^2\right]^{1/2}, \tag{13}$$

which is *stronger* than the consequence of the Robertson inequality (1):

$$X_{11} + X_{22} + X_{33} \geqslant |Y_{12}| + |Y_{23}| + |Y_{13}|. \tag{14}$$

In the special case of three canonical observables, x, p, and $\xi = x + p$, inequality (13) was found in [13]. Its generalization to the case of N observables, which are *linear combinations* of the canonical coordinate and momentum operators, was found in [23]. Other inequalities for traces of the covariance matrix for N observables can be found in [15,22,26,27].

The choice $\alpha_k^2 = X_{kk}^n$ results in the inequality

$$X_{11}^{n+1} + X_{22}^{n+1} + X_{33}^{n+1} \geqslant 2\left[Y_{12}^2 X_{11}^n X_{22}^n + Y_{23}^2 X_{33}^n X_{22}^n + Y_{13}^2 X_{11}^n X_{33}^n\right]^{1/2}. \tag{15}$$

Of course, to use inequalities (13) or (15) one must preliminarily rescale observables z_k in such a way that all of them acquire the same physical dimensions.

Wishing to find an inequality for the triple product $X_{11}X_{22}X_{33}$, let us choose $\alpha_1^2 = X_{22}X_{33}$, $\alpha_2^2 = X_{11}X_{33}$, and

$\alpha_3^2 = X_{22}X_{11}$. Then, the following correct inequality arises instead of (8):

$$X_{11}X_{22}X_{33} \geqslant \tfrac{4}{9}\left(X_{11}Y_{23}^2 + X_{22}Y_{13}^2 + X_{33}Y_{12}^2\right). \tag{16}$$

It turns into the equality for the state (9) with $\sigma = \hbar/\sqrt{3}$ and $r = -\frac{1}{2}$. A similar (but weaker) inequality, with coefficient $\frac{1}{3}$ instead of correct $\frac{4}{9}$, was derived in [45].

If three commutator mean values coincide, $|Y_{12}| = |Y_{23}| = |Y_{31}| = Y$, then combining (16) with (13) one obtains the inequality

$$X_{11}X_{22}X_{33} \geqslant (4/3)^{3/2}Y^3. \tag{17}$$

For the triple $(x,p,x+p)$ inequality (17) yields immediately the main result of Ref. [13].

Applying the inequality $a + b \geqslant 2\sqrt{ab}$ to the right-hand side of (16), we get the inequality $\zeta^2 - 2B\zeta - \frac{4}{9}Y_{12}^2 \geqslant 0$, where $\zeta \equiv \sqrt{X_{11}X_{22}}$ and $B \equiv 4|Y_{13}Y_{23}|/(9X_{33})$. Resolving this inequality with respect to the non-negative variable ζ, we arrive at the following inequality for the uncertainty product $\Delta z_1\Delta z_2 \equiv \sqrt{X_{11}X_{22}}$:

$$\Delta z_1\Delta z_2 \geqslant \sqrt{(2Y_{12}/3)^2 + B^2} + B. \tag{18}$$

This inequality is especially important if $Y_{12} = 0$, when the standard Robertson uncertainty relation (1) $\Delta z_1\Delta z_2 \geqslant |Y_{12}|$ becomes useless. In this case, a better inequality is

$$\Delta z_1\Delta z_2 \geqslant 8|Y_{13}Y_{23}|/(9X_{33}). \tag{19}$$

But, the coefficient 8/9 in this relation can be improved, as shown at the end of the next subsection.

B. Partially symmetric inequalities

Maintaining the symmetry with respect to the variables z_1 and z_2 only, let us choose $\alpha_1 = \alpha_2$, but $\alpha_3 = b\alpha_1$ in the basic inequality (12). Then, one gets instead of (13) a more general inequality

$$X_{11} + X_{22} + b^2X_{33} \geqslant 2\left[Y_{12}^2 + b^2\left(Y_{23}^2 + Y_{13}^2\right)\right]^{1/2}. \tag{20}$$

It seems reasonable to try to find the "best" value of parameter b, that would result in the strongest inequality. To do this, let us rewrite (20) in the form $f(b) \equiv C + b^2X_{33} - 2[Y_{12}^2 + b^2D]^{1/2} \geqslant 0$, with $C = X_{11} + X_{22}$ and $D = Y_{23}^2 + Y_{13}^2$, and try to find the minimum of function $f(b)$ for fixed non-negative coefficients C and D. The local extremum at $b = 0$ yields the inequality $X_{11} + X_{22} \geqslant 2|Y_{12}|$, which is the consequence of the Robertson relation $X_{11}X_{22} \geqslant Y_{12}^2$. The problem is that the second extremal point b_*, given by the formula

$$b_*^2 = \frac{Y_{23}^2 + Y_{13}^2}{X_{33}^2} - \frac{Y_{12}^2}{Y_{23}^2 + Y_{13}^2}, \tag{21}$$

exists under the restriction

$$\left(Y_{23}^2 + Y_{13}^2\right)^2 > Y_{12}^2X_{33}^2, \tag{22}$$

which cannot be satisfied for arbitrary values of the variance X_{33} and mean values of commutators Y_{jk}. If condition (22) is satisfied, then inequality $f(b_*) \geqslant 0$ can be written as some kind of the "sum uncertainty relation"

$$X_{11} + X_{22} \geqslant \frac{Y_{23}^2 + Y_{13}^2}{X_{33}} + \frac{Y_{12}^2X_{33}}{Y_{23}^2 + Y_{13}^2}. \tag{23}$$

Note that the right-hand side of inequality (23) is bigger than $2|Y_{12}|$ under condition (22). If condition (22) is not satisfied, then $df/db > 0$ for all values of $b > 0$. In this case, one can only say that $X_{11} + X_{22} \geqslant 2|Y_{12}|$.

In the case of equal mean values of three commutators, $|Y_{12}| = |Y_{23}| = |Y_{31}| = Y$, condition (22) reads as $X_{33} < 2|Y|$. Then inequality (23) assumes the form

$$X_{11} + X_{22} \geqslant \frac{2Y^2}{X_{33}} + \frac{1}{2}X_{33}. \tag{24}$$

The equality sign is achieved, e.g., for the Kechrimparis-Weigert triple $(x,p,x+p)$ in the correlated coherent state (9) with the variances given by Eq. (10).

To generalize inequality (16) for the triple product $X_{11}X_{22}X_{33}$, let us choose now $\alpha_1^2 = X_{22}X_{33}$, $\alpha_2^2 = X_{11}X_{33}$, but $\alpha_3^2 = b^2X_{22}X_{11}$ in inequality (12). Then,

$$X_{11}X_{22}X_{33} \geqslant \frac{4(C + b^2D)}{(2 + b^2)^2}, \tag{25}$$

where $C = X_{33}Y_{12}^2$ and $D = X_{11}Y_{23}^2 + X_{22}Y_{13}^2$. Note once again that inequality (25) must hold for *any real value* of parameter b. For $b = 0$ we have the standard inequality $X_{11}X_{22} \geqslant Y_{12}^2$. The right-hand side of (25) decreases with b, if $D \leqslant C$. In this case, the standard inequality is the best one. But, if $D \geqslant C$, i.e.,

$$X_{11}Y_{23}^2 + X_{22}Y_{13}^2 \geqslant X_{33}Y_{12}^2, \tag{26}$$

then the right-hand side of (25) attains the maximal value at

$$b_*^2 = \frac{2\left(X_{11}Y_{23}^2 + X_{22}Y_{13}^2 - X_{33}Y_{12}^2\right)}{X_{11}Y_{23}^2 + X_{22}Y_{13}^2},$$

and we get a stronger inequality

$$X_{11}X_{22}X_{33} \geqslant \frac{\left(X_{11}Y_{23}^2 + X_{22}Y_{13}^2\right)^2}{2\left(X_{11}Y_{23}^2 + X_{22}Y_{13}^2\right) - X_{33}Y_{12}^2}. \tag{27}$$

The equality in this relation is attained again for the triple $(x,p,x+p)$ in the correlated coherent state (9) with the variances given by Eq. (10).

In the special case of $Y_{12} = 0$ [when condition (26) is certainly fulfilled] we have

$$X_{11}X_{22}X_{33} \geqslant \tfrac{1}{2}\left(X_{11}Y_{23}^2 + X_{22}Y_{13}^2\right) \geqslant |Y_{13}Y_{23}|\sqrt{X_{11}X_{22}}.$$

Thus, we arrive at the improved version of inequality (19):

$$\Delta z_1\Delta z_2 \geqslant |Y_{13}Y_{23}|/X_{33}. \tag{28}$$

C. Examples

1. Three components of the angular momentum

A natural example of three observables is the set of three components L_x, L_y, and L_z of the angular momentum vector $\mathbf{L}$. Inequality (13) reads in this case as

$$\langle \mathbf{L}^2\rangle - \langle \mathbf{L}\rangle^2 \geqslant \hbar|\langle \mathbf{L}\rangle| \equiv \hbar\sqrt{L_x^2 + L_y^2 + L_z^2}, \tag{29}$$

where $L_j \equiv \langle \hat{L}_j \rangle$. Inequality (16) reads as

$$L_{xx}L_{yy}L_{zz} \geqslant \hbar^2\left(L_{xx}L_x^2 + L_{yy}L_y^2 + L_{zz}L_z^2\right)/9, \quad (30)$$

where $L_{jj} \equiv \langle \hat{L}_j^2 \rangle - \langle \hat{L}_j \rangle^2$. If $L_z = 0$, then inequality (28) assumes the form

$$\Delta L_x \Delta L_y \geqslant \hbar^2 |L_x L_y/(4L_{zz})|. \quad (31)$$

Recently, many new uncertainty relations for the angular momentum operators were found in [30,48–50] (in addition to the set of inequalities collected in [27]). But, inequalities (29)–(31) seem to be new. To illustrate (31), we have considered various superpositions of the angular momentum p states $|1,m\rangle$ with $m = 1,0,-1$, resulting in $L_z = 0$. It appears that the minimal ratio of the left- and right-hand sides of (31) is achieved for the state

$$|\psi\rangle = \tfrac{1}{2}[|1,1\rangle + i|1,-1\rangle + (1+i)|1,0\rangle], \quad (32)$$

possessing the following mean values and variances:

$$L_x = L_y = \hbar/\sqrt{2}, \quad \Delta L_x = \Delta L_y = \hbar/2, \quad L_{zz} = \hbar^2/2.$$

Then, the right-hand side of (31) equals $\hbar^2/4$, and this value coincides with the value of the left-hand side. The same result in this special case can be deduced from the Schrödinger inequality (2) because $L_{xy} = -\hbar^2/4$ for the state (32), so that $\Delta L_x \Delta L_y = |L_{xy}|$.

2. Three independent products of canonical operators

Another interesting triple is

$$\hat{z}_1 = (\delta\hat{p})^2, \quad \hat{z}_2 = (\delta\hat{x})^2, \quad \hat{z}_3 = \tfrac{1}{2}(\delta\hat{p}\delta\hat{x} + \delta\hat{x}\delta\hat{p}). \quad (33)$$

The choice of *shifted* operators $\delta\hat{x} = \hat{x} - \langle\hat{x}\rangle$ and $\delta\hat{p} = \hat{p} - \langle\hat{p}\rangle$ implies that the non-negative average value $\langle\Psi|\hat{F}^\dagger\hat{F}|\Psi\rangle \geqslant 0$ should be calculated with respect to the same state $|\psi\rangle$ that was used for the calculation of the first-order mean values $\langle\hat{x}\rangle$ and $\langle\hat{p}\rangle$. But, this restriction does not influence the results. Now, we have the following expressions for the quantities Y_{kl} and X_{jj}:

$$Y_{12} = -2\hbar\sigma_{px}, \quad Y_{23} = \hbar\sigma_{xx}, \quad Y_{13} = -\hbar\sigma_{pp},$$
$$X_{11} = \langle(\delta p)^4\rangle - \sigma_{pp}^2 \equiv \sigma_{4p}, \quad X_{22} = \langle(\delta x)^4\rangle - \sigma_{xx}^2 \equiv \sigma_{4x},$$
$$X_{33} = \tfrac{1}{4}\langle(\delta\hat{p}\delta\hat{x} + \delta\hat{x}\delta\hat{p})^2\rangle - \sigma_{px}^2.$$

The simple Robertson inequality (1) $\sigma_{4p}\sigma_{4x} \geqslant 4\hbar^2\sigma_{px}^2$ becomes useless for states with $\sigma_{px} = 0$ (in particular, for any real wave function). In this case, it is better to use inequality (28), which assumes the form

$$\sqrt{\sigma_{4p}\sigma_{4x}} \geqslant 4\hbar^2\sigma_{pp}\sigma_{xx}/\langle(\delta\hat{p}\delta\hat{x} + \delta\hat{x}\delta\hat{p})^2\rangle. \quad (34)$$

The right-hand side of (34) is obviously nonzero. A simple illustration of the strength of inequality (34) can be done for arbitrary Gaussian states because all higher-order statistical moments can be expressed in terms of (co)variances for this kind of state (see Appendix B for details). In particular,

$$\sigma_{4p} = 2\sigma_{pp}^2, \quad \sigma_{4x} = 2\sigma_{xx}^2,$$
$$\langle(\delta\hat{p}\delta\hat{x} + \delta\hat{x}\delta\hat{p})^2\rangle = 4\sigma_{pp}\sigma_{xx} + 8(\sigma_{xp})^2 + \hbar^2.$$

Consequently, the left-hand side of (34) equals $2\sigma_{pp}\sigma_{xx}$. On the other hand, if $\sigma_{px} = 0$, then the right-hand side equals $4\hbar^2\sigma_{pp}\sigma_{xx}/(4\sigma_{pp}\sigma_{xx} + \hbar^2)$. For pure uncorrelated quantum Gaussian states we have $\sigma_{pp}\sigma_{xx} = \hbar^2/4$, so that (34) turns into an equality. Moreover, inequality (34) turns out better than the Schrödinger inequality (2) in this special case. Indeed, for any Gaussian state one has (see Appendix B)

$$X_{12} \equiv \tfrac{1}{2}\langle(\delta\hat{p})^2(\delta\hat{x})^2 + (\delta\hat{x})^2(\delta\hat{p})^2\rangle$$
$$= \sigma_{pp}\sigma_{xx} + 2\sigma_{px}^2 - \hbar^2/2,$$

so that the value $\sqrt{X_{11}X_{22}} = \hbar^2/2$ is *twice* bigger than $|X_{12}| = \hbar^2/4$ for all pure Gaussian states with $\sigma_{px} = 0$. Note that $X_{12} < 0$ in this case, due to the noncommutativity of the coordinate and momentum operators.

3. Symmetric triple of operators in the phase plane

Let us consider, following [16], the triple of operators, which are obtained by rotations in the phase plane (x-p) by 120°:

$$z_k = x\cos(2k\pi/3) + p\sin(2k\pi/3), \quad k = 0,1,2. \quad (35)$$

In this case, we have $Y_{01} = Y_{12} = Y_{20} = \sqrt{3}/4$ (in the dimensionless units with $\hbar = 1$), so that inequality (17) yields $X_{00}X_{11}X_{22} \geqslant 1/8$. This result was obtained in [16] in a more complicated way.

4. Bounds on the uncertainty product for commuting operators in specific quantum states

Inequalities (31) and (34) give lower bounds of the uncertainty products for noncommuting operators in the specific states possessing zero mean values of their commutators. Now, let us consider an example of *commuting* operators $\hat{z}_1$ and $\hat{z}_2$. The Robertson inequality (1) tells us that the product $\Delta z_1 \Delta z_2$ can be made as small as desired, if one has in mind *all admissible quantum states*. But, if the family of quantum states is *restricted* somehow, then it is better to replace (1) with inequality (28), which shows explicitly that the product $\Delta z_1 \Delta z_2$ can be limited from below. For example, let us take $z_1 = x, z_2 = y$ (two Cartesian coordinates), but $z_3 = p_x \pm p_y$. Then, $Y_{12} \equiv 0$, and two inequalities (19) (for two choices of z_3) lead to the inequality

$$\Delta x \Delta y \geqslant \hbar^2/[4(\sigma_{p_xp_x} + \sigma_{p_yp_y} - 2|\sigma_{p_xp_y}|)]. \quad (36)$$

On the other hand, the standard Heisenberg uncertainty relation yields

$$\Delta x \Delta y \geqslant \hbar^2/[4\sqrt{\sigma_{p_xp_x}\sigma_{p_yp_y}}]. \quad (37)$$

Consequently, inequality (36) provides a much higher lower bound for the product $\Delta x \Delta y$ than inequality (37), for states with strongly correlated momenta, when $\sigma_{p_xp_x} \approx \sigma_{p_yp_y}$ and $|\sigma_{p_xp_y}| \approx \sqrt{\sigma_{p_xp_x}\sigma_{p_yp_y}}$. The simplest example is the pure Gaussian quantum state described by the wave function

$$\psi(x,y) = \mathcal{N}\exp\left(-\frac{a}{2}x^2 - bxy - \frac{a}{2}y^2\right), \quad (38)$$

with real coefficients a and b, satisfying the restrictions $a > 0$ and $D \equiv a^2 - b^2 > 0$. $\mathcal{N}$ is the normalization factor. Then, $\Delta x = \Delta y = \sqrt{a/(2D)}$, $\sigma_{p_xp_x} = \sigma_{p_yp_y} = \hbar^2 a/2$, and $\sigma_{p_xp_y} = \hbar^2 b/2$. The product $\Delta x \Delta y = a/(2D)$ can be very big, if $|b|$ is close to a, but the right-hand side of (37) [which is equal to $(2a)^{-1}$] does not feel the presence of parameter b. On the contrary, the right-hand side of (36) equals $[4(a-|b|)]^{-1}$, and this value tends to $a/[2(a^2-b^2)]$ when $|b|\to a$.

III. FOUR OBSERVABLES

The scheme used in the preceding section can be generalized to sets of four arbitrary Hermitian operators, if one replaces three 2×2 Pauli's matrices σ_k with four anticommuting Hermitian 4×4 Dirac's matrices, satisfying the relations

$$\gamma_m\gamma_n+\gamma_n\gamma_m=2I_4\delta_{mn},\quad m,n=1,2,3,4 \tag{39}$$

where I_n is the $n\times n$ unit matrix. Consider the operator $\hat{f}=\sum_{k=1}^{4}\xi_k\delta\hat{z}_k\gamma_k$, where ξ_k are arbitrary *real* coefficients and $\hat{z}_k$ arbitrary Hermitian operators. It acts in the extended Hilbert space of states $|\Psi\rangle=|\psi\rangle\otimes|\chi\rangle$, where $|\chi\rangle$ is an auxiliary bispinor. Then, the condition $\langle\Psi|\hat{f}^\dagger\hat{f}|\Psi\rangle\geqslant 0$ can be written as the condition of positive semidefiniteness of the Hermitian 4×4 matrix

$$F=gI_4+i\sum_{j<k}\gamma_j\gamma_k y_{jk}, \tag{40}$$

where

$$g=\sum_{k=1}^{4}\xi_k^2X_{kk},\quad y_{jk}=2\xi_j\xi_kY_{jk}=-y_{kj}. \tag{41}$$

The covariances X_{jk} with $j\neq k$ go out due to the anticommutation relations (39).

Choosing matrices γ_k in the form

$$\gamma_k=\begin{Vmatrix}0 & \sigma_k\\ \sigma_k & 0\end{Vmatrix},\quad \gamma_4=\begin{Vmatrix}I_2 & 0\\ 0 & -I_2\end{Vmatrix}, \tag{42}$$

we have

$$\gamma_j\gamma_k=i\epsilon_{jkl}\begin{Vmatrix}\sigma_l & 0\\ 0 & \sigma_l\end{Vmatrix},\quad \gamma_k\gamma_4=\begin{Vmatrix}0 & -\sigma_k\\ \sigma_k & 0\end{Vmatrix}. \tag{43}$$

Here, $j,k,l=1,2,3$, and ϵ_{jkl} is totally antisymmetric tensor with $\epsilon_{123}=1$. Then, matrix (40) has the form

$$F=\begin{Vmatrix}g-y_{12} & y_{32}-iy_{13} & -iy_{34} & y_{42}-iy_{14}\\ y_{32}+iy_{13} & g+y_{12} & y_{24}-iy_{14} & iy_{34}\\ iy_{34} & y_{24}+iy_{14} & g-y_{12} & y_{32}-iy_{13}\\ y_{42}+iy_{14} & -iy_{34} & y_{32}+iy_{13} & g+y_{12}\end{Vmatrix}. \tag{44}$$

The non-negativeness of diagonal elements of this matrix, $g\geqslant|y_{12}|$, is the consequence of the Robertson inequality $X_{11}X_{22}\geqslant|Y_{12}|$. The non-negativeness of principal minors of the second order results in four inequalities, whose typical form is (other ones are obtained by means of changes of indices)

$$g^2\geqslant 4[(\xi_1\xi_2Y_{12})^2+(\xi_1\xi_3Y_{13})^2+(\xi_2\xi_3Y_{23})^2]. \tag{45}$$

But, all such inequalities are consequences of (12).

The non-negativeness of the third-order principal minors yields two inequalities:

$$(g\pm y_{12})(g^2-v)\geqslant\mp 2uy_{34}, \tag{46}$$

where

$$v=\sum_{j<k}y_{jk}^2,\quad u=y_{12}y_{34}+y_{23}y_{14}+y_{31}y_{24}. \tag{47}$$

Since $g\pm y_{12}>0$, the consequence of (46) is the inequality

$$g^2\geqslant v, \tag{48}$$

which must hold for arbitrary sets of real parameters ξ_k. In particular, taking all $\xi_k=1$ (this means that the dimensions of operators $\hat{z}_k$ should be made equal by means of some scaling transformations), we get the inequality, generalizing (13) to the case of four observables,

$$\sum_{k=1}^{4}X_{kk}\geqslant 2\left[\sum_{j<k}Y_{jk}^2\right]^{1/2}. \tag{49}$$

The generalization of (13) and (49) to the case of N observables was found in [23], although under the restriction that these observables are arbitrary *linear combinations* of the canonical coordinate and momentum operators.

Inequalities (13) and (49) can be written in the nice matrix form

$$[\mathrm{Tr}(X)]^2\geqslant 2\,\mathrm{Tr}(Y\tilde{Y}), \tag{50}$$

where $\tilde{Y}$ means the transposed matrix. It would be interesting to prove (50) for any dimension $N>4$ and *arbitrary* Hermitian operators.

The inequalities (48) and (49) can be strengthened, if one considers the most general condition of positive semidefiniteness of matrix F, namely, $\det F\geqslant 0$. After some algebra, it can be written in the following compact form:

$$\det F=(g^2-v)^2-4u^2\geqslant 0. \tag{51}$$

Since $g^2\geqslant v$, the consequences of (51) are the inequalities

$$g^2\geqslant v+2|u| \tag{52}$$

and

$$\left(\sum_{k=1}^{4}X_{kk}\right)^2\geqslant 4\sum_{j<k}Y_{jk}^2+8\Lambda, \tag{53}$$

where

$$\Lambda=|Y_{12}Y_{34}+Y_{23}Y_{14}+Y_{31}Y_{24}|. \tag{54}$$

Note that Λ (as well as a more general coefficient u) is invariant with respect to the ordering of indices, due to the property $Y_{jk}=-Y_{kj}$.

The choice $\xi_j^2=X_{kk}X_{mm}X_{nn}$ (with $j\neq k\neq m\neq n$) transforms (52) to the following inequality, containing the "uncertainty product" $\Pi=\sqrt{X_{11}X_{22}X_{33}X_{44}}$:

$$4\Pi^2-2\Lambda\Pi-\Psi\geqslant 0, \tag{55}$$

where

$$\begin{aligned}\Psi=&\,Y_{12}^2X_{33}X_{44}+Y_{13}^2X_{22}X_{44}+Y_{14}^2X_{33}X_{22}\\ &+Y_{23}^2X_{11}X_{44}+Y_{24}^2X_{33}X_{11}+Y_{34}^2X_{11}X_{22}.\end{aligned}$$

Resolving inequality (55) with respect to variable Π, we arrive at the inequality

$$4\sqrt{X_{11}X_{22}X_{33}X_{44}}\geqslant\Lambda+\sqrt{4\Psi+\Lambda^2} \tag{56}$$

(the second solution, giving an upper bound for Π, is unphysical). Note that

$$\Psi\geqslant\Psi_*=2\big(Y_{12}^2Y_{34}^2+Y_{23}^2Y_{14}^2+Y_{31}^2Y_{24}^2\big) \tag{57}$$

as a consequence of the standard uncertainty relation (2). Therefore, one can get rid of variances in the right-hand side of (56), replacing Ψ with Ψ_*. If $Y_{12} = Y_{13} = 0$, then the right-hand side of (28) turns into zero, even if $Y_{23} \neq 0$. In this case, inequality (56) with Ψ_* yields a nonzero lower bound for the product of two uncertainties (provided $Y_{14} \neq 0$):

$$\Delta z_1 \Delta z_2 \geqslant |Y_{23}Y_{14}|/(\Delta z_3 \Delta z_4). \tag{58}$$

At this point, it is worth comparing (56) with Robertson's inequality (6). It is useless for $N = 3$ (because $\det Y \equiv 0$ for any antisymmetric 3×3 matrix Y). But, for $N = 4$ we have

$$\det(Y) = Y_{12}^2 Y_{34}^2 + Y_{23}^2 Y_{14}^2 + Y_{31}^2 Y_{24}^2 - 2Y_{12}Y_{13}Y_{24}Y_{34} - 2Y_{12}Y_{14}Y_{23}Y_{43} - 2Y_{13}Y_{14}Y_{32}Y_{42} \equiv \Lambda^2. \tag{59}$$

Consequently, (56) is not weaker than (6) if $\Psi \geqslant 2\Lambda^2$. The sufficient condition $\Psi_* \geqslant 2\Lambda^2$ holds provided

$$Y_{12}Y_{13}Y_{24}Y_{34} + Y_{12}Y_{14}Y_{23}Y_{43} + Y_{13}Y_{14}Y_{32}Y_{42} \geqslant 0. \tag{60}$$

This condition is fulfilled automatically for systems with only two nonzero mean values of commutators. In particular, two independent sets of coordinates and momenta belong to this family. In this case, the right-hand side of (6) coincides with the right-hand side of (56), *provided* Ψ is replaced by Ψ_*. But, if such a replacement is not done, then inequality (56) is *stronger* than (6). One can check that the equality sign in (56) is achieved, e.g., for the coordinates and momenta in the minimum uncertainty states of two uncoupled harmonic oscillators.

The concrete choice of γ matrices does not influence the final results. For example, if one chooses, instead of (42), matrices

$$\gamma_k = \begin{Vmatrix} \sigma_k & 0 \\ 0 & -\sigma_k \end{Vmatrix}, \quad \gamma_4 = -\begin{Vmatrix} 0 & I_2 \\ I_2 & 0 \end{Vmatrix}, \tag{61}$$

then matrices $\gamma_j \gamma_k$ have exactly the same form as in (43), so that matrix F maintains the form (44).

Example

For example, let us consider the set $z_1 = x, z_2 = p_x, z_3 = y, z_4 = p_y$, and the pure quantum state

$$\psi(x,y) = \mathcal{N} \exp\left(-\frac{a}{2}x^2 - bxy - \frac{c}{2}y^2\right), \tag{62}$$

where $\mathcal{N}$ is the normalization factor, and all coefficients a, b, and c are real numbers, satisfying the restrictions $a > 0, c > 0$, and $D \equiv ac - b^2 > 0$. Then,

$$X_{11} = \frac{c}{2D}, \quad X_{33} = \frac{a}{2D}, \quad X_{22} = \frac{1}{2}a\hbar^2, \quad X_{44} = \frac{1}{2}c\hbar^2,$$

so that

$$\sqrt{X_{11}X_{22}X_{33}X_{44}} = \frac{(ac)\hbar^2}{4D}, \quad \Psi = \frac{ac\hbar^4}{8D}, \quad \Lambda = \frac{\hbar^2}{4}.$$

If $b = 0$, then we have equalities in both relations (56) and (6). But, if $b \neq 0$, and especially if $D \ll ac$, then (6) becomes very weak because its right-hand side equals always $\hbar^2/4$. On the other hand, inequality (56) shows that the uncertainty product $\sqrt{X_{11}X_{22}X_{33}X_{44}}$ must be much bigger than $\hbar^2/4$ in this case [although the right-hand side of (56) appears much smaller than the left-hand side, so that this inequality is not the best possible].

IV. CONCLUSION

The main results of this paper are inequalities (12), (13), and (16) for arbitrary sets of three Hermitian operators, and inequalities (52), (53), and (56) for arbitrary sets of four Hermitian operators. These inequalities give lower bounds for the sums and products of three or four variances in terms of the mean values of commutators, but they do not contain explicitly the covariances between the observables. Therefore, the inequalities are simpler than the known Robertson's inequalities for several observables. (Note, however, that covariances can enter the inequalities implicitly, through the mean values of commutators.) The "magic numbers" 3 and 4 arise due to the existence of three anticommuting 2×2 Pauli's matrices and four anticommuting 4×4 Dirac's matrices.

Important consequences of relations (12) and (56) are inequalities (28) and (58), which show that even if the mean value of the commutator between two operators is zero, nonetheless, the product of variances of the corresponding observables must be nonzero, if these observables are parts of some extended system. Perhaps, it is worth mentioning that all inequalities derived in this article remain valid, if one uses unshifted operators $\hat{z}_j$ instead of $\delta\hat{z}_j$ in the construction of operator $\hat{F}$. Then, simplified versions of the inequalities can be written, using the definition $X_{jj} = \langle \hat{z}_j^2 \rangle$.

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APPENDIX A: WHY THE TREATMENT OF REF. [47] RESULTED IN A WRONG INEQUALITY

The 2×2 matrix (11) must be positive semidefinite for arbitrary real parameters $\alpha_1,\alpha_2,\alpha_3$. This means that function $\varphi(\alpha_1,\alpha_2,\alpha_3,\chi_1,\chi_2) = \langle \chi|\mathcal{A}|\chi\rangle$ (which depends on *five* variables) must be non-negative for an arbitrary spinor $\langle \chi| = (\chi_1^*,\chi_2^*)$. Considering $\langle \chi|\mathcal{A}|\chi\rangle$ as a bilinear form with respect to components of *six-dimensional* vector $\mathbf{v} = (\alpha_1\chi_1,\alpha_2\chi_1,\alpha_3\chi_1,\alpha_1\chi_2,\alpha_2\chi_2,\alpha_3\chi_2)$ (where χ_1 and χ_2 are the complex components of the auxiliary spinor $|\chi\rangle$), one could write (following [47]) $\langle \chi|\mathcal{A}|\chi\rangle = \mathbf{v}^*\Phi\mathbf{v}$ with the 6×6 Hermitian matrix

$$\Phi = \begin{Vmatrix} X_{11} & Y_{21} & 0 & 0 & 0 & iY_{31} \\ Y_{21} & X_{22} & 0 & 0 & 0 & Y_{32} \\ 0 & 0 & X_{33} & iY_{31} & Y_{32} & 0 \\ 0 & 0 & iY_{13} & X_{11} & Y_{12} & 0 \\ 0 & 0 & Y_{32} & Y_{12} & X_{22} & 0 \\ iY_{13} & Y_{32} & 0 & 0 & 0 & X_{33} \end{Vmatrix}. \tag{A1}$$

Then, the main condition of positive semidefiniteness of matrix Φ, namely $\det \Phi \geqslant 0$, would result in the inequality (8), which is not correct, as was shown at the end of Sec. I. A possible origin of mistake based on using matrix (A1) is that condition $\det \Phi \geqslant 0$ guarantees the inequality $\mathbf{v}^*\Phi\mathbf{v} \geqslant 0$ for *all possible* choices of vector $\mathbf{v}$, whereas not all components of this vector are independent in the case involved since $v_4/v_1 = v_5/v_2 = v_6/v_3$.

APPENDIX B: FOURTH-ORDER MOMENTS IN TERMS OF THE SECOND-ORDER ONES FOR THE GAUSSIAN STATES

The variances of the triple (33) correspond to the fourth-order moments of coordinates and momenta. Such moments can be calculated easily for any Gaussian state because its Wigner functions (in the single space dimension for simplicity) $W(x,p)$ is also Gaussian, so that one can use classical formulas for average values of the Gauss distributions (with some modifications due to the noncommutativity of the coordinate and momentum operators). The details can be found, e.g., in [51].

Consider four operators (not necessarily different) $\hat{A}$, $\hat{B}$, $\hat{C}$, and $\hat{D}$ (with zero mean values), where each of them can be either $\delta\hat{x}$ or $\delta\hat{p}$. Then, the mean value of the *symmetrical* (or Wigner-Weyl) product of these operators is given by the formula (see, e.g., [52])

$$\langle ABCD\rangle_W = \int W(x,p)\, ABCD\, dx\, dp/(2\pi\hbar). \tag{B1}$$

The meaning of symbol $\langle ABCD\rangle_W$ is the following: this is the quantum mechanical mean value of the sum of all different products of operators $\hat{A}$, $\hat{B}$, $\hat{C}$, and $\hat{D}$, taken in all possible orders, divided by the number of terms. For example, if $\langle\hat{x}\rangle = \langle\hat{p}\rangle = 0$, then

$$\langle x^2p^2\rangle_W = \tfrac{1}{6}\langle\hat{x}^2\hat{p}^2 + \hat{p}^2\hat{x}^2 + \hat{x}\hat{p}\hat{x}\hat{p} + \hat{p}\hat{x}\hat{p}\hat{x} + \hat{x}\hat{p}^2\hat{x} + \hat{p}\hat{x}^2\hat{p}\rangle.$$

Mean values of concrete products of operators in predefined orders can be expressed in terms of symmetrical mean values with the aid of commutation relations. For example (if $\langle\hat{x}\rangle = \langle\hat{p}\rangle = 0$),

$$\langle x^2p^2\rangle_W = \tfrac{1}{2}\langle\hat{x}^2\hat{p}^2 + \hat{p}^2\hat{x}^2\rangle + \frac{\hbar^2}{2},$$
$$\langle(\hat{x}+\hat{p})^2\rangle = 2\langle\hat{x}^2\hat{p}^2 + \hat{p}^2\hat{x}^2\rangle + 3\hbar^2.$$

Since the Gaussian Wigner function is positive, one can consider it as a classical probability distribution and apply the classical formulas for the Gaussian probabilities to the right-hand side of (B1). The final result is the following formula of decoupling the fourth-order moments into the sums of products of the second-order moments:

$$\langle ABCD\rangle_W = \overline{AB}\cdot\overline{CD} + \overline{AC}\cdot\overline{BD} + \overline{AD}\cdot\overline{BC}, \tag{B2}$$

where $\overline{AB} \equiv \frac{1}{2}\langle\hat{A}\hat{B} + \hat{B}\hat{A}\rangle$ (remember that we suppose here that $\langle\hat{A}\rangle = \langle\hat{B}\rangle = 0$). In particular, taking $\hat{A} = \hat{B} = \hat{C} = \hat{D} = \delta\hat{x}$ we arrive at the known formula

$$\langle(\delta\hat{x})^4\rangle = 3(\sigma_{xx})^2. \tag{B3}$$

Another formula used in the main text is

$$\langle(\delta x)^2(\delta p)^2\rangle_W = \sigma_{xx}\sigma_{pp} + 2(\sigma_{xp})^2. \tag{B4}$$

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