Quantum description of radiative decay in optical cavities

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We derive the quantum mechanical description of light-matter interactions in optical cavities characterized solely by radiative decay. Unique to radiative decay is the conservation of photon number and coherence, in stark contrast to absorptive losses. This prohibits the description of such cavities by traditional means, e.g., coupling it to a bath of harmonic oscillators into which energy is dissipated. Here, we propose a description of cavities with radiative decay by introducing cavity and noise operators in terms of scattering modes. A multimode input-output formalism to predict measurable far field quantities arises naturally. We apply our general model to the special case of the single excitation regime. We find dynamics reminiscent of the dissipative Jaynes-Cummings model, but with vanishing backaction and a rich temporal and spectral structure of the output modes.

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I. INTRODUCTION

The proper treatment of radiative decay in optical cavities is an important issue. Over the course of time, optical cavities have been shown to be of use to achieve strong-coupling regime [1–5] of light and matter, to generate squeezed states of light [6–8], entanglement [9–14], single photons [15–19], and nonclassical light in general [20–22]. All these rely on influencing atomic transition properties in quantum systems through tailored electromagnetic field modes [23–26], which in turn can be controlled by the geometry of the optical cavity [27–29].

Cavity quantum electrodynamics (QED) models prototypical systems to study the quantum physical aspects of light-matter interaction [30]. In the absence of decay, the joint system of light and matter is closed and, therefore, its temporal evolution can be described by the Hamiltonian of the system alone. The canonical system consists of a single electromagnetic cavity mode coupled to an atomic transition. Such a system is described by the Rabi Hamiltonian, which in many cases can be further simplified to yield the Jaynes-Cummings Hamiltonian [31].

The disregard of any decay leads to a simple and reliable theoretical model. However, it does not reflect realistic light-matter interaction scenarios, that are always characterized by some kind of decay, e.g., radiative decay or absorptive losses. This decay is usually introduced by coupling system operators to a bath of other excitations [32], e.g., phonons. Since realistic systems contain a large number of such unwanted excitations, the energy of the system is eventually distributed somewhat evenly across all of them and is therefore effectively lost. In the context of optical processes, this is clearly the way to model absorption losses [33]. In contrast, radiative decay, e.g., photons emitted into the far field, constitute a different category of decay that is based on a different mechanism and, hence, requires a different modeling.

Radiative decay does not lead to the loss of electromagnetic energy. Instead, the energy is just displaced over time to a place far away from the system of interest, i.e., the far field. Modeling this as outlined above would be equivalent to placing a perfect absorber around the cavity, effectively leading to the far field being integrated out. Since experimental detection mostly takes place in the far field, this would mean that information about the observables of the theory is lost. Therefore, an accurate theory is needed that can accommodate these specific aspects of radiative decay. In this work, we solve the problem by introducing a formalism that splits the electromagnetic field up into one system operator and multiple bath operators (Fig. 1). This leads to dissipative dynamics and provides insights concerning the relation between scattering and cavity modes. Apart from providing a more rigorous description of open cavity QED, our formalism also introduces an unconventional type of exactly soluble bath dynamics (i.e., relying neither on Born nor Markov approximations) and paves the way towards a unified description of classical and quantum processes in arbitrary nanooptical structures, that can act as cavities.

This work is organized as follows: In Sec. II we derive the quantum mechanical description of light-matter interaction in optical cavities characterized by radiative decay. The resulting model represents the core result of this paper. In Sec. III we show how a multimode input-output formalism emerges naturally within the model. This allows us to describe complete experimental setups in a unified manner. In Sec. IV we apply the general formalism to an exemplary single-photon scenario. We find dynamics reminiscient of the dissipative Jaynes-Cummings model, but with backaction from the bath vanishing completely and a rich temporal and spectral signature in the far field. In Sec. V we summarize our findings and provide a universal procedure for the quantum description of nonabsorbing cavities with Lorentzian spectrum.

II. RADIATIVE LOSSES IN CQED

We consider the problem of a quantized electromagnetic field coupled to a single two-level system (TLS) in electric

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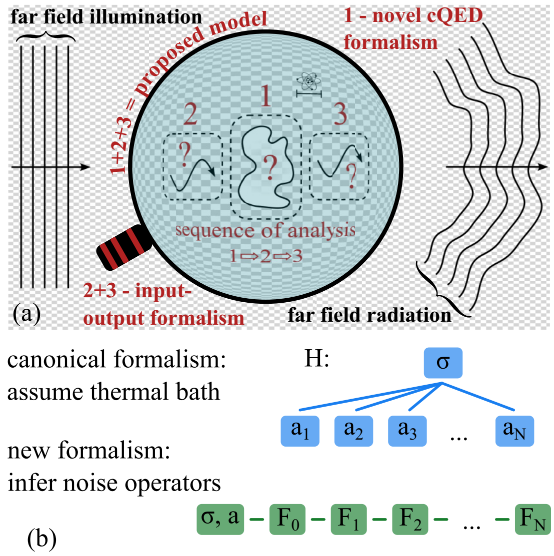

FIG. 1. (a) System under investigation: An arbitrary optical cavity is exposed to arbitrary incident illumination, which results in emission into the far field. (b) Methodological change in formulation of emission: Coupling to thermal bath is replaced by new coupling to chain of noise operators.

dipole and rotating wave approximation. These approximations are justified whenever the TLS is small compared to the size scale of electric-field modulations, typically given by the wavelength, and whenever the light-matter coupling strength is much smaller than all resonant frequencies of the system. Both these approximations are typically very well applicable. The latter condition can be checked using the interaction strength derived below.

We assume the presence of a localized and lossless dielectric structure giving rise to classical light scattering. The Hamiltonian of the system can be found in many textbooks on quantum optics and reads [34]

$$H = \sum_{\lambda} \int d^3k \hbar\omega_{\mathbf{k}} a^\dagger_{\mathbf{k},\lambda} a_{\mathbf{k},\lambda} + \hbar\omega_a \frac{\sigma_z}{2} + \sum_{\lambda} \int d^3k (\mathbf{E}_{\mathbf{k},\lambda}(\mathbf{r}_a) \cdot \mathbf{d} a^\dagger_{\mathbf{k},\lambda} \sigma_- + \text{H.c.}), \quad (1)$$

where $\omega_{\mathbf{k}} = c_0|\mathbf{k}|$, c_0 is the speed of light in vacuum, $\mathbf{d}$ and $\mathbf{r}_a$ are the transition dipole moment and spatial position of the TLS, σ_z is a Pauli matrix, σ_- is the corresponding Pauli lowering operator, and $a_{\mathbf{k},\lambda}$ are photonic annihilation operators for scattering modes of wave vector $\mathbf{k}$ and polarization λ. For the rest of this work we drop the mode index $\mathbf{k}$ from $\omega_{\mathbf{k}}$, making the dispersion relation implicit. We note here that we do not consider any nonradiative losses of the TLS, as we assume them to be negligible compared to radiative decay. However, they can be introduced in the usual manner via a thermal bath without affecting the following considerations. The electromagnetic field modes are of the form [35],

$$\mathbf{E}_{\mathbf{k},\lambda}(\mathbf{r}) = \sqrt{\frac{\hbar\omega}{(2\pi)^3 2\epsilon_0\epsilon_b}} \hat{\mathbf{e}}_{\mathbf{k},\lambda} e^{i\mathbf{k}\cdot\mathbf{r}} + \mathbf{E}^{(s)}_{\mathbf{k},\lambda}(\mathbf{r}), \quad (2)$$

where ϵ_b is the permittivity of the background medium and $\mathbf{E}^{(s)}_{\mathbf{k},\lambda}(\mathbf{r})$ is the scattered field, which arises due to the spatially inhomogeneous dielectric function $\epsilon(\mathbf{r},\omega)$. The photonic operators satisfy the usual harmonic oscillator commutation relations,

$$[a_{\mathbf{k},\lambda}, a^\dagger_{\mathbf{k}',\lambda'}] = \delta(\mathbf{k} - \mathbf{k}')\delta_{\lambda\lambda'}, \quad (3)$$

with all other commutators vanishing. The description offered by the Hamiltonian in (1) is exact and general, but usually not tractable. Luckily, many problems of practical interest feature electromagnetic resonances, i.e., the electromagnetic field at the position of the TLS only takes on appreciable values over a set of comparatively narrow frequency ranges. In the following we will assume the existence of a single resonant electromagnetic mode in the vicinity of the transition frequency ω_a of the TLS. All other modes are assumed to be far detuned when compared to the characteristic resonance linewidth and, therefore, can be ignored. In practice most resonances are well described by a Lorentzian lineshape [36–38] and we, therefore, assume

$$\mathbf{E}_{\mathbf{k},\lambda}(\mathbf{r}_a) = \mathbf{E}_0(\mathbf{r}_a)\delta_{\lambda,\lambda_0}\sqrt{\frac{\Gamma}{2\pi}}\frac{g(\hat{\mathbf{k}})}{\omega - \omega_0 - \mathrm{i}\Gamma/2}, \quad (4)$$

where ω_0 is the central frequency, Γ is the linewidth of the mode, and $g(\hat{\mathbf{k}})$ describes the angular dependence. In the following, we always assume that the resonance is sharp, i.e., $\Gamma \ll \omega_0$. This is the case for most practical scenarios involving nonabsorptive cavities, since long lifetimes are generally advantageous for cQED applications and very high-Q cavities are widely available [39–42]. The Kronecker delta $\delta_{\lambda,\lambda_0}$ signifies that there is indeed only one electromagnetic mode and not two degenerate ones of different polarization. This constitutes no limitation since in practice one can describe the polarizations in the coupled-uncoupled basis. Since only one polarization couples to the TLS we shall drop polarization indices λ from here on. In the following, we will show that the above assumptions allow us to reduce the exact Hamiltonian to an approximate but tractable form, with the only condition for the approximation to hold being $\Gamma \ll \omega_0$. Consequently, the error introduced by this approximation is of order $O(\Gamma/\omega_0)$, i.e., inversely proportional to the cavity Q factor. We note here without providing explicit proof, that our formalism can be generalized to scenarios that involve either multiple cavity modes, multiple TLSs, multilevel systems, different multipolar atomic transitions, ultrastrong coupling, or combinations thereof. It is even possible to consider non-Lorentzian modes, if the modified spectrum can be described as the product of a Lorentzian with an analytical function. However, such generalizations are beyond the scope of the present work.

Taking (4) into account, we can derive evolution equations in the Heisenberg picture for the operators from (1) that read as

$$\dot{a}_{\mathbf{k}} = -i\omega a_{\mathbf{k}} - i\frac{\mathbf{E}_0(\mathbf{r}_a)\cdot\mathbf{d}}{\hbar}\sqrt{\frac{\Gamma}{2\pi}}\frac{g(\hat{\mathbf{k}})}{\omega - \omega_0 - i\Gamma/2}\sigma_-, \quad (5)$$

$$\dot{\sigma}_- = -i\omega_a\sigma_- + i\frac{\mathbf{E}^*_0(\mathbf{r}_a)\cdot\mathbf{d}^*}{\hbar} \times \int d^3k \sqrt{\frac{\Gamma}{2\pi}}\frac{g^*(\hat{\mathbf{k}})}{\omega - \omega_0 + i\Gamma/2}\sigma_z a_{\mathbf{k}}. \quad (6)$$

Inspection of (6) motivates the following definition of a resonant mode annihilation operator:

$$a := \int \frac{c_0^{3/2} d^3k}{\sqrt{G\omega}} \sqrt{\frac{\Gamma}{2\pi}} \frac{g^*(\hat{\mathbf{k}})}{\omega - \omega_0 + i\Gamma/2} a_{\mathbf{k}}, \tag{7}$$

$$G := \int d\Omega_k |g(\hat{\mathbf{k}})|^2, \tag{8}$$

where $d\Omega_k = \sin(\theta_k) d\theta_k d\varphi_k$ denotes integration over solid angles. Please note that the normalization constant G in (8) is chosen in such a way that the harmonic oscillator commutation relations are satisfied,

$$[a, a^\dagger] = 1, \tag{9}$$

and all other commutators vanishing. The original evolution Eqs. (5) and (6) can now be rewritten in terms of the newly defined resonant mode operators. The process is detailed in Appendix A and the results read

$$\dot{a} = (-i\omega_0 - \Gamma/2)a - i\kappa\sigma_- - iF_0, \tag{10}$$

$$\dot{\sigma}_- = -i\omega_a\sigma_- + i\kappa^*\left(1 - i\frac{\Gamma}{2\omega_0}\right)\sigma_z a + i\frac{\kappa^*}{\omega_0}\sigma_z F_0, \tag{11}$$

where the effective light-matter coupling constant κ is defined as

$$\kappa = \frac{\sqrt{G}}{c_0^{3/2}}\omega_0 \frac{\mathbf{E}_0(\mathbf{r}_a)\cdot\mathbf{d}}{\hbar} = \sqrt{\frac{G}{4\pi}\frac{\pi}{c_0^{3/2}}\omega_0}\sqrt{2\Gamma}\frac{\mathbf{E}_{\max}(\mathbf{r}_a)\cdot\mathbf{d}}{\hbar}, \tag{12}$$

with $\mathbf{E}_{\max}(\mathbf{r}_a)$ being the field strength at resonance at the location of the TLS. To the best of our knowledge this is the first time that the coupling strength between an open cavity mode and a quantum emitter has been calculated from first principles rather than from phenomenological considerations, e.g., by defining a mode volume for open cavities *ad hoc* [43]. The new operator F_0 appearing in (10) and (11) belongs to a family of operators defined as

$$F_n := \int \frac{d^3k}{\omega} \frac{c_0^{3/2}}{\sqrt{G}} \sqrt{\frac{\Gamma}{2\pi}} g^*(\hat{\mathbf{k}})(\omega - \omega_0)^n a_{\mathbf{k}}. \tag{13}$$

Comparing (7) and (13), we notice that the operators F_n are not associated with the resonance mode at ω_0, but rather with a broad range of frequencies. For this reason we will call F_n noise operators from here on. Please note that these noise operators were retrieved without having introduced a thermal bath.

In order to construct a closed system of equations, we need equations of motion for F_n. The derivation of these is detailed in Appendix B. The results read

$$\dot{F}_n = -i\omega_0 F_n - iF_{n+1}. \tag{14}$$

The set of Eqs. (10), (11), and (14) is closed in the sense that equations of motion for all operators of the system can be inferred and can, therefore, be used in its current form to describe the system dynamics. In the spirit of the theory of open quantum systems, however, we wish to find a set of equations that only contains the system operators a and σ_- as dynamic quantities.

As is detailed in Appendix C, the equations of motion of the noise operators (14) can be formally solved without further approximations to yield

$$F_n(t) = e^{-i\omega_0 t} \sum_{m=0}^{\infty} \frac{(-it)^m}{m!} F_{m+n}(0), \tag{15}$$

where the form of $F_m(0)$ defines the type of illumination, as can be seen from (13) and the examples of Sec. III. Please note that the operators $F_n(t)$ can be interpreted as input parameters, because (15) tells us that there exists no backaction from the system. Insertion of (15) into (10) and (11) now yields a set of equations of motion for the operators a and σ_-, which is closed in the sense that equations of motion for all system operators can be derived from them. The lowest order input operator serves as a pump term,

$$\dot{a}(t) = (-i\omega_0 - \Gamma/2)a(t) - i\kappa\sigma_-(t) - iF_0(t), \tag{16}$$

$$\dot{\sigma}_-(t) = -i\omega_a\sigma_-(t) + i\kappa^*\left(1 - i\frac{\Gamma}{2\omega_0}\right)\sigma_z(t)a(t) + i\frac{\kappa^*}{\omega_0}\sigma_z(t)F_0(t). \tag{17}$$

Using (16) and (17) allows one to find equations of motion for all observables of the system, i.e., expectation values of arbitrary operators. The initial values of operators containing a and F_n can be inferred from the initial state of the quantized electrodynamic field using the definitions (7) and (13). Instead of the simultaneous coupling of the system to a larger number of bath operators, the operators that describe the evolution of our actual system are only coupled to one noise operator. In a sequential type of process, each noise operator then couples to the next.

We have now succeeded not only at describing the internal quantum dynamics of a cavity with radiative decay, but also at linking it to the external field via the noise operator $F_0(t)$. In the following section, this link will be used to formulate an input-output scheme for scattering modes capable of describing real experimental setups.

III. MULTIMODE INPUT-OUTPUT FORMALISM

At this point we have formulated the problem of evolution of a cavity-matter system in a rigorous manner. However, this is not yet sufficient to make predictions about actual experiments. A typical quantum optical experiment consists of probing an optical system with a beam of light and measuring the outgoing radiation. In the following, we will establish a quantitative relation between the different parts of the systems. What makes this challenging is the infinite amount of possible input and output channels, one for each photon momentum. However, since the cavity and noise operators of our model can be expanded in terms of scattering operators according to Eqs. (7) and (13), we are able to bridge the gap between internal and far field states directly without introducing additional coupling mechanisms or determining transfer operators. This enables an accurate calculation of any given output channel for illuminating fields of arbitrary angular and spectral composition.

A. Input operators

Most experimental illumination schemes in optics use a light beam with a waist diameter much larger than the system under consideration. For all practical purposes such a light beam can be considered as a plane wave with the wave vector pointing along the beam axis. For this reason, the following subsections describe how an illumination with a coherent plane wave can be described by means of the noise operator $F_0(t)$.

1. Continuous pumping

One of the most common pumping schemes is excitation by a continuous laser beam, i.e., the incident light is monochromatic, coherent, and polarized. In terms of the scattering eigenmode operators this means

$$\langle a_{\mathbf{k}}(t=0)\rangle = \alpha_P \delta(\mathbf{k}-\mathbf{k}_P), \tag{18}$$

where α_P is the amplitude and $\mathbf{k}_P$ is the wave vector of the laser beam. Using (13) one finds

$$\langle F_n(0)\rangle = C(\omega_P-\omega_0)^n, \quad C := \frac{c_0^{3/2}}{\omega_P}\sqrt{\frac{\Gamma}{2\pi G}}\, g(\hat{\mathbf{k}}_P)\alpha_P. \tag{19}$$

Substitution of (19) into the expectation value of (15) now yields

$$\langle F_0(t)\rangle = Ce^{-i\omega_0 t}\sum_{n=0}^{\infty}\frac{(-i(\omega_P-\omega_0)t)^n}{n!} = Ce^{-i\omega_0 t}e^{-i(\omega_P-\omega_0)t} = Ce^{-i\omega_P t}. \tag{20}$$

Equation (20) implies that the equation of motion for a contains a pump term of constant amplitude, which oscillates at the pump laser frequency. Terms of this form have been routinely employed when discussing driven quantum systems, but now we actually have the means to quantify the relation between laser intensity and pump strength.

2. Pulsed pumping

Assume now that the laser used to pump the system is not continuous, but pulsed. This means that a range of frequencies centered around the laser frequency is excited according to

$$\langle a_{\mathbf{k}}\rangle = \alpha_P\delta(\hat{\mathbf{k}}-\hat{\mathbf{k}}_P)e^{-\Delta^2(\omega-\omega_0)^2}. \tag{21}$$

Inserting this into the expectation value of (13) leads to

$$\langle F_n(0)\rangle = C\int_0^{\infty} d\omega\omega(\omega-\omega_0)^n e^{-\Delta^2(\omega-\omega_0)^2}, \quad C = c_0^{3/2}\sqrt{\frac{\Gamma}{2\pi G}}\, g^*(\hat{\mathbf{k}}_P)\alpha_P. \tag{22}$$

The frequency integral can be evaluated after extending the lower integration boundary to $-\infty$ to yield [44]

$$\langle F_{2n}(0)\rangle = C\omega_0\sqrt{\pi}\frac{(2n-1)!!}{2^n\Delta^{2n+1}}, \tag{23}$$

$$\langle F_{2n+1}(0)\rangle = C\sqrt{\pi}\frac{(2n+1)!!}{2^{n+1}\Delta^{2n+3}}, \tag{24}$$

where the double factorial is defined as

$$n!! = \prod_{k=0}^{\lceil n/2\rceil -1}(n-2k). \tag{25}$$

Insertion of (23) and (24) into the expectation value of (15) now leads to

$$\begin{aligned}\langle F_0(t)\rangle &= e^{-i\omega_0 t}\sum_{n=0}^{\infty}\frac{(-it)^n}{(2n)!}\langle F_{2n}(0)\rangle + e^{-i\omega_0 t}\sum_{n=0}^{\infty}\frac{(-it)^{2n+1}}{(2n+1)!}\langle F_{2n+1}(0)\rangle \\ &= e^{-i\omega_0 t}\frac{\sqrt{\pi}C\omega_0}{\Delta}\sum_{n=0}^{\infty}\left(-\frac{t^2}{2\Delta^2}\right)\frac{(2n-1)!!}{(2n)!} + e^{-i\omega_0 t}\sqrt{\pi}C\left(\frac{-it}{2\Delta^3}\right)\sum_{n=0}^{\infty}\left(-\frac{t^2}{2\Delta^2}\right)\frac{(2n+1)!!}{(2n+1)!} \\ &= e^{-i\omega_0 t}\frac{\sqrt{\pi}C\omega_0}{\Delta}\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{t^2}{(2\Delta)^2}\right) + e^{-i\omega_0 t}\sqrt{\pi}C\left(\frac{-it}{2\Delta^3}\right)\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{t^2}{(2\Delta)^2}\right) \\ &= e^{-i\omega_0 t}\frac{\sqrt{\pi}C\omega_0}{\Delta}\left[1-i\frac{t}{2\omega_0\Delta^2}\right]e^{\left(-\frac{t^2}{(2\Delta)^2}\right)} \\ &\approx \frac{\sqrt{\pi}C\omega_0}{\Delta}e^{\left(-\frac{t^2}{(2\Delta)^2}\right)}e^{-i\left(\omega_0-\frac{1}{2\omega_0\Delta^2}\right)t},\end{aligned} \tag{26}$$

where in the last step $\omega\Delta \ll 1$ was assumed, i.e., the Gaussian envelope contains many oscillations of the field. We once again arrive at the expected result, namely that the pump term gains a Gaussian envelope with a temporal spread equal to the inverse of the frequency spread.

B. Output operators

Up to this point we have only been concerned with the internal dynamics of the system under external irradiation. But in order to describe actual experiments, we also need to consider the dynamics of output states, that are experimentally accessible via detectors. If all detectors are placed in the far field, have sufficiently small apertures, and are sensitive to only a narrow frequency range, the output modes we are interested in are of the form,

$$\mathbf{E}^{(\mathrm{out})}_{\mathbf{k},\lambda}(\mathbf{r}) = C_{\mathbf{k},\lambda}\hat{\mathbf{e}}_{\mathbf{k},\lambda}e^{i\mathbf{k}\cdot\mathbf{r}} + \mathbf{E}^{(i)}_{\mathbf{k},\lambda}(\mathbf{r}), \tag{27}$$

where $\mathbf{E}^{(i)}_{\mathbf{k},\lambda}(\mathbf{r})$ only contains incoming field components. A solution to Maxwell's equations of this form describes a complicated scenario, in which the scattering responses of all incident fields interfere destructively in all but one spatial direction. This leads to a single plane wave as an outgoing field.

One should of course ask whether solutions of the form given by (27) exist and how one can find them. In order to answer these questions, we first recall that the macroscopic Maxwell equations are invariant under time reversal, if no absorption losses are present [45]. This means that the system

Hamiltonian is invariant under the antiunitary time-reversal operator T [46]:

$$THT = H. \tag{28}$$

We can, therefore, exchange the electric field for its time-reversed counterpart, without changing the structure of the Hamiltonian. Therefore, the Heisenberg equations of motion also keep their form under time reversal. The time-reversed electric field operator reads

$$\begin{aligned} T\mathbf{E}(\mathbf{r})T &= \int d^3k[T\mathbf{E}_{\mathbf{k},\lambda}(\mathbf{r})a_{\mathbf{k}}T + \text{H.c.}] \\ &= \int d^3k[\mathbf{E}^*_{\mathbf{k},\lambda}(\mathbf{r})Ta_{\mathbf{k}}T + \text{H.c.}] \\ &= \int d^3k[\mathbf{E}^*_{-\mathbf{k},\lambda}(\mathbf{r})Ta_{-\mathbf{k}}T + \text{H.c.}]. \end{aligned} \tag{29}$$

We now define the time-reversed photon operators,

$$a^{(\text{out})}_{\mathbf{k}} = Ta_{-\mathbf{k}}T, \quad \left(a^{(\text{out})}_{\mathbf{k}}\right)^\dagger = Ta^\dagger_{-\mathbf{k}}T, \tag{30}$$

which satisfy harmonic oscillator commutation relations. Inserting the scattering eigenmodes from (2) into (29), yields the time-reversed field modes,

$$\mathbf{E}^*_{-\mathbf{k},\lambda}(\mathbf{r}) = C^*_{-\mathbf{k},\lambda}\hat{\mathbf{e}}^*_{-\mathbf{k},\lambda}e^{i\mathbf{k}\cdot\mathbf{r}} + \mathbf{E}^{(s)*}_{-\mathbf{k},\lambda}(\mathbf{r}). \tag{31}$$

We now see that the time-reversed scattering modes (31) are indeed of the form given by (27), since the complex conjugate of an incoming incident field is an outgoing scattered field. The relevant output modes are therefore the ones described by the operators $a^{(\text{out})}_{\mathbf{k}}$, which obey the Heisenberg equations,

$$\dot{a}^{(\text{out})}_{\mathbf{k}} = -i\omega a^{(\text{out})}_{\mathbf{k}} - i\frac{\mathbf{E}^*_0\cdot\mathbf{d}}{\hbar}\sqrt{\frac{\Gamma}{2\pi}}\frac{g^*(-\hat{\mathbf{k}})}{\omega-\omega_0+i\Gamma/2}\sigma_-. \tag{32}$$

The important thing to notice now is that the dynamics of σ_- can be calculated without referring to any output mode, so that σ_- can be treated as an external parameter in (32). This enables us to make output calculations without being required to keep track of an infinite number of operators.

To summarize this section, we have established a way to treat the internal dynamics of an open cavity under arbitrary illumination with only a small number of operators. Furthermore, we have established a way to relate the results of the internal calculations to the temporal dynamics of individual Fourier components of the far field. Examples of this procedure are demonstrated in the following section.

IV. EXAMPLE: RELATION TO JAYNES-CUMMINGS MODEL

In order to offer a verification of the theory developed above, we now turn to the task of retrieving the well-established Jaynes-Cummings model from our formalism. Consider an input state of the form,

$$|\Phi\rangle := \int d^3k\frac{c_0^{3/2}}{\omega\sqrt{G'}}g'(\hat{\mathbf{k}})\sqrt{\frac{\Gamma}{2\pi}}\frac{1}{\omega-\omega_0-\mathrm{i}\Gamma'/2}a^\dagger_{\mathbf{k}}|0\rangle, \tag{33}$$

where the normalization factor G' is defined as

$$G' := \int d\Omega_k|g'(\hat{\mathbf{k}})|^2. \tag{34}$$

It is easy to check that the initial state $|\Phi\rangle$ is properly normalized, i.e., $\langle\Phi|\Phi\rangle = 1$. We will now proceed to derive, from (16) and (17), the Heisenberg equations for the number operators $a^\dagger a$ and $\sigma_+\sigma_-$, as well as the appropriate initial conditions arising from the initial state $|\Phi\rangle$.

We first want to determine the effect of the zero-time noise operators $F_n(0)$ on the initial state $|\Phi\rangle$:

$$\begin{aligned} F_n(0)|\Phi\rangle = \int d^3k\frac{c_0^3}{\omega^2\sqrt{GG'}}g'(\hat{\mathbf{k}})g^* \\ \times(\hat{\mathbf{k}})\frac{\sqrt{\Gamma\Gamma'}}{2\pi}(\omega-\omega_0)^n\frac{1}{\omega-\omega_0-\mathrm{i}\Gamma'/2}|0\rangle. \end{aligned} \tag{35}$$

Comparing (35) with (B1), we see that the frequency integrals are formally identical. Since the frequency integral in (B1) vanishes, as is demonstrated in Appendix B, we conclude that

$$F_n(0)|\Phi\rangle = 0. \tag{36}$$

But since the zero order noise operator $F_0(t)$ is of the form (15) at all times, we conclude that

$$F_0(t)|\Phi\rangle = 0. \tag{37}$$

Next we consider the action of the cavity operator at zero time on the initial state:

$$\begin{aligned} a(0)|\Phi\rangle &= \int d\Omega_k\frac{g'(\hat{\mathbf{k}})}{\sqrt{G'}}\frac{g^*(\hat{\mathbf{k}})}{\sqrt{G}}\frac{\sqrt{\Gamma\Gamma'}}{2\pi} \\ &\quad\times\int d\omega\frac{1}{\omega-\omega_0-i\Gamma'/2}\frac{1}{\omega-\omega_0+i\Gamma/2}|0\rangle \\ &= \left(\frac{g}{\sqrt{G}} * \frac{g'}{\sqrt{G'}}\right)\frac{\sqrt{\Gamma\Gamma'}}{(\Gamma+\Gamma')/2}|0\rangle, \end{aligned} \tag{38}$$

where the scalar product between two angular functions is defined as

$$a * b := \int d\Omega_k[a(\hat{\mathbf{k}})]^* b(\hat{\mathbf{k}}). \tag{39}$$

From (38) it is now easy to obtain the initial photon number in the cavity,

$$\langle\Phi|a^\dagger(0)a(0)|\Phi\rangle = \left[\frac{g}{\sqrt{G}} * \frac{g'}{\sqrt{G'}}\right]^2\left[\frac{\sqrt{\Gamma\Gamma'}}{(\Gamma+\Gamma')/2}\right]^2. \tag{40}$$

Please note that due to the Cauchy-Schwarz inequality [47],

$$\left[\frac{g}{\sqrt{G}} * \frac{g'}{\sqrt{G'}}\right]^2 \leqslant \left(\frac{g}{\sqrt{G}} * \frac{g}{\sqrt{G}}\right)\left(\frac{g'}{\sqrt{G'}} * \frac{g'}{\sqrt{G'}}\right) = 1, \tag{41}$$

and that the geometric mean of two numbers is always smaller than the algebraic mean [47],

$$\frac{\sqrt{\Gamma\Gamma'}}{(\Gamma+\Gamma')/2} \leqslant 1. \tag{42}$$

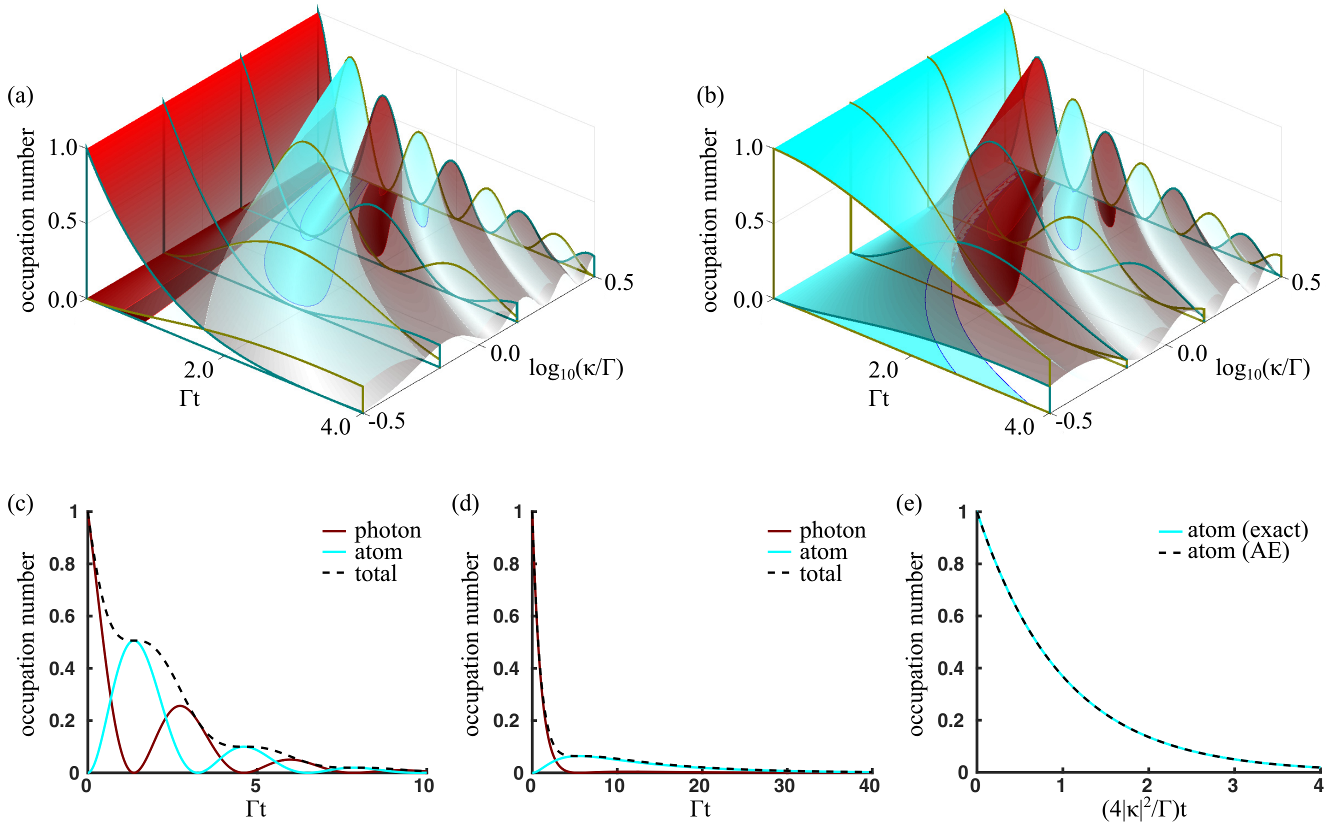

FIG. 2. Results of numerical simulations of (44) with a single-excitation input state. (a) Temporal system dynamics for different values of the coupling-to-decay-rate ratio κ/Γ when a single photon is injected into the system. Red (dark gray) colors correspond to the photonic and blue (light gray) colors to the atomic excited state. (b) Same as (a) but with the atom initially excited. (c) Strong coupling case $\kappa = \Gamma = 10^{-3}\omega_0$. The characteristic Rabi oscillations are clearly visible. (d) Weaker coupling $\kappa = 10^{-4}\omega_0$ and $\Gamma = 10^{-3}\omega_0$. Please note the different time scales of photonic and atomic decay. (e) Atomic decay for $\kappa = 10^{-5}\omega_0$ and $\Gamma = 10^{-3}\omega_0$ with the atom initially excited. The solid line refers to the numerical solution of the complete equations of motion and the dashed line is the approximate result obtained from adiabatic elimination (AE).

Therefore, it follows that the initial number of cavity photons in (40) is smaller than 1, as is required from the fact that only a single photon is incident.

We now turn to the problem of deriving equations of motion for the number operator expectation values $a^\dagger a$ and $\sigma_+\sigma_-$. To this end, we use $\sigma_z\sigma_- = -\sigma_-$, which follows from the Pauli algebra, and $\langle\sigma_z a^\dagger a\rangle = -\langle a^\dagger a\rangle$, which relies on the single photon condition. Along with (16), (17), and (37) this leads to

$$\frac{d}{dt}\langle a^\dagger a\rangle = -\Gamma\langle a^\dagger a\rangle + 2\mathrm{Im}[\kappa\langle a^\dagger\sigma_-\rangle], \tag{43}$$

$$\frac{d}{dt}\langle\sigma_+\sigma_-\rangle = -2\mathrm{Im}\left[\kappa\left(1 + i\frac{\Gamma}{2\omega_0}\right)\langle a^\dagger\sigma_-\rangle\right], \tag{44}$$

$$\frac{d}{dt}\langle a^\dagger\sigma_-\rangle = \left[-i(\omega_a - \omega_0) - \frac{\Gamma}{2}\right]\langle a^\dagger\sigma_-\rangle + i\kappa^*\langle\sigma_+\sigma_-\rangle - i\kappa^*\left(1 - \mathrm{i}\frac{\Gamma}{2\omega_0}\right)\langle a^\dagger a\rangle\,, \tag{45}$$

which indeed form a closed set of equations.

The above equations of motion are very similar to the ones derived from the Jaynes-Cummings (JC) Hamiltonian in the single photon case, except for the damping terms and the small asymmetry in the light-matter coupling. We, therefore, expect to find the well-known phenomena of the JC model when solving them. In order to see if these expectations are accurate, we present a number of numerical simulations for different system parameters. We assume without loss of generality that $g' = g$ and $\Gamma' = \Gamma$, so that the initial photon number in the cavity is exactly 1. Figure 2(a) shows the result for a system in the strong coupling regime, i.e., $\kappa > \Gamma/2$. The Rabi oscillations are clearly visible. The result for a weakly coupled system is shown in Fig. 2(b). The cavity photon is seen to decay rapidly, exciting the atom only weakly. The radiative decay of the atom is also visible.

In order to further investigate the decay of the atom via the cavity, we turn to the case where the atom is initially excited, while the electromagnetic field is in its ground state:

$$|\Phi\rangle = |0,e\rangle\,. \tag{46}$$

One can easily convince oneself that the equations of motion derived above still hold for the initial state in (46). The initial conditions, however, have to be changed to $\langle\sigma_+\sigma_-\rangle = 1$ and $\langle a^\dagger a\rangle = 0$. If we restrict ourselves to the case $\kappa \ll \Gamma$, i.e., weak coupling, we can obtain an approximate solution by adiabatic elimination (AE) of the cavity mode [48]. The AE result reads

$$\langle\sigma_+(t)\sigma_-(t)\rangle = \exp\left(-4\frac{|\kappa|^2}{\Gamma}t\right). \tag{47}$$

The numerical result for the complete set of equations of motion is shown in Fig. 2(c) and compared to the AE result. We see an excellent agreement between the approximate and the complete solution. The de-excitation of the atom due to its coupling to the electromagnetic vacuum is, of course, just the well-known phenomenon of spontaneous emission. The dependence of the emission time on the coupling strength is a manifestation of the Purcell effect [49].

If the decay of an initially excited atom is indeed due to spontaneous emission, then the energy should be transferred to the output modes described in Sec. III. In order to verify this, we first derive from (16), (17), and (32) the single-photon equations of motion,

$$\frac{d}{dt}\langle a_{\mathbf{k}}^{(\mathrm{out})\dagger}a_{\mathbf{k}}^{(\mathrm{out})}\rangle = 2\mathrm{Im}[\zeta_{\mathbf{k}}\langle a^{(\mathrm{out})\dagger}\sigma_-\rangle], \tag{48}$$

$$\frac{d}{dt}\langle a_{\mathbf{k}}^{(\mathrm{out})\dagger}\sigma_-\rangle = -i\Delta_{\mathbf{k}}\langle a_{\mathbf{k}}^{(\mathrm{out})\dagger}\sigma_-\rangle + i\zeta_{\mathbf{k}}^*\langle\sigma_+\sigma_-\rangle - i\kappa^*\langle a_{\mathbf{k}}^{(\mathrm{out})\dagger}a\rangle, \tag{49}$$

$$\frac{d}{dt}\langle a_{\mathbf{k}}^{(\mathrm{out})\dagger}a\rangle = \left(-i\Delta_{\mathbf{k}} - \frac{\Gamma}{2}\right)\langle a_{\mathbf{k}}^{(\mathrm{out})\dagger}a\rangle + i\zeta_{\mathbf{k}}^*\langle\sigma_+ a\rangle - i\kappa\langle a_{\mathbf{k}}^{(\mathrm{out})\dagger}\sigma_-\rangle, \tag{50}$$

where the definitions,

$$\zeta_{\mathbf{k}} := \frac{\mathbf{E}_0^*\cdot\mathbf{d}}{\hbar}\sqrt{\frac{\Gamma}{2\pi}}\frac{g^*(-\mathbf{k})}{\omega_{\mathbf{k}} - \omega_0 + i\Gamma/2}, \tag{51}$$

$$\Delta_{\mathbf{k}} := \omega_0 - \omega_{\mathbf{k}}, \tag{52}$$

were used. Since we assume $\kappa \ll \Gamma$, Eq. (50) can be adiabatically eliminated. Equation (49) then becomes

$$\frac{d}{dt}\langle a_{\mathbf{k}}^{(\mathrm{out})\dagger}\sigma_-\rangle = \left[-i\Delta_{\mathbf{k}} - \frac{|\kappa|^2}{i\Delta_{\mathbf{k}} + \Gamma/2}\right]\langle a_{\mathbf{k}}^{(\mathrm{out})\dagger}\sigma_-\rangle + i\zeta_{\mathbf{k}}^*\langle\sigma_+\sigma_-\rangle + \zeta_{\mathbf{k}}^*\frac{\kappa^*}{i\Delta_{\mathbf{k}} + \Gamma/2}\langle\sigma_+ a\rangle. \tag{53}$$

Using the AE result,

$$\langle\sigma_+ a\rangle \approx -i\frac{\kappa}{\Gamma/2}\langle\sigma_+\sigma_-\rangle \ll \langle\sigma_+\sigma_-\rangle, \tag{54}$$

Eq. (53) can be solved to yield

$$\langle a_{\mathbf{k}}^{(\mathrm{out})\dagger}\sigma_-\rangle_t = i\zeta_{\mathbf{k}}^*\int_0^t dt'\exp\left(\left[-i\Delta_{\mathbf{k}} - \frac{|\kappa|^2}{i\Delta_{\mathbf{k}} + \Gamma/2}\right][t-t']\right)\langle\sigma_+\sigma_-\rangle_{t'}, \tag{55}$$

where the indices of the expectation values denote the time of evaluation. Inserting (55) into (48) and integrating the resulting equation leads to

$$\langle a_{\mathbf{k}}^{(\mathrm{out})\dagger}a_{\mathbf{k}}^{(\mathrm{out})}\rangle = 2|\zeta_{\mathbf{k}}|^2\mathrm{Re}\left\{\int_0^t dt'\int_0^{t'} dt''\exp\left[\left(-i\Delta_{\mathbf{k}} - \frac{|\kappa|^2}{i\Delta_{\mathbf{k}} + \Gamma/2}\right)(t'-t'')\right]\langle\sigma_+\sigma_-\rangle_{t''}\right\}. \tag{56}$$

Using the AE result (47), the integrations in (56) can be easily performed. The result reads

$$\begin{aligned}\langle a_{\mathbf{k}}^{(\mathrm{out})\dagger}a_{\mathbf{k}}^{(\mathrm{out})}\rangle = |\zeta_{\mathbf{k}}|^2\mathrm{Re}\Bigg\{&\frac{\Gamma/2}{|\kappa|^2}\frac{1}{i\Delta_{\mathbf{k}} + |\kappa|^2/(i\Delta_{\mathbf{k}} + \Gamma/2) - 2|\kappa|^2/(\Gamma/2)}\left[1 - \exp\left(-2\frac{|\kappa|^2}{\Gamma/2}t\right)\right]\\ &+2\frac{1}{-i\Delta_{\mathbf{k}} - |\kappa|^2/(i\Delta_{\mathbf{k}} + \Gamma/2)}\frac{1}{i\Delta_{\mathbf{k}} + |\kappa|^2/(i\Delta_{\mathbf{k}} + \Gamma/2) - 2|\kappa|^2/(\Gamma/2)}\left[1 - \exp\left(-i\Delta_{\mathbf{k}}t - \frac{|\kappa|^2}{i\Delta_{\mathbf{k}} + \Gamma/2}t\right)\right]\Bigg\}.\end{aligned} \tag{57}$$

Considering the denominators in (57), it becomes clear that the number of photons with frequency detuning $\Delta_{\mathbf{k}}$ is small, unless $\Delta_{\mathbf{k}} \lesssim |\kappa|^2/(\Gamma/2)$. But due to the adiabatic assumption this means $\Delta_{\mathbf{k}} \ll \Gamma/2$, which allows for the following simplification of (57):

$$\langle a_{\mathbf{k}}^{(\mathrm{out})\dagger}a_{\mathbf{k}}^{(\mathrm{out})}\rangle \approx |\zeta_{\mathbf{k}}|^2\frac{1}{\Delta_{\mathbf{k}}^2 + [|\kappa|^2/(\Gamma/2)]^2}\left[1 - 2\cos(\Delta_{\mathbf{k}}t)\exp\left(-\frac{|\kappa|^2}{\Gamma/2}t\right) + \exp\left(-2\frac{|\kappa|^2}{\Gamma/2}t\right)\right]. \tag{58}$$

Likewise, the expression (51) for $\zeta_{\mathbf{k}}$ can be simplified to yield

$$|\zeta_{\mathbf{k}}|^2 \approx \left|\frac{\mathbf{E}_0\cdot\mathbf{d}}{\hbar}\right|\frac{\Gamma}{2\pi}\frac{|g(-\hat{\mathbf{k}})|^2}{(\Gamma/2)^2} = \frac{c_0^3}{\omega_0^2}\frac{|g(-\hat{\mathbf{k}})|^2}{G}\frac{|\kappa|^2}{\pi(\Gamma/2)}. \tag{59}$$

To acquire the total output photon number, the result in (58) has to be integrated over the wave vector. Using (59) and writing the cosine in terms of exponential functions, one finds

$$\int d^3k \left\langle a_{\mathbf{k}}^{(\mathrm{out})\dagger} a_{\mathbf{k}}^{(\mathrm{out})} \right\rangle \approx \int_{-\infty}^{\infty} d\omega \frac{\omega^2}{\omega_0^2} \frac{|\kappa|^2}{\pi\Gamma/2} \frac{1}{\Delta_{\mathbf{k}}^2 + [|\kappa|^2/(\Gamma/2)]^2} \times \left[1 + \exp\left(-2\frac{|\kappa|^2}{\Gamma/2}t\right) - \exp\left(-\frac{|\kappa|^2}{\Gamma/2}t + i\Delta_{\mathbf{k}}t\right) - \exp\left(-\frac{|\kappa|^2}{\Gamma/2}t - i\Delta_{\mathbf{k}}t\right)\right] \approx \left[1 - \exp\left(-2\frac{|\kappa|^2}{\Gamma/2}t\right)\right] = 1 - \langle\sigma_+\sigma_-\rangle_t, \tag{60}$$

where the integrations are performed with standard contour integral techniques. We, therefore, find that the total excitation number is a constant of motion and that the number of photons asymptotically reaches 1 for large times. This is in perfect agreement with the requirements of the original Hamiltonian (1), as well as the physical intuition regarding spontaneous emission.

V. CONCLUSION

We demonstrated the extension of the unitary internal cavity dynamics in QED by a rigorous quantum description of radiative decay. Unlike the canonical formulation based on a phenomenological coupling to a thermal bath, we have derived a description employing a chain of noise operators. Furthermore, we added input and output channels to the formalism that allow for a complete description of the dynamics: starting from an incident far field illumination, incorporating all unitary cavity related processes, and culminating in far field emission. Over the course of the derivation we made the following assumptions and approximations: Electric dipole and rotating wave approximation, single Lorentzian cavity mode, absence of nonradiative losses, and cavity Q factor large compared to one. The formalism can be generalized to lift all but the last of these requirements, but this is beyond the scope of this work.

The procedure suggested here consists of the following steps:

(1) Characterize the cavity mode classically by determining the resonance frequency and linewidth as well as the dependence on the illumination direction according to Eq. (4).

(2) Calculate the light-matter coupling constant from the properly normalized field strength (according to [35]), the cavity parameters, and the emitter's transition dipole moment according to Eq. (12).

(3) Evaluate $F_0(t)$ by means of the zero-time noise operators $F_n(0)$ and the initial photonic state according to Eq. (15).

(4) Calculate the internal dynamics of the cavity mode according to Eqs. (16) and (17).

(5) Solve the equations of motion for the output modes of interest according to Eq. (32).

We discussed the example of single-photon dynamics in a leaky cavity coupled to a single atom and retrieved the familiar Jaynes-Cummings model, but with the added possibility of calculating the far field dynamics. However, the formalism presented here can be employed to describe a multitude of different scenarios of light-matter interaction, which go beyond the simple Jaynes-Cummings model. We hope that this work will pave the way towards a more rigorous description of open optical cavities and their interaction with the far field.

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APPENDIX A: DERIVATION OF SYSTEM OPERATOR HEISENBERG EQUATIONS

Using the definition (7) together with the Heisenberg equation of motion (5), one derives

$$\begin{aligned}\dot{a} &= \int d^3k \frac{c_0^{3/2}}{\sqrt{G\omega}} \sqrt{\frac{\Gamma}{2\pi}} \frac{g^*(\hat{\mathbf{k}})}{\omega - \omega_0 + i\Gamma/2} \dot{a}_{\mathbf{k}} \\ &= \int d^3k \frac{c_0^{3/2}}{\sqrt{G\omega}} \sqrt{\frac{\Gamma}{2\pi}} \frac{g^*(\hat{\mathbf{k}})}{\omega - \omega_0 + i\Gamma/2} (-i\omega) a_{\mathbf{k}} - i\frac{\mathbf{E}_0 \cdot \mathbf{d}}{\hbar}\sigma_- \int d^3k \frac{c_0^{3/2}}{\sqrt{G\omega}} \frac{\Gamma}{2\pi} \frac{|g(\hat{\mathbf{k}})|^2}{(\omega-\omega_0)^2 + (\Gamma/2)^2} \\ &= -i \int d^3k \frac{c_0^{3/2}}{\sqrt{G\omega}} \sqrt{\frac{\Gamma}{2\pi}} g^*(\hat{\mathbf{k}}) a_{\mathbf{k}} \frac{\omega - \omega_0 + i\Gamma/2 + \omega_0 - i\Gamma/2}{\omega - \omega_0 + i\Gamma/2} - i\frac{\sqrt{G}}{c_0^{3/2}} \frac{\mathbf{E}_0 \cdot \mathbf{d}}{\hbar} \sigma_- \int_0^{\infty} d\omega \frac{\Gamma}{2\pi} \frac{\omega - \omega_0 + \omega_0}{(\omega-\omega_0)^2 + (\Gamma/2)^2}.\end{aligned} \tag{A1}$$

Since we assume $\Gamma \ll \omega_0$, the lower integration boundary in the second term can be approximately shifted to $-\infty$. Noticing that the part of the second integral antisymmetric in $\omega - \omega_0$ vanishes and splitting up the fracture under the first integral, one arrives at

$$\dot{a} = -i\int d^3k \frac{c_0^{3/2}}{\sqrt{G\omega}}\sqrt{\frac{\Gamma}{2\pi}}g^*(\hat{\mathbf{k}})a_{\mathbf{k}} + \left(-i\omega_0 - \frac{\Gamma}{2}\right)\int d^3k \frac{c_0^{3/2}}{\sqrt{G\omega}}\sqrt{\frac{\Gamma}{2\pi}}\frac{g^*(\hat{\mathbf{k}})}{\omega-\omega_0+i\Gamma/2}a_{\mathbf{k}} - i\frac{\sqrt{G}}{c_0^{3/2}}\frac{\mathbf{E}_0\cdot\mathbf{d}}{\hbar}\sigma_-. \tag{A2}$$

Employing the definitions in (7), (12), and (13) this becomes

$$\dot{a} \approx (-i\omega_0 - \Gamma/2)a - i\kappa\sigma_- - iF_0. \tag{A3}$$

Turning now to the atom dynamics, the equation of motion (6) can be written

$$\begin{aligned}\dot{\sigma}_- &= -i\omega_a\sigma_- + i\frac{\mathbf{E}_0^*\cdot\mathbf{d}^*}{\hbar}\int\frac{d^3k}{\omega}\sqrt{\frac{\Gamma}{2\pi}}\frac{g^*(\hat{\mathbf{k}})\omega}{\omega-\omega_0+i\Gamma/2}\sigma_z a_{\mathbf{k}}\\ &= -i\omega_a\sigma_- + i\frac{\mathbf{E}_0^*\cdot\mathbf{d}^*}{\hbar}\int\frac{d^3k}{\omega}\sqrt{\frac{\Gamma}{2\pi}}g^*(\hat{\mathbf{k}})\sigma_z a_{\mathbf{k}}\frac{\omega-\omega_0+i\Gamma/2+\omega_0-i\Gamma/2}{\omega-\omega_0+i\Gamma/2}\\ &= -i\omega_a\sigma_- + i\frac{\mathbf{E}_0^*\cdot\mathbf{d}^*}{\hbar}\sigma_z\int\frac{d^3k}{\omega}\sqrt{\frac{\Gamma}{2\pi}}g^*(\hat{\mathbf{k}})a_{\mathbf{k}} + i\left(\omega_0 - i\frac{\Gamma}{2}\right)\frac{\mathbf{E}_0^*\cdot\mathbf{d}^*}{\hbar}\sigma_z\int\frac{d^3k}{\omega}\sqrt{\frac{\Gamma}{2\pi}}\frac{g(\hat{\mathbf{k}})}{\omega-\omega_0+i\Gamma/2}a_{\mathbf{k}}.\end{aligned} \tag{A4}$$

Using once again the definitions in (7), (12), and (13) this can be written

$$\dot{\sigma}_- = -i\omega_a\sigma_- + i\frac{\mathbf{E}_0^*\cdot\mathbf{d}^*}{\hbar}\int d^3k\sqrt{\frac{\Gamma}{2\pi}}\frac{g^*(\hat{\mathbf{k}})}{\omega-\omega_0+i\Gamma/2}\sigma_z a_{\mathbf{k}}. \tag{A5}$$

APPENDIX B: DERIVATION OF NOISE OPERATOR HEISENBERG EQUATIONS

Using the definition in (13) together with the Heisenberg Eqs. (5) and (6) we arrive at

$$\begin{aligned}\dot{F}_n &= -i\omega_0 F_n - iF_{n+1} - i\frac{\kappa}{\omega_0}\frac{\Gamma}{2\pi}\sigma_-\\ &\quad\times\int_0^\infty d\omega\omega(\omega-\omega_0)^n\frac{1}{\omega-\omega_0-i\Gamma/2}.\end{aligned} \tag{B1}$$

As can be easily seen, the above frequency integral is highly divergent. This is due to the fact that we assumed a perfect Lorentzian frequency dependence of the electromagnetic field at the emitter position. In a real system, however, one would not expect this assumption to hold for frequencies far off-resonance. Especially for very high frequencies one expects rapid oscillations of the field strength, so that the high frequency contributions average out to zero. Equation (4) therefore has to be modified to take the off-resonance contributions into account. We do this by adding a Gaussian envelope that decays on time scales large compared to the Lorentzian linewidth Γ, but small compared to ω_0:

$$\begin{aligned}&\mathbf{E}_{\mathbf{k},\lambda}(\mathbf{r}_a) = \mathbf{E}_0\delta_{\lambda,\lambda_0}\sqrt{\frac{\Gamma}{2\pi}}\frac{g(\hat{\mathbf{k}})e^{-(\omega-\omega_0)^2/\beta^2}}{\omega-\omega_0-i\Gamma/2},\\ &\Gamma \ll \beta \ll \omega_0.\end{aligned} \tag{B2}$$

Using (B2) instead of (4), the integral in (B1) becomes

$$\begin{aligned}&\int_0^\infty d\omega\omega(\omega-\omega_0)^n\frac{e^{-(\omega-\omega_0)^2/\beta^2}}{\omega-\omega_0+i\Gamma/2}\\ &\approx\int_{-\infty}^\infty d\omega\omega(\omega-\omega_0)^n\frac{e^{-(\omega-\omega_0)^2/\beta^2}}{\omega-\omega_0+i\Gamma/2},\end{aligned} \tag{B3}$$

where the lower integration boundary has been approximately extended to $-\infty$, since the exponential function decays much faster than any polynomial can grow.

The integral in (B3) can now be solved by contour integration techniques, if one introduces an auxiliary factor of $\exp(\pm i\epsilon\omega)$. But while ϵ can just be chosen to be infinitesimally small, the choice of sign in the exponent leads to very different results. This is due to the fact that the integrand only possesses a pole in the upper half-plane. Hence, in order to find a meaningful result we need to eliminate one of the two possibilities by physical reasoning. This is similar to choosing the retarded instead of the advanced Green's function, since the latter violates causality. However, in the current case it is not immediately obvious which solution is the unphysical one. After obtaining the solutions for both possible equations, it will be obvious which one to choose. For this reason we consider, for the moment, both possible solutions:

$$\dot{F}_n = -i\omega_0 F_n - iF_{n+1} \quad \text{(lower half-plane)}, \tag{B4}$$

$$\begin{aligned}\dot{F}_n &= -i\omega_0 F_n - iF_{n+1} - i\kappa\left[1 + i\frac{\Gamma}{2\omega_0}\right]\\ &\quad\times 2\left(i\frac{\Gamma}{2}\right)^{n+1} \quad \text{(upper half-plane)}.\end{aligned} \tag{B5}$$

The details of solving both of these equations will be presented in Appendix C, where we show that the solution for integration over the lower half-plane is the physical one.

APPENDIX C: SOLUTION OF NOISE OPERATOR HEISENBERG EQUATIONS

We start by considering (B4), for which the formal solution reads

$$F_n(t) = e^{-i\omega_0 t}F_n(0) - i\int_0^t dt' e^{-i\omega_0(t-t')}F_{n+1}(t'). \tag{C1}$$

Iteration of (C1) then leads to the form,

$$F_n(t) = e^{-i\omega_0 t}\sum_{m=0}^{\infty} F_{n+m}(t)(-i)^m I_m(t), \tag{C2}$$

$$I_m(t) = \int_0^t dt_1 \dots \int_0^{t_{m-1}} dt_m. \tag{C3}$$

The elements of the series $I_m(t)$ can be easily calculated by induction. First we notice that the base case,

$$I_0(t) = 1, \tag{C4}$$

is in agreement with the assumption,

$$I_m(t) = \frac{t^m}{m!}. \tag{C5}$$

We now proceed with the inductive step,

$$I_{m+1}(t) = \int_0^t dt_1 I_m(t_1) = \int_0^t dt_1 \frac{t_1^m}{m!} = \frac{t^{m+1}}{(m+1)!}, \tag{C6}$$

hence proving our assumption. Substitution of (C6) into (C2) gives the final result,

$$F_n(t) = e^{-i\omega_0 t}\sum_{m=0}^{\infty}\frac{(-it)^m}{m!}F_{n+m}(0). \tag{C7}$$

Now we need to solve the equation of motion (B5) in order to demonstrate its unphysical nature. Formally solving and then iterating the equation leads to

$$F_n(t) = e^{-i\omega_0 t}\sum_{m=0}^{\infty}\frac{(-it)^m}{n!}F_{n+m}(0) + \Gamma\kappa\left[1 + i\frac{\Gamma}{2\omega_0}\right]\sum_{m=0}^{\infty}\left(\frac{\Gamma}{2}\right)^m J_m(t;t), \tag{C8}$$

where the operator valued terms $J_n(t;t_0)$ read

$$J_n(t;t_0) = \int_0^t dt_1 \dots \int_0^{t_n} dt_{n+1} e^{i\omega_0(t_{n+1}-t_0)}\sigma_-(t_{n+1}). \tag{C9}$$

We can now use induction to calculate the values of $J_n(t;t)$. Since $J_0(t;t)$ is of the form,

$$J_0(t;t) = \int_0^t dt_1 e^{i\omega_0(t_1-t)}\sigma_-(t_1), \tag{C10}$$

the following induction hypothesis is consistent with the base case:

$$J_n(t;t_0) = \int_0^t dt' e^{i\omega_0(t'-t_0)}\sigma_-(t')\frac{(t-t')^n}{n!}. \tag{C11}$$

Performing the induction step is now straightforward,

$$\begin{aligned} J_{n+1}(t;t) &= \int_0^t dt'' J_n(t'';t) \\ &= \int_0^t dt'' \int_0^{t''} dt' e^{i\omega_0(t'-t)}\sigma_-(t')\frac{(t''-t')^n}{n!} \\ &= \int_0^t dt' e^{i\omega_0(t'-t)}\sigma_-(t')\int_0^t dt'' \frac{(t''-t')^n}{n!}\Theta(t''-t') \\ &= \int_0^t dt' e^{i\omega_0(t'-t)}\sigma_-(t')\int_{t'}^t dt'' \frac{(t''-t')^n}{n!} \\ &= \int_0^t dt' e^{i\omega_0(t'-t)}\sigma_-(t')\frac{(t-t')^{n+1}}{(n+1)!}, \end{aligned} \tag{C12}$$

which is of the required form. Inserting (C12) into (C8) now yields

$$F_n(t) = e^{-i\omega_0 t}\sum_{m=0}^{\infty}\frac{(-it)^m}{n!}F_{n+m}(0) + \kappa\left[1 + i\frac{\Gamma}{2\omega_0}\right]\Gamma\int_0^t dt' e^{(-i\omega_0+\Gamma/2)(t-t')}\sigma_-(t'), \tag{C13}$$

where the infinite sum was performed to yield an exponential function. Close inspection of (C13) reveals that the second term is divergent in time due to the factor $\exp[(\Gamma/2)t]$, which can be pulled in front of the integral. But this would mean that the noise operators grow without limit, driving the temperature of the system towards infinity. The equation of motion (B5) is therefore clearly unphysical.

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