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Schrödinger equation for general linear velocity-dependent forces

L. Gonçalves and L. S. F. Olavo[*]

Instituto de Física, Universidade de Brasília - UnB, Campus Universitário Darcy Ribeiro, Asa Norte, Brasília, DF. 70910-900, Brazil

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Quantization processes generally assume the Hamiltonian formulation of classical mechanics. The notion of dissipation does not fit easily into the Hamiltonian structure. There appeared in the literature a quantization process that circumvents the use of the Hamiltonian approach and derives the Schrödinger equation from first principles. Thus, the usual approach of assuming a dissipative force of the type $f_d = k(t)v$, where $k(t)$ is a scalar depending only on the time and v is the velocity, can be approached using this quantization process to mathematically derive a Schrödinger equation. The derivation for this simple case was already performed by the authors elsewhere. The full generalization of a dissipative force in the context of a linear response theory, as is usual in classical mechanics, would be to write this force as $f_i(\mathbf{x},t) = \mathcal{K}_{ij}(\mathbf{x},t)v_j$, where $\mathcal{K}(\mathbf{x},t)$ is a tensor giving the nonhomogeneity and nonisotropy of the process. In this paper, we present this generalization and connect our results with nonlinear Schrödinger-like equations already known from the literature.

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I. INTRODUCTION

The introduction of dissipative physical systems within the formal structure of quantum mechanics has attracted attention since the beginning of the theory [1–9].

To tackle this problem, some [6–14] assumed an *isolated* physical system with two subsystems, one of them functioning as a heat bath and taking energy from the other because of its much larger number of accessible states. The advantage of this approach comes from the fact that it can be framed into a Hamiltonian structure, making quantization easier. One of its disadvantages is that one has to model the heat bath in some concrete way, generally in terms of harmonic oscillators.

Others considered [1–5] *open* physical systems being acted by a dissipative velocity-dependent force of the type $-kv$. One advantage of this approach is that the details of the dissipation process are put into the phenomenological coefficient k. The disadvantage of this approach comes from the fact that it cannot be put into a Hamiltonian structure, making the quantization process more difficult.

Since the quantization of such open systems cannot depart from the usual Hamiltonian classical background, one of the possibilities would be to try to generalize the Hamilton's classical formalism to encompass dissipation [15]. Quantization could then depart from this generalization in the usual terms. However, there has been no general agreement on what extension of the classical formalism one should use. The very classical problem is quite complex in its own terms, let alone its extension to cope with quantization.

This paper presents the extension of a method [16] to perform the direct quantization of *open* physical systems, which does not make any recourse to a Hamiltonian support.

Indeed, in a previous article [16], we showed that one can mathematically derive a Schrödinger equation for a dissipative system in which dissipation comes from a velocity-dependent force of the type

$$f_d(t) = -k(t)v, \tag{1}$$

where v is the velocity and $k(t)$ is the phenomenological term that encompasses all the physical intricacies in the context of a linear response theory.

The derivation was based on a quantization method [17] which has already proved to be very reliable. This quantization method has already been applied to quantize physical systems directly in generalized curvilinear coordinates, to quantize physical systems in which the canonical momentum encompasses the electromagnetic vector potential, and to make quantization in the realm of the special theory of relativity [18]. The method provided the correct equation in each one of these cases, the one related to curvilinear orthogonal coordinates being the most noteworthy, given the complexities of the intermediary equations [17].

The method uses an axiomatic approach that departs from the Liouville equation as the equation to be *extended* so that one ends up with a Schrödinger equation representing the quantum mechanical counterpart of the physical problem. It is noteworthy that being an axiomatic approach, all the eventual modifications of the theory *must be performed only in the axioms*. This can be very restrictive but, on the other side, gives one a quite good control of the extension of the modifications.

In its original form [17], the quantization method could deal only with Hamiltonian systems and the only velocity-dependent forces it could encompass were those derived from a potential that keeps the Hamiltonian structure of the problem intact—such as the electromagnetic force.

However, despite the difficulties inherent to formulate a Hamiltonian approach for dissipative systems [19], the Liouville equation can be easily generalized [20] to refer to non-Hamiltonian systems of which dissipative systems are but an example. The original quantization method can then be

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[*] olavolsf@unb.br

reformulated to take into account this extension of the Liouville equation to deal with forces of the form (1).

In Sec. II, we present the quantization of velocity-dependent forces that keep the Hamiltonian structure unaltered. This derivation is important to show the applicability of the present derivation method for more involved physical situations and also because one of its results will become important later on. The generalization of the Liouville equation is thus presented.

Section III is devoted to very briefly present a previous more restricted result [16] related to a dissipative force $f_i = -k(t)v_i$ with a phenomenological factor $k(t)$ which can be a function of the time alone.

In Sec. IV, we extend our approach to embrace the most general dissipative force, in the context of a linear response theory, given by $f_i(\mathbf{x},t) = -\mathcal{K}_{ij}(\mathbf{x},t)v_j$, where $\mathcal{K}(\mathbf{x},t)$ is a tensor.

Section V presents the connection of our approach and the one of Caldeira and Leggett, and in Sec. VI we present our conclusions.

It should be emphasized that this paper also intends to present a very general quantization method that extends much beyond traditional ones, being able to cope with both Hamiltonian and non-Hamiltonian systems. To show this generality once more, we include in the Appendix a derivation showing that other results obtained in the past in relation to dissipation, and related to a nonlinear Schrödinger-like equation can be addressed by the same method adopted here—which is solely *based on first principles*.

The application of this quantization method to a dissipative system is but a good example of the applicability of the method. The necessity of making an explicit assessment comes from the fact that non-Hamiltonian systems do not constitute a universality class, and must be treated in its specificities.

II. HAMILTONIAN VELOCITY-DEPENDENT POTENTIALS

When one deals with velocity-dependent potentials, there is one important class of potentials consisting of those that keep the form of Lagrange's equations unaltered. This class of potentials is comprised of all potential functions $U(\mathbf{x},\dot{\mathbf{x}},t)$, such that the generalized force is given by [21]

$$Q_j = -\frac{\partial U(\mathbf{x},\dot{\mathbf{x}},t)}{\partial x_j} + \frac{d}{dt}\left[\frac{\partial U(\mathbf{x},\dot{\mathbf{x}},t)}{\partial \dot{x}_j}\right]. \tag{2}$$

A very important example is furnished by the electromagnetic field for which the Lorentz force is given by ($c = 1$ and Einstein's summation convention assumed)

$$Q_i = e(E_i + \epsilon_{ijk}v_j B_k),$$

where e is the electric charge and ϵ_{ijk} is the fully antisymmetric tensor, with the electric and magnetic fields (respectively, $\mathbf{E}$ and $\mathbf{B}$) given by

$$E_i = -\partial_i\varphi - \frac{\partial A_i}{\partial t}, \quad B_i = \epsilon_{ijk}\partial_j A_k,$$

where φ is the scalar potential and $\mathbf{A}$ is the vector potential. Indeed, if we write

$$U = e\varphi - eA_i v_i, \tag{3}$$

we recover the electromagnetic Lorentz force $\mathbf{Q}$ showing that the potential U is a member of our class of velocity-dependent potentials that leaves the Lagrangian and Hamiltonian structure of the mathematical problem unaltered [21].

The electromagnetic field is surely the best example of a field that satisfies condition (2) for a velocity-dependent force. The fact that physical systems presenting such a velocity-dependent force *can* be quantized by the present method should give the reader the necessary confidence on the same quantization method applied for other more general forces. We thus present, in what follows, this quantization.

For a physical system presenting electromagnetic interactions, one has the Liouville equation

$$\frac{dF}{dt} = \frac{\partial F}{\partial x_j}\left(\frac{p_j - eA_j}{m}\right) - \frac{\partial F}{\partial p_j}\left[\frac{e}{m}\frac{\partial A_k}{\partial x_j}(eA_k - p_k) + e\frac{\partial \phi}{\partial x_j}\right] + \frac{\partial F}{\partial t} = 0. \tag{4}$$

The quantization method comes simply from the application of the following two axioms:

Axiom 1. The Liouville equation (4) must be extended for us to arrive at a generalized Schrödinger equation.

Axiom 2. The marginal characteristic function, given by

$$\zeta(\mathbf{x},\mathbf{y};t) = \int \exp\left(\frac{i}{\hbar}p_i\delta x_i\right)F(\mathbf{x},\mathbf{p};t)d\mathbf{p}, \tag{5}$$

should be written as

$$\zeta(\mathbf{x},\delta\mathbf{x};t) = \psi^*(\mathbf{x} - \delta\mathbf{x}/2;t)\psi(\mathbf{x} + \delta\mathbf{x}/2;t), \tag{6}$$

and must be expanded only up to second order in $\delta\mathbf{x}$ as a method to extend (4).

We may now apply the integral transform (5) to (4) to get, after some straightforward calculations, the equation

$$i\hbar\frac{\partial\zeta}{\partial t} = -\frac{i\hbar}{m}\frac{\partial}{\partial x_j}\left[-i\hbar\frac{\partial\zeta}{\partial(\delta x_j)} - eA_j\zeta\right] + e\delta x_j\frac{\partial\phi}{\partial x_j}\zeta + \delta x_j\frac{e}{m}\frac{\partial A_k}{\partial x_j}\left[-i\hbar\frac{\partial\zeta}{\partial(\delta x_k)} - eA_k\zeta\right]. \tag{7}$$

Equation (7) will be very important for the results to follow later on.

We now use (6) of our quantization process and write

$$\psi(\mathbf{x},t) = R(\mathbf{x},t)e^{iS(\mathbf{x},t)/\hbar}, \tag{8}$$

where $R(\mathbf{x},t)$ and $S(\mathbf{x},t)$ are real functions, since the amplitudes $\psi(\mathbf{x},t)$ are generally complex functions. We thus expand the characteristic function in second order in $\delta\mathbf{x}$ to get

$$\zeta(\mathbf{x},\delta\mathbf{x},t) = \left(R^2(\mathbf{x},t) + \frac{(\delta x_i\delta x_k)}{4}\left\{R(\mathbf{x},t)\frac{\partial^2 R(\mathbf{x},t)}{\partial x_i\partial x_k} - \left[\frac{\partial R(\mathbf{x},t)}{\partial x_i}\right]\left[\frac{\partial R(\mathbf{x},t)}{\partial x_k}\right]\right\}\right)\exp\left[\frac{i}{\hbar}\frac{\partial S(\mathbf{x},t)}{\partial x_j}\delta x_j\right]. \tag{9}$$

If we take ζ into (7) and keep terms up to first order in δx_k, we get an equation with real and imaginary parts. Separating them, we get, for the imaginary part (which is also the coefficient of $\delta\mathbf{x}^0$),

$$\frac{\partial R^2(\mathbf{x},t)}{\partial t} + \partial_i \left\{ R^2(\mathbf{x},t) \frac{[\partial_i S(\mathbf{x},t) - eA_i(\mathbf{x},t)]}{m} \right\} = 0, \quad (10)$$

while for the real part (which is also the coefficient of $\delta\mathbf{x}^1$), we get

$$R^2 \delta x_i \partial_i \left\{ \frac{\partial S(\mathbf{x},t)}{\partial t} + \frac{1}{2m} [\nabla S(\mathbf{x},t) - e\mathbf{A}(\mathbf{x},t)]^2 + e\phi - \frac{\hbar^2 \nabla^2 R(\mathbf{x},t)}{2m R(\mathbf{x},t)} \right\} = 0. \quad (11)$$

These two equations (10) and (11) are equivalent to the Schrödinger equation (encompassing the electromagnetic potentials) if we write

$$\frac{1}{2m}[-i\hbar\nabla - e\mathbf{A}(\mathbf{x},t)]^2 \psi(\mathbf{x},t) + \phi(\mathbf{x},t)\psi(\mathbf{x},t) = i\hbar \frac{\partial \psi(\mathbf{x},t)}{\partial t}, \quad (12)$$

and use the polar form (8). Indeed, substitution of this polar form into (12) gives two equations which are exactly (10) and the term inside curly brackets in (11), when we separate the real and the imaginary parts.

Equations (10) and (11) in connection to (12) are reminiscent of the Bohm-Madelung hydrodynamic approach [22,23]. However, it should be stressed here that the present method is a quantization process—a way to mathematically derive the Schrödinger equation from more basic postulates—while the Bohm-Madelung approach is simply a way to rewrite the Schrödinger equation and this approach already assumes, rather than derives, this equation.

Thus, the previous derivation shows that we can use our method to derive the Schrödinger equation for a velocity-dependent potential of the type (2). However, if we want to move to non-Hamiltonian forces, we must generalize the first axiom of the quantization process, which is the Liouville equation.

We made such a generalization in [16] and showed that one should depart, *in the general case of non-Hamiltonian systems*, from the generalized Liouville equation given by

$$\frac{dF(\eta,t)}{dt} = -F(\eta,t)\partial_i \dot{\eta}_i, \quad (13)$$

where $\eta = (\mathbf{x},\mathbf{p})$, and ∂_i runs over this $6N$-dimensional space. The right-hand term in the last expression is called the *compression factor* and is identically zero if the system is Hamiltonian.

III. EXTENSION OF THE QUANTIZATION METHOD

In [16], we generalized our set of axioms to mathematically derive a time-dependent Schrödinger equation that can encompass forces of the type

$$f_i(\mathbf{x},\dot{\mathbf{x}};t) = -\partial_i \varphi(\mathbf{x}) + k(t)\dot{x}_i,$$

where $\varphi(\mathbf{x})$ is the velocity-independent potential, and $k(t)$ can depend on time only. Since we are now considering physical situations in which the phase-space volume element can shrink or expand, and, in quantum mechanics, this volume is proportional to Planck's constant, we must assume the possibility

$$\bar{\hbar} \to \hbar e^{\sigma(t)}, \quad (14)$$

so that the new quantum mechanical elementary volume element becomes $\Delta\mathbf{x}\Delta\mathbf{p} \sim \hbar e^{\sigma(t)}$, which can shrink or expand, depending on the behavior of $\sigma(t)$.

This implies changing the second axiom to write it as follows:

Axiom 2′. The marginal characteristic function, given by

$$\zeta(\mathbf{x},\delta\mathbf{x};t) = \int \exp\left(\frac{i}{\hbar e^{f(t)}}\mathbf{p}\cdot\delta\mathbf{x}\right) F(\mathbf{x},\mathbf{p};t)d\mathbf{p}, \quad (15)$$

should be written as

$$\zeta(\mathbf{x},\delta\mathbf{x};t) = \psi^*(\mathbf{x}-\delta\mathbf{x}/2;t)\psi(\mathbf{x}+\delta\mathbf{x}/2;t), \quad (16)$$

and must be expanded only up to second order in $\delta\mathbf{x}$.

Now, if we exactly apply the same steps of the derivation of the previous section, we find the dissipative Schrödinger equation [16],

$$-\frac{\hbar^2 e^{2\sigma(t)}}{2m}\nabla^2\psi(\mathbf{x},t) + \varphi(\mathbf{x})\psi(\mathbf{x},t) = i\hbar e^{\sigma(t)}\frac{\partial\psi(\mathbf{x},t)}{\partial t}. \quad (17)$$

Equation (17) is simply the Schrödinger equation for a time-dependent Planck's "constant," where

$$\sigma(t) = \frac{1}{m}\int^t k(t')dt'. \quad (18)$$

From the very derivation process, the dissipative Schrödinger equation reduces itself to the usual Schrödinger equation when there is neither dissipation nor absorption [$k(t) = 0$].

This derivation is important to make it explicit the need to change Axiom 2. However, it is quite limited since it assumes a very particular form of the dissipation force. In the next section, we extend the previous results to a quite general dissipative force related to a dissipative linear response theory.

IV. EXTENSION OF THE QUANTIZATION METHOD FOR $\mathcal{K}(\mathbf{x},t)$

We now use our generalized Liouville equation to take into account non-Hamiltonian systems encompassing dissipation of the type

$$f_i(\mathbf{x},\mathbf{p};t) = -\partial_i\varphi(\mathbf{x}) + \mathcal{K}_{ij}(\mathbf{x},t)v_j, \quad (19)$$

where $\varphi(\mathbf{x})$ is the velocity-independent potential and $\mathcal{K}(\mathbf{x},t)$ is a tensorial function of the position and time, in principle—and thus is a very general linear response phenomenological dissipation coefficient.

The Liouville equation becomes

$$\underbrace{\frac{\partial F}{\partial t}}_{A} + \underbrace{\frac{p_n}{m}\frac{\partial F}{\partial x_n}}_{B} + \underbrace{-\frac{\partial F}{\partial p_n}\frac{\partial\varphi}{\partial x_n}}_{C} + \underbrace{\frac{\mathcal{K}_{nl}}{m}\frac{\partial(p_l F)}{\partial p_n}}_{D} = 0, \quad (20)$$

whose terms were labeled for future reference. Equation (20) is the equation we will use to quantize the non-Hamiltonian velocity-dependent force (19) from first principles.

Following our previous discussion, we generalize again our Axiom 2, putting

$$\begin{aligned}\zeta(\mathbf{x},\delta\mathbf{x},t) &= \int F(\mathbf{x},\mathbf{p},t)\exp\left\{\frac{i}{\hbar e^{\sigma(t)}}[p_k - g_k(\mathbf{x},t)]\delta x_k\right\}d\mathbf{p}\\ &= \psi^*\left(\mathbf{x}-\frac{\delta\mathbf{x}}{2};t\right)\psi\left(\mathbf{x}+\frac{\delta\mathbf{x}}{2};t\right),\end{aligned} \tag{21}$$

where $\mathbf{g}(\mathbf{x},t)$ is a yet *unknown function* (our *gauge*) that will be discovered in the process of quantization, in ways analogous to $\sigma(t)$ in the previous derivation.

Applying the kernel $\exp[i(p_n+g_n)\delta x_n/\hbar e^{\sigma(t)}]$ to Eq. (20) and integrating it in $\mathbf{p}$, we get the following:

For A,

$$A = \frac{\partial\zeta}{\partial t} + \frac{i\dot{\sigma}\delta x_n}{\hbar e^{\sigma}}\left[-i\hbar e^{\sigma}\frac{\partial\zeta}{\partial(\delta x_n)} + g_n\zeta\right] + \frac{i\delta x_n}{\hbar e^{\sigma}}(\dot{g}_n - \dot{\sigma}g_n)\zeta. \tag{22}$$

For B,

$$\begin{aligned}B =& \frac{1}{m}\frac{\partial}{\partial x_n}\left[-i\hbar e^{\sigma}\frac{\partial\zeta}{\partial(\delta x_n)} + g_n\zeta\right]\\ &+ \frac{i\delta x_n}{m\hbar e^{\sigma}}\frac{\partial g_n}{\partial x_l}\left[-i\hbar e^{\sigma}\frac{\partial\zeta}{\partial(\delta x_l)} + g_l\zeta\right].\end{aligned} \tag{23}$$

For C,

$$C = \frac{i\delta x_l}{\hbar e^{\sigma}}\frac{\partial\varphi}{\partial x_l}\zeta. \tag{24}$$

For D,

$$D = -\frac{i\mathcal{K}_{nl}(x,t)\delta x_n}{m\hbar e^{\sigma}}\left[-i\hbar e^{\sigma}\frac{\partial\zeta}{\partial(\delta x_l)} + g_l\zeta\right]. \tag{25}$$

After some rearrangements, we end up with Eq. (20) transformed into

$$\begin{aligned}&-i\hbar e^{\sigma}\frac{\partial\zeta}{\partial t} - \frac{i\hbar e^{\sigma}}{m}\frac{\partial}{\partial x_l}\left[-i\hbar e^{\sigma}\frac{\partial\zeta}{\partial(\delta x_l)} + g_l\zeta\right] + \delta x_l\frac{\partial\varphi}{\partial x_l}\zeta\\ &+\delta x_n\left[\dot{\sigma}\delta_{nl} + \frac{1}{m}\frac{\partial g_n}{\partial x_l} - \frac{\mathcal{K}_{nl}}{m}\right]\left[-i\hbar e^{\sigma}\frac{\partial\zeta}{\partial(\delta x_l)} + g_l\zeta\right]\\ &+\delta x_n(\dot{g}_n - \dot{\sigma}g_n)\zeta = 0.\end{aligned} \tag{26}$$

We want to find a Schrödinger-like equation such as (17). If we look at (7) for the electromagnetic field, we know in advance that it would be *sufficient* to write

$$\dot{g}_n - \dot{\sigma}g_n = 0 \tag{27}$$

and

$$\dot{\sigma}\delta_{nl} + \frac{1}{m}\frac{\partial g_n}{\partial x_l} - \frac{\mathcal{K}_{nl}}{m} = \frac{1}{m}\frac{\partial g_l}{\partial x_n}. \tag{28}$$

The first constraint means that

$$g_n(\mathbf{x},t) = h_n(\mathbf{x})e^{\sigma(t)}, \tag{29}$$

which, looking back into the characteristic function ζ, simply means an adjustment to remove the factor $e^{\sigma(t)}$ from the second term in the exponential, while making it just a term depending on the position.

It is interesting to rewrite (28) in a more intuitive format. Indeed, we can put it as

$$\dot{\sigma}\delta_{nl} + \frac{1}{m}\left(\frac{\partial g_n}{\partial x_l} - \frac{\partial g_l}{\partial x_n}\right) = \frac{\mathcal{K}_{nl}}{m}, \tag{30}$$

which implies some constraints on the form of $\mathcal{K}$ that can be taken into account. This means that a *linear* Schrödinger equation encompassing dissipation can be obtained only for tensor coefficients of the form (30) (but see the Appendix).

Indeed, if we substitute the characteristic function (9) in (26), we get two equations: one as the coefficient of $(\delta\mathbf{x})^0$ and one as the coefficient of $(\delta\mathbf{x})^1$. The equation that comes as the coefficient of $(\delta\mathbf{x})^0$, which appears multiplied by the imaginary term $i\hbar$, can be written as

$$\frac{\partial R(\mathbf{x},t)^2}{\partial t} + \partial_i\left\{R(\mathbf{x},t)^2\left[e^{\sigma(t)}\frac{\partial_i S(\mathbf{x},t) + h_i(\mathbf{x})}{m}\right]\right\} = 0, \tag{31}$$

while for $(\delta\mathbf{x})^1$ we get

$$\begin{aligned}R(\mathbf{x},t)^2\delta x_i\partial_i\Bigg\{&e^{\sigma(t)}\frac{\partial S(\mathbf{x},t)}{\partial t} - \frac{\hbar^2 e^{2\sigma(t)}}{2mR(\mathbf{x},t)}\nabla^2 R(\mathbf{x},t)\\ &+ e^{2\sigma(t)}\frac{[\nabla S(\mathbf{x},t) + \mathbf{h}(\mathbf{x})]^2}{2m} + \varphi(\mathbf{x},t)\Bigg\} = 0,\end{aligned} \tag{32}$$

so that if we write

$$\frac{e^{2\sigma(t)}}{2m}[-i\hbar\nabla + \mathbf{h}(\mathbf{x})]^2\psi(\mathbf{x},t) + \varphi(\mathbf{x})\psi(\mathbf{x},t) = i\hbar e^{\sigma(t)}\frac{\partial\psi(\mathbf{x},t)}{\partial t}, \tag{33}$$

and substitute on it again, $\psi(\mathbf{x},t) = R(\mathbf{x},t)\exp[iS(\mathbf{x},t)/\hbar]$, we get exactly (31) and the term within curly brackets in (32). Thus, Eq. (33) is the dissipative quantum mechanical equation for a dissipative force of the type shown in (19) with the form (30) for the multiplicative tensor of the linear response approach.

The approach of considering a time-dependent "Planck's" constant, as the result (14) points to, is closely related to one presented in [24]. However, in that work, the authors use *heuristic* arguments connected to the Bohm-Madelung equation to introduce, somewhat *ad hoc*, this time dependence. In this paper, we have derived all the results from first principles and presented a direct physical argument for such a time dependence, which is different from [24]. In any case, we end with a similar, although quite general, conclusion that can be found in that work: from a more general nonlinear Schrödinger equation that can be derived for dissipative systems, in general, there is a class of solutions that can keep the equation *linear*, even in the presence of dissipation. The structure (30) is exactly the one to allow for such a preservation of the linearity of the Schrödinger equation.

In the Appendix, we derive, from the same formal structure presented here, a nonlinear Schrödinger equation close in form to others already known in the literature [25–28]. We then make explicit the above-mentioned result and, again, reassure ourselves of the soundness of the quantization process.

V. CONNECTION WITH THE CALDEIRA-LEGGETT APPROACH

One of the main objectives in the field of quantum dissipation is to explain the way classical dissipation can be described from a quantum mechanical point of view.

The Caldeira–Leggett model [29] gives us a glimpse of how this is done within its framework: one takes the average over all the possible realizations of the bath and looks for the effective dynamics of the quantum system. Then, the limit $\hbar \to 0$ must be taken to recover the classical equations of motion as

$$m\frac{d^2}{dt^2}x(t) = -\frac{\partial V(x)}{\partial x} - \int_0^T dt'\alpha(t-t')[x(t)-x(t')], \tag{34}$$

where

$$\alpha(t-t') = \frac{1}{2\pi}\int_0^\infty J(\omega)e^{-\omega|t-t'|}d\omega \tag{35}$$

is a kernel which characterizes the effective force that affects the motion of the particle in the presence of dissipation, while $J(\omega)$ is the bath spectral function. When this function has the form $J(\omega) = \eta\omega$, the corresponding classical kind of dissipation can be shown to be ohmic. One can generalize this and put $J(\omega) \propto \omega^s$. In this case, if $s > 1$, the dissipation is called superohmic, while if $s < 1$, it is called subohmic.

For Markovian baths and ohmic dissipation, the equations of motion simplify to the classical equations of motion of a particle with friction,

$$m\frac{d^2}{dt^2}x(t) = -\frac{\partial V(x)}{\partial x} - \eta\frac{dx(t)}{dt}. \tag{36}$$

This is how the Caldeira–Leggett model represents classical dissipation within quantum mechanics.

The present work uses an inverse strategy, which seems to be a more *direct* approach. It begins with a usual dissipative classical system and ends with the dissipative Schrödinger equation (33). One can see from the last equations that the connection between the two approaches is given by the bath spectral function $J(\omega)$, which is a central quantity in the Caldeira-Leggett model.

VI. CONCLUSION

In this paper, we derived, from first principles, a generalized Schrödinger equation that encompasses dissipative phenomena. The quantization method may be considered to be very reliable since it delivers all the expected results for Hamiltonian systems. In fact, the method applies to the quantization of non-Hamiltonian systems in general, of which dissipative systems are a particular case. Even a nonlinear Schrödinger-like equation, close to equations already known in the literature, could be obtained, as we show in the Appendix.

We have also shown that there is a connection between the Caldeira-Leggett approach, based on the interaction of the system with a heat bath, and the present one, meaning that the latter can be applied to all physical systems to which the Caldeira-Leggett approach applies.

We think, however, that this more direct approach presents some advantages from the Caldeira-Leggett one since it gives a general Schrödinger equation that can be solved by usual methods, without the need to appeal to renormalization and other more sophisticated techniques.

We hope the present approach can be useful to the study of dissipative quantum mechanical systems, such as the mesoscopic system.

APPENDIX: DERIVATION OF A NONLINEAR SCHRÖDINGER EQUATION

One can find in the literature a number of results regarding the expression of dissipation in the quantum realm as a counterpart of some nonlinear feature of a Schrödinger-like equation. Thus, Kostin [28] used arguments from Langevin equations applied to Heisenberg operators to suggest the equation

$$\begin{aligned} i\hbar\frac{\partial\psi}{\partial t} = &-\frac{\hbar^2}{2m}\nabla^2\psi + \varphi\psi + V_R\psi \\ &+ \frac{\hbar f}{2im}\left[\ln\left(\frac{\psi}{\psi^*}\right) - W(t)\right]\psi, \end{aligned} \tag{A1}$$

where f is a friction constant, V_R is a random potential—two terms akin to Langevin equations—and $W(t) = \langle\ln(\psi/\psi^*)\rangle$.

Another result was obtained from heuristic considerations about the possibility of a diffusion term in the flux of probability. This argument is somewhat close to the one we used in the generalization of the Liouville equation, but was made on the configuration-space probability density function, while ours is made on the classical phase space in connection to the Liouville equation. The authors [25] presented their nonlinear counterpart of the Schrödinger equation as

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + \varphi\psi + iD\hbar\nabla^2\psi + iD\hbar|\nabla\ln\psi|^2\psi, \tag{A2}$$

where D is a diffusion constant coming from a term included from physical arguments in the probability continuity equation (to make it a Fokker-Planck equation). Result (A2) resembles (A1), despite being different from it [25]. This leaves us with different equations for the same physical processes. The problem here is that those equations were obtained from heuristic arguments based on (good but different) physical intuitions, and we all know how elusive quantum mechanics can be on such matters.

In the derivation made in the main text, we used the kernel of the Fourier transform that defines the characteristic function with a term e^σ, which introduced the time dependence of Planck's constant reflecting the possibility of phase-space volumes to shrink or expand.

However, if we refrain from changing this kernel, we must write the second axiom as [one dimension for simplicity and for a dissipation coefficient $k(t)$]

$$\begin{aligned} \zeta(x,\delta x,t) &= \int e^{ip\delta x/\hbar}F(x,p;t)dp \\ &= \psi^*\left(x-\frac{\delta x}{2};t\right)\psi\left(x+\frac{\delta x}{2};t\right). \end{aligned} \tag{A3}$$

If we now use the generalized Liouville equation (13), we get

$$i\hbar\frac{\partial\zeta}{\partial t} = -\frac{\hbar^2}{m}\frac{\partial^2\zeta}{\partial x\,\partial(\delta x)} + \delta x\frac{d\varphi}{dx}\zeta + \frac{i\hbar k}{m}\frac{\partial}{\partial(\delta x)}[(\delta x)\zeta]. \tag{A4}$$

We also use (5) to write ζ, as usual. If we take (5) into (A4), and keep terms up to δx^1, we end up with real and imaginary equations (or, equivalently, one equation depending on δx^1 and one depending on δx^0, respectively). Separating them, we get, for the real part (or the one depending on δx^1),

$$\frac{\partial}{\partial t}\left(\frac{\partial S}{\partial x}\right) = -\frac{\partial}{\partial x}\left[\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 - \frac{\hbar^2}{2mR}\frac{\partial^2 R}{\partial x^2} + \varphi(x)\right] + \frac{k(t)}{m}\frac{\partial S}{\partial x}, \tag{A5}$$

and if we put, as usual, $p(x,t) = \partial S/\partial x$, (A5) becomes simply Bohm's equation, with an extra term, given by

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial H_B(x,t)}{\partial x} + \frac{k(t)}{m}p(x,t), \tag{A6}$$

where $H_B(x,t)$ is "Bohm's Hamiltonian" (or Bohmian),

$$H_B(x,t) = \frac{p(x,t)^2}{2m} + \varphi(x) - \frac{\hbar^2}{2mR(x,t)}\frac{\partial^2 R(x,t)}{\partial x^2}.$$

Equation (A6) resembles the classical dissipative Newtonian equation (as would be expected within Bohm's approach).

However, (A5) can also be written as

$$\frac{\partial}{\partial x}\left[\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 - \frac{\hbar^2}{2mR}\frac{\partial^2 R}{\partial x^2} + \varphi(x) + \frac{k(t)}{m}S\right] = 0, \tag{A7}$$

and since $S(x,t) = (\hbar/2i)\ln(\psi/\psi^*)$, we may use all the previous formal results to get the nonlinear Schrödinger equation,

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + \varphi\psi + \left[\frac{\hbar k(t)}{2im}\ln\left(\frac{\psi}{\psi^*}\right)\right]\psi. \tag{A8}$$

This is exactly (A1) with the dissipation coefficient $f = k(t)$ and $V_R = W(t) = 0$. Had we begun with an equation including fluctuation (such as the Klein-Kramers equation), we may had found (A1) or an equation closely related to it.

The fact is that all the results we found in the literature, to the best of our knowledge, were obtained from heuristic arguments sometimes based on Langevin equations and sometimes based on an assumed diffusion of the probability density function. They give, for each approach, a different Schrödinger-like equation.

The present derivation uses the same framework from which it derives the usual Schrödinger equation. It does that using an axiomatic approach. All the necessary generalizations are then made on the axioms to encompass non-Hamiltonian forces. In the main text, we show that for a wide class of dissipation coefficients $\mathcal{K}$, this nonlinear equation can be linearized by assuming a time-dependent Planck's constant.

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