Dynamic generation of light states with discrete symmetries

S. Cordero,* E. Nahmad-Achar, O. Castaños, and R. López-Peña

Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Apartado Postal 70-543, 04510 México Cd. Mx., Mexico

(Received 17 August 2017; published 5 January 2018)

A dynamic procedure is established within the generalized Tavis–Cummings model to generate light states with discrete point symmetries, given by the cyclic group $\mathcal{C}_n$. We consider arbitrary dipolar coupling strengths of the atoms with a one-mode electromagnetic field in a cavity. The method uses mainly the matter-field entanglement properties of the system, which can be extended to any number of three-level atoms. An initial state constituted by the superposition of two states with definite total excitation numbers, $|\psi\rangle_{M_1}$, and $|\psi\rangle_{M_2}$, is considered. It can be generated by the proper selection of the time of flight of an atom passing through the cavity. We demonstrate that the resulting Husimi function of the light is invariant under cyclic point transformations of order $n = |M_1 - M_2|$.

DOI: 10.1103/PhysRevA.97.013808

I. INTRODUCTION

QED in cavities is a very useful tool to explore, measure, and control quantized radiation fields and atomic systems coupled coherently within an electromagnetic resonator. Typically, a resonator is formed by two mirrors inside of which there is an atom at rest or passing through the cavity. For a single two-level atom, the system has four parameters, viz., the spontaneous emission, the quality factor of the cavity, the atom-field coupling strength, and the travel time in the cavity. The observed effects in QED cavities include modifications in the spontaneous emission rates, changes in the atomic energy spectrum, and the continuous matter-field interchange of energy [1–3].

Systems of N_a noninteracting two-level atoms or molecules confined in a small container compared with the radiation wavelength have been studied within the Dicke model. In this model, the dipolar interaction between the atoms and the field is considered [4]. For the one-atom case, the model (called the Jaynes–Cummings model) is exactly soluble with and without the rotating wave approximation (RWA) [5,6]. For an arbitrary number of two-level atoms or molecules in the RWA, the Tavis–Cummings (TC) model has been also solved analytically under resonant conditions [7].

The Jaynes–Cummings (JC) Hamiltonian takes into account only the reversible coherent evolution and describes a single two-level system strongly coupled to a single-mode electromagnetic field [5]. The energy spectrum consists of an infinite ladder of doublets additionally to the ground state. For a total number of excitations, M, it is found that, in resonant conditions, the splitting of the levels is equal to $2\sqrt{M}\mu$, where μ denotes the atom-field coupling strength, and this allows us to observe the antibunching phenomenon. Other nonlinear effects in optics have been observed in the strong-coupling regime between multilevel atoms and a radiation field, within the electromagnetic induced transparency (EIT) medium scheme [8]. Another example in which it is necessary to consider three-level atoms is in the area of quantum information theory for the establishment of a quantum communication protocol in a quantum network [9].

Infinite superpositions of Fock photon states with discrete symmetries appear naturally as symmetry-adapted states in the TC and Dicke models [10]. Since the foundations of quantum mechanics, the coherent states have attracted the attention of the community because they minimize the Heisenberg uncertainty relations [11]. Later works have considered superpositions of even or odd coherent states [12] and of squeezed states [13–16], all of them describing nonclassical states of light because they have different statistical properties than the coherent states (which are usually called classical states of light [17–19]). The statistical properties of these infinite superpositions of Fock photon states carry irreducible representations of a finite group (called crystallized Schrödinger cats) have also been studied [20]. Their evolution with respect to quadratic Hamiltonians in the quadratures of the electromagnetic field has been investigated [21]. The proposals to generate this type of states can be grouped as follows: (i) nonlinear processes [22,23], (ii) nondemolition measurements [24,25], and (iii) field-atom interactions [26–28].

Our purpose in this work is to consider the dynamic generation of nonclassical states of light in a cavity which exhibits a cyclic point symmetry, because this will be reflected in their Husimi Q function. For a superposition of states with different excitation numbers, the point symmetry will be given by the group $\mathcal{C}_n$, where n is the (absolute value of the) difference between the total excitation numbers. This is shown to be valid for arbitrary coupling strengths between matter and radiation, detuning conditions, and an arbitrary number of atoms. The fact that this symmetry exists and is retained during evolution is important to the measuring the Q function: it may even be taken as a precision measurement test, in that the measurement must reflect the said symmetry.

Seeing photons as carriers of quantum information, a way to manipulate nonclassical states of light is to couple them to atoms in a strong-coupling regime, as is cavity-QED. Their time evolution will be a consequence of the quantization of the field inside the cavity and of their interaction with the atoms, and the best way to visualize this is through the Husimi Q

*sergio.cordero@nucleares.unam.mx

function. Techniques for measuring this function have been described, and the time evolution of Q for a coherent state in a nonlinear cavity is given [29].

Furthermore, the Wigner function of a quantum mode of light has been measured [30] to reveal the formation of nonclassical field states by implementing quantum Zeno dynamics in a quantum electrodynamics circuit architecture. This allows for a better control of the system phase space, a dynamic tailoring of the Hilbert space, so to speak. In circuits, signals are carried as currents and voltages; however, by coupling them to microwave signals one may distinguish a difference of a single photon [31], and may thus resolve photon number states. This is the circuit equivalent to atom-photon interactions in cavity QED and it would be interesting to test our results in this context, with the advantage of the possibility for having a stronger atom-photon coupling strength as compared with, say, quantum dots, alkaline atoms, and even Rydberg atoms.

The exploitation of the above possibilities should enable an easier path for quantum computing and quantum information processing. Not only that, it is well known that what can be measured, at best, in any attempt to obtain information on the simultaneous values of two conjugate variables, is the Q function [32].

More specifically, in this work we generate *finite* superpositions of photon number operator states with discrete symmetries given by the cyclic group $\mathcal{C}_n$ ($\mathcal{C}_n$ states), within the generalized Tavis–Cummings model (GTC), independently of the dipolar strengths and of the number of atoms. A possible experimental setup is also presented to generate these finite superposition states with a fixed difference of the total excitation number, $\Delta M = |M_1 - M_2|$, whose field part is invariant under point transformations of a cyclic group in n dimensions, $\mathcal{C}_n$, with $n = |M_1 - M_2|$. The stationary states of the GTC model, for the one-particle case, are given in analytic form together with the corresponding evolution operator. This operator is then used to study the dynamics of an arbitrary initial state by properly selecting the time of flight (tof) of the atom within the resonant cavity. In particular, we consider a superposition of two states with M_1 and M_2 values for the total excitation number. The results for the one-atom case can be extended to any number of atoms, under resonant conditions as well as under detuning.

The paper is organized as follows: in Sec. II we discuss the generalization of the TC model to consider atoms of three levels in each of the three configurations Ξ, Λ, and V. For the one-particle case, the dressed states are obtained in analytic form together with their atom-field quantum correlations measured through the calculation of the linear entropy. In Sec. III the evolution operator for the single-particle case is determined in analytic form for a given total number of excitations. This is used to study the behavior of the light sector for two different initial conditions, which requires the reduced density matrix for the light, and from this the calculation of the Husimi function. The formation of $\mathcal{C}_n$ states is established with this procedure. Section IV is dedicated to the evolution of an atom traversing a cavity, and to establishing the procedure to dynamically generate $\mathcal{C}_n$ states of light. In Sec. V the extension to consider any number of particles is considered. The conclusions are presented in Sec. VI together with some additional remarks.

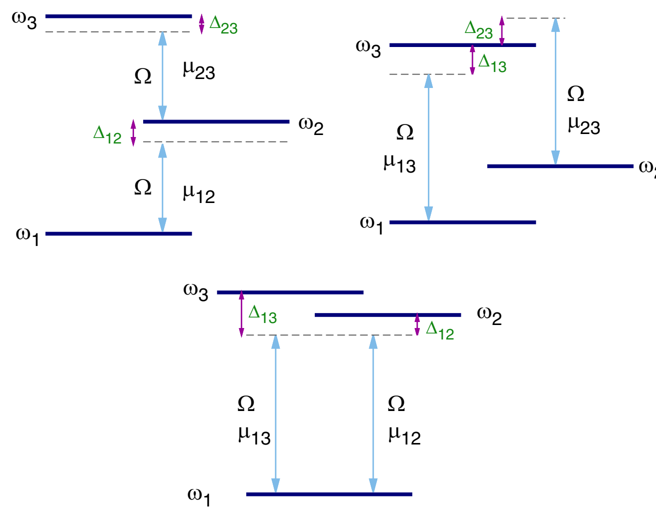

FIG. 1. Detuning parameters Δ_{ij} for the three-level atomic configurations Ξ, Λ, and V, respectively (see text for details).

II. TAVIS–CUMMINGS MODEL FOR THREE-LEVEL ATOMS

The Tavis–Cummings model describes a system of two-level atoms or molecules interacting dipolarly with a one-mode electromagnetic field of frequency Ω in the RWA [7]. This model has been used extensively to study quantum phase transitions as well as for applications in quantum information theory [33–36]. A natural extension of the model is to consider three-level atoms, and the Hamiltonian in this case takes the form [37,38]

$$\boldsymbol{H} = \hbar\Omega \boldsymbol{a}^\dagger \boldsymbol{a} + \hbar \sum_{j=1}^{3} \omega_j \boldsymbol{A}_{jj} - \frac{1}{\sqrt{N_a}} \sum_{i<j=2}^{3} \mu_{ij}(\boldsymbol{a}^\dagger \boldsymbol{A}_{ij} + \boldsymbol{a}\boldsymbol{A}_{ji}), \tag{1}$$

with the convention $\omega_1 \leqslant \omega_2 \leqslant \omega_3$ for the atomic frequencies. $\boldsymbol{A}_{jk}$ denotes the raising, lowering, and weight generators of the unitary algebra in three dimensions, which for identical particles can be realized in terms of bosonic operators as $\boldsymbol{A}_{jk} = \boldsymbol{b}_j^\dagger \boldsymbol{b}_k$. Thus, $\boldsymbol{A}_{jj}$ is the population operator for level j. $\boldsymbol{a}^\dagger, \boldsymbol{a}$ are the creation and annihilation photon operators, respectively. μ_{jk} is the matter-field coupling parameter between levels ω_j and ω_k, which is given by $\mu_{ij} = \pm\hbar|\omega_i - \omega_j| d_{jk}\sqrt{2\pi\rho_m/\hbar\Omega}$, with d_{jk} being the matrix elements of the dipolar operator.

In the nonresonant case we define detuning parameters Δ_{ij} in terms of the atomic frequencies $\omega_{ij} = \omega_i - \omega_j$ as (cf. Fig. 1)

$$\Xi : \omega_{21} = \Omega + \Delta_{12}, \quad \omega_{32} = \Omega + \Delta_{23}, \quad \omega_{31} = 2\Omega + (\Delta_{12} + \Delta_{23});$$
$$V : \omega_{21} = \Omega + \Delta_{12}, \quad \omega_{31} = \Omega + \Delta_{13}, \quad \omega_{32} = \Delta_{13} - \Delta_{12};$$
$$\Lambda : \omega_{31} = \Omega + \Delta_{13}, \quad \omega_{32} = \Omega - \Delta_{23}, \quad \omega_{21} = \Delta_{13} + \Delta_{23}.$$

In the RWA the operator that measures the total number of excitations

$$\boldsymbol{M}_X = \boldsymbol{a}^\dagger \boldsymbol{a} + \lambda_2 \boldsymbol{A}_{22} + \lambda_3 \boldsymbol{A}_{33} \tag{2}$$

is a constant of motion. It is dependent on the atomic configuration: the label X indicates the atomic configuration Ξ, V, and Λ, each one with $(\lambda_2,\lambda_3) = \{(1,2),(1,1),(0,1)\}$, respectively. In particular, in this work we will be interested in the case $\boldsymbol{M}_X \geqslant \lambda_3 N_a$, then the dimension of the Hilbert space for all the configurations is $(N_a+1)(N_a+2)/2$ [39]. This is equivalent to the Hilbert space of a three-dimensional harmonic oscillator with N_a quanta of energy.

The basis states

$$|\nu; N_a q r\rangle = |\nu\rangle \otimes \sqrt{\frac{r!}{(N_a-q)!(q-r)!N_a!}} \times \boldsymbol{A}_{31}^{N_a-q}\boldsymbol{A}_{21}^{q-r}|N_a N_a N_a\rangle \tag{3}$$

are the tensorial product of a Fock state $|\nu\rangle$, associated with the number of photons, and the Gelfand–Tsetlin state of N_a identical particles, $|N_a q r\rangle$. The lowest energy state is determined by $\boldsymbol{A}_{jk}|N_a N_a N_a\rangle = 0$, for all $k > 1$. Here, r denotes the eigenvalue of $\boldsymbol{A}_{11}$, i.e., the population of the lowest energy level ω_1, and q denotes the sum of the populations of the two lowest energy levels. Notice that these are eigenstates of $\boldsymbol{M}_X$ with eigenvalue $M_X = \nu + \lambda_2(q-r) + \lambda_3(N_a-q)$. Thus M_X determines the maximum number of photons that the cavity may contain.

A. One-particle case

For $N_a = 1$ all matter and field observables may be written as 3×3 matrices in the basis

$$|1\rangle := |M_X-\lambda_3; 1\,0\,0\rangle, \quad |2\rangle := |M_X-\lambda_2; 1\,1\,0\rangle, \quad |3\rangle := |M_X; 1\,1\,1\rangle, \tag{4}$$

where $M_X \geqslant \lambda_3$.

In general, analytic expressions for the eigensystem of the Hamiltonian can be obtained for several cases, including

(i) resonant conditions:

$$\Delta_{12} = \Delta_{23} = \Delta_{13} = 0 \Rightarrow \omega_{31} = \lambda_3\Omega, \quad \omega_{21} = \lambda_2\Omega.$$

(ii) the following detuning conditions:

$$\Xi : \Delta_{12} + \Delta_{23} = 0, \quad V : \Delta_{12} - \Delta_{13} = 0, \quad \Lambda : \Delta_{13} - \Delta_{23} = 0.$$

The obtained energy spectrum is constituted by an infinite ladder of three-level steps. Defining

$$\mathcal{E}_X = \sqrt{(\Delta_X/2)^2 + \Omega_X^2}, \tag{5}$$

each step is determined by

$$E_\pm = M_X + \Delta_X/2 \pm \mathcal{E}_X, \quad E_0 = M_X, \tag{6}$$

with detuning values $(\Delta_\Xi,\Delta_V,\Delta_\Lambda) \equiv (\Delta_{12},\Delta_{12},\Delta_{13})$, and the frequencies Ω_X for the different configurations are given by

$$\Omega_\Xi = \sqrt{M_\Xi \mu_{12}^2 + (M_\Xi - 1)\mu_{23}^2}, \tag{7a}$$

$$\Omega_V = \sqrt{M_V\left(\mu_{12}^2 + \mu_{13}^2\right)}, \tag{7b}$$

$$\Omega_\Lambda = \sqrt{M_\Lambda\left(\mu_{13}^2 + \mu_{23}^2\right)}. \tag{7c}$$

The eigenstates of the Hamiltonian, called dressed states, can be determined in analytic form as a linear combination of $|\nu\rangle \otimes |1qr\rangle \equiv |\nu; 1qr\rangle$. Expressions for the dressed states for the Ξ, V, and Λ configurations are given in Appendix A.

In contrast to the bare states, which are separable, the dressed states exhibit entanglement between matter and field. It is customary to measure this entanglement through the von Neumann entropy of the reduced density matrix; for our purposes, in order to show entanglement it is sufficient to calculate whether the linear entropy $S_L = 1 - \mathrm{Tr}(\rho_F^2)$, where ρ_F is the reduced density matrix for the field, is different from zero. For the Ξ configuration under the condition of opposite detunings $\Delta_{23} = -\Delta_{12}$, using the expressions in Appendix A, we get

$$S_L^{(\pm)} = 1 - \frac{1}{\mathcal{E}_\Xi^4\left(2 \pm \frac{\Delta_{12}}{\mathcal{E}_\Xi}\right)^2}\left\{(M_\Xi - 1)^2\mu_{23}^4 + M_\Xi^2\mu_{12}^4 + \left(\frac{\Delta_{12}}{2} \pm \mathcal{E}_\Xi\right)^4\right\}, \tag{8a}$$

$$S_L^{(0)} = \frac{2M_\Xi(M_\Xi-1)\mu_{12}^2\mu_{23}^2}{\left(M_\Xi\mu_{12}^2 + (M_\Xi-1)\mu_{23}^2\right)^2}. \tag{8b}$$

Notice that $S_L^+ \to S_L^-$ under the change of sign of the detuning parameter.

For the V configuration, with $\Delta_{13} = \Delta_{12}$, we get $S_L^{(0)} = 0$ and

$$S_L^{(\pm)} = 1 - \frac{1}{\mathcal{E}_V^4\left(2 \mp \frac{\Delta_{12}}{\mathcal{E}_V}\right)^2}\left\{M_V^2\left(\mu_{13}^2 + \mu_{12}^2\right)^2 + \left(\mp\frac{\Delta_{12}}{2} + \mathcal{E}_V\right)^4\right\}, \tag{9}$$

while for the Λ configuration, under the condition of equal detuning $\Delta_{23} = \Delta_{13}$, one has only to do the replacements $V \to \Lambda$, $\mu_{13} \to \mu_{23}$, and $\Delta_{12} \to -\Delta_{13}$ in expression (9). Under resonant conditions both configurations yield the simplified value $S_L^{(\pm)} = \frac{1}{2}$, independent of the total number of excitations. These expressions may also allow us to distinguish between the different configurations.

III. EVOLUTION OPERATOR

For the one-atom case, the evolution operator associated with the Hamiltonian (1) can be obtained in analytic form as $U(\tau) = e^{-iM\tau}U_I(\tau)$, with $\tau = \Omega t$, and $U_I(\tau)$ being the evolution operator in the interaction picture, given by

$$\begin{aligned}\boldsymbol{U}_I(\tau) = {}& U_{11}(\tau)|M-\lambda_3;100\rangle\langle M-\lambda_3;100| + U_{12}(\tau)|M-\lambda_3;100\rangle\langle M-\lambda_2;110| \\ &+ U_{13}(\tau)|M-\lambda_3;100\rangle\langle M;111| \\ &+ U_{21}(\tau)|M-\lambda_2;110\rangle\langle M-\lambda_3;100| + U_{22}(\tau)|M-\lambda_2;110\rangle\langle M-\lambda_2;110| \\ &+ U_{23}(\tau)|M-\lambda_2;110\rangle\langle M;111| \\ &+ U_{31}(\tau)|M;111\rangle\langle M-\lambda_3;100| \\ &+ U_{32}(\tau)|M;111\rangle\langle M-\lambda_2;110| \\ &+ U_{33}(\tau)|M;111\rangle\langle M;111|.\end{aligned} \tag{10}$$

The explicit form of the evolution operator depends of the considered atomic configuration through λ_2 and λ_3, and the corresponding matrix elements are given in Appendix B.

Given the experimental and technological advances, it is possible to prepare a resonant cavity with a definite number of photons [1]. It is then relevant to consider the evolution of an atom in its ground state in a QED cavity with ν_0 photons; at time τ the state is given by the expression

$$|\phi_{\nu_0}(\tau)\rangle = U_{13}(\tau)|\nu_0-\lambda_3;100\rangle + U_{23}(\tau)|\nu_0-\lambda_2;110\rangle + U_{33}(\tau)|\nu_0;111\rangle. \tag{11}$$

From this state we construct the corresponding density matrix and, by taking the partial trace with respect to the matter sector, one gets the reduced density matrix for the electromagnetic field

$$\rho_F(\nu_0,\tau) = |U_{13}(\tau)|^2|\nu_0-\lambda_3\rangle\langle\nu_0-\lambda_3| + |U_{23}(\tau)|^2|\nu_0-\lambda_2\rangle\langle\nu_0-\lambda_2| + |U_{33}(\tau)|^2|\nu_0\rangle\langle\nu_0|. \tag{12}$$

The probability to have ν photons in the cavity at time τ then takes the form

$$\mathcal{P}(\nu,\tau) = |U_{13}(\tau)|^2\delta_{\nu,\nu_0-\lambda_3} + |U_{23}(\tau)|^2\delta_{\nu,\nu_0-\lambda_2} + |U_{33}(\tau)|^2\delta_{\nu,\nu_0}. \tag{13}$$

This yields a Husimi function for the state of light which is a linear combination of Poissonian distributions, weighted by the corresponding probability to find ν photons. An animation of the evolution of the state of light inside the cavity, as given by this Husimi function, for a total excitation number M, is given in the supplemental material [40] for the cases $\nu_0 = 2$ and $\nu_0 = 7$ and an atom in the Ξ configuration.

A. Husimi function: Electromagnetic field

If we consider a superposition of matter-field states of the form

$$|\Phi(0)\rangle = \cos\theta|\nu_1;1\,1\,1\rangle + e^{i\xi}\sin\theta|\nu_2;1\,1\,1\rangle, \tag{14}$$

with $\nu_1 \neq \nu_2$; the reduced density matrix can be obtained as before and the Husimi function is calculated by taking the expectation value with respect to the Glauber coherent state of light $|\alpha\rangle$:

$$Q_H(\varrho,\phi,\theta,\xi) = \frac{e^{-\varrho^2}}{\pi}\left(\cos^2\theta\frac{\varrho^{2\nu_1}}{\nu_1!} + \sin^2\theta\frac{\varrho^{2\nu_2}}{\nu_2!} + \varrho^{\nu_1+\nu_2}\frac{\sin 2\theta}{\sqrt{\nu_1!\nu_2!}}\cos[(\nu_1-\nu_2)\phi-\xi]\right), \tag{15}$$

where $\alpha = \varrho e^{i\phi}$. This quasidistribution probability function is invariant under the transformation $\phi \to \phi + 2\pi/(\nu_1-\nu_2)$. That is, the Husimi function exhibits a cyclic point symmetry, $C_{|\nu_1-\nu_2|}$. Notice that, at this stage, this result can be extended to any number of particles because we have only to replace the matter sector of the state $|1\,1\,1\rangle \to |N_a\,N_a\,N_a\rangle$, which disappears after the tracing operation.

The dynamics of state (14) can be calculated through the action of the unitary evolution operator; in other words,

$$|\Phi(\tau)\rangle = \cos\theta|\phi_{\nu_1}(\tau)\rangle + e^{i\xi}\sin\theta|\phi_{\nu_2}(\tau)\rangle, \tag{16}$$

with $|\phi_{\nu_a}(\tau)\rangle$, and $a = 1,2$ given by Eq. (11). Calculating its density operator and taking the partial trace with respect to the matter sector, the reduced density matrix for the electromagnetic field takes the form

$$\rho_F(\nu_1,\nu_2,\tau) = \cos^2\theta\rho_F(\nu_1,\tau) + \sin^2\theta\rho_F(\nu_2,\tau) + \sin\theta\cos\theta\,\mathrm{Tr_M}[e^{-i\xi}\,|\phi_{\nu_1}(\tau)\rangle\langle\phi_{\nu_2}(\tau)| + e^{i\xi}\,|\phi_{\nu_2}(\tau)\rangle\langle\phi_{\nu_1}(\tau)|], \tag{17}$$

where $\rho_F(\nu,\tau)$ is obtained from expression (12), and

$$\mathrm{Tr_M}(\ldots) = |U_{13}(\tau)|^2[e^{-i\xi}|\nu_1-\lambda_3\rangle\langle\nu_2-\lambda_3| + \mathrm{H.c.}] + |U_{23}(\tau)|^2[e^{-i\xi}|\nu_1-\lambda_2\rangle\langle\nu_2-\lambda_2| + \mathrm{H.c.}] + |U_{33}(\tau)|^2[e^{-i\xi}|\nu_1\rangle\langle\nu_2| + \mathrm{H.c.}]. \tag{18}$$

From this expression one calculates the Husimi function by taking the expectation value with respect to the coherent states

$$Q_H(\nu_1,\nu_2,\alpha,\theta,\xi,\tau) = \frac{1}{\pi}\langle\alpha|\rho_F(\nu_1,\nu_2,\tau)|\alpha\rangle, \tag{19}$$

which again is invariant under point transformations of the cyclic group $\mathcal{C}_{|\nu_1-\nu_2|}$. The extension of this result to any number of particles is not straightforward. However, as we will see below, this result is maintained.

B. Ξ configuration

Here we apply the previous results for a three-level atom in the ladder configuration. We start by calculating the photon number probabilities, which are shown in Fig. 2. Since the probability amplitude of finding $\nu_0 - 1$ photons is given by

$$U_I(\tau)_{23} = i\sqrt{M_\Xi}\,\mu_{12}\frac{\sin\mathcal{E}_\Xi\tau}{\mathcal{E}_\Xi}e^{-i\Delta_{12}\tau/2}, \tag{20}$$

we can always make it vanish by appropriately choosing the time

$$t_s = n\pi/\mathcal{E}_\Xi, \quad n \in \mathbb{N}. \tag{21}$$

We may then have a resonant cavity in a superposition of states with a difference of two photons; that is, with $\mathcal{P}(\nu_0-1,t_s) = 0$. Furthermore, by requiring

$$\mathcal{P}(\nu_0-2,t_s) = \mathcal{P}(\nu_0,t_s),$$

we have

$$\cos\left(\frac{\Delta_{12}\pi}{\sqrt{4\nu_0(\mu_{12}^2+\mu_{23}^2)+\Delta_{12}^2-4\mu_{23}^2}}\right) = \frac{[\nu_0\mu_{12}^2-(\nu_0-1)\mu_{23}^2]^2}{4\nu_0(\nu_0-1)\mu_{12}^2\mu_{23}^2}, \tag{22}$$

where Δ_{12} is still a free parameter. When in resonance, $\Delta_{12} = 0$,

$$\mu_{12}^{(s\pm)} = (\sqrt{2}\pm 1)\sqrt{\frac{\nu_0-1}{\nu_0}}\mu_{23}\,. \tag{23}$$

This result will play an important role in Sec. IV.

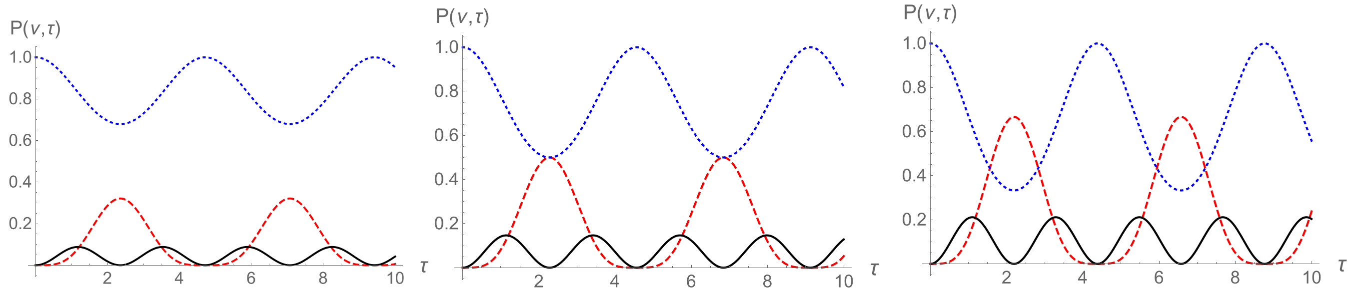

FIG. 2. The probability to find $\nu-2$ (red, dashed line), $\nu-1$ (black, solid line), and ν (blue, dotted line) photons, in the state $|\psi_{\nu_0}(\tau)\rangle$ as a function of dimensionless time $\tau=\Omega t$, for the Ξ configuration. From left to right: $\mu_{12}<\mu_{12}^{(s-)}$, $\mu_{12}=\mu_{12}^{(s-)}$, and $\mu_{12}>\mu_{12}^{(s-)}$.

Note that it is not necessary to require $\mathcal{P}(\nu_0-2,t_s)=\mathcal{P}(\nu_0,t_s)$; having them different from zero is enough, and this may be achieved for any value of μ_{12}, as shown in Fig. 2, allowing us to write any linear superposition of these two-photon states as

$$|\psi(0)\rangle_{\mathrm{F}}=\cos\theta|\nu_0-2\rangle+e^{i\xi}\sin\theta|\nu_0\rangle.$$

We will establish below a robust process for building such a superposition state.

IV. EVOLUTION OF ONE ATOM TRAVERSING A CAVITY

In this section, we show how to generate dynamically a $\mathcal{C}_n$ state of light by using resonant or near-resonant atoms with the mode field of the cavity.

Although the work described in this paper is a theoretical work, we will venture into proposing an experimental setup for the creation of the superposed state. It should be taken as a thought experiment (*Gedankenexperiment*), rather than an actual experimental setup, and nothing more. Our proposal has the intention of showing that, in principle, a superposition of states with different excitation numbers could be (now, or in the future) obtained. Any experimental obstacles in the proposed scenario have, however, not been foreseen.

The group of Rempe [41] has reported the trapping of an atom in a microcavity containing a single photon. The fact that the optical force of one photon is sufficient gives much food for thought. But Kimble and collaborators [42] have further followed the dynamics of single atoms bound in orbit by single photons.

In our thought experiment, we do not consider the trapping of atoms in a cavity, but rather the dynamical preparation of a cavity with a definite number of photons, through which (afterwards) an atom will fly by and leave the field entangled with its state. The cavity may be prepared *á la* Haroche [43], with two-level Dicke atoms, for instance, or by following the method described by Walther [1,44] for obtaining Fock states on demand inside a cavity.

A possible experimental setup could be the following: A resonant cavity is prepared to have a definite number of photons, $\nu=\nu_0$, with $\nu_0>2$ [1,27]. Then a three-level atom in the Ξ configuration, in the ground state, is sent through the cavity. Correlations between matter and field are established, which allow us to select the exit time t_s of the atom to leave the cavity with an electromagnetic state in a superposition with ν_0-2 and ν_0 photons, approximately with the same probability, i.e., t_s is chosen so that the amplitude of the contribution with ν_0-1 photons vanishes (cf. Fig. 2). The cavity is left in a state with a Husimi function which has $\mathcal{C}_2$ symmetry. Immediately, a second atom, in its lower level, is sent through the cavity. As it enters, it feels a time-dependent matter-field coupling forming a superposition of the following form as initial state:

$$|\psi(0)\rangle=(\cos\theta|\nu_0-2\rangle+e^{i\xi}\sin\theta|\nu_0\rangle)\otimes|1\,1\,1\rangle, \qquad (24)$$

where ξ is an additional phase. This state will evolve while the atom traverses the cavity, until it leaves. As we have shown in the previous section, the Husimi function that describes the light maintains a $\mathcal{C}_2$ point symmetry at all times; that is, the state in the cavity is a $\mathcal{C}_2$ state.

So much for the mental picture. We now establish the Schrödinger equation with time-dependent matter-field coupling interaction strengths. The time-dependent coupling $\mu(\tau)$ for a time of flight $t=t_{\mathrm{tof}}$ (in units of the frequency of the field) through the cavity may be described as

$$\mu(t_{\mathrm{tof}},t)=\begin{cases}\dfrac{\exp\left[-\frac{t_{\mathrm{tof}}}{t(t_{\mathrm{tof}}-t)}\right]}{\left(\exp\left[-\frac{1}{t}\right]+\exp\left[-\frac{1}{1-t}\right]\right)\left(\exp\left[-\frac{1}{t_{\mathrm{tof}}-t}\right]+\exp\left[-\frac{1}{1-t_{\mathrm{tof}}+t}\right]\right)} & \text{for } 0<t<t_{\mathrm{tof}}\\ 0 & \text{otherwise.}\end{cases}$$

The explicit expressions for μ_{ij} will be given in terms of the previous function, which is differentiable to all orders because it is a partition of unity.

For an arbitrary initial state $|\psi(0)\rangle=|\psi_0\rangle$ one may write its evolution in terms of the states $|\nu;\,1\,q\,r\rangle$. The matrix elements of the different atomic operators can be calculated [38]. Thus the time-dependent solution to the Schrödinger equation with initial condition $|\psi(0)\rangle=|\psi_0\rangle$ may be written as

$$|\psi(\tau)\rangle_M=\sum_{q=0}^{1}\sum_{r=0}^{q}\phi_{M+q+r-2,qr}(\tau)e^{-i\mathcal{E}_{M+q+r-2,\,q\,r}\tau}|M+q+r-2;\,1\,q\,r\rangle, \qquad (25)$$

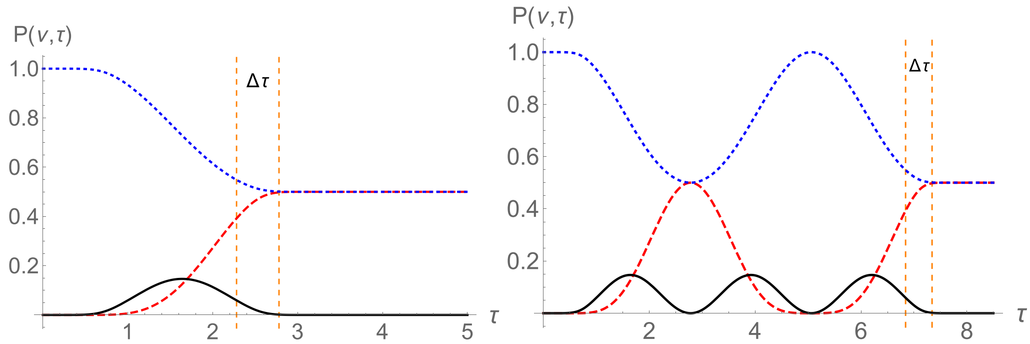

FIG. 3. Photon number probability during the time inside the cavity for the first atom. Two different times have been chosen which annihilate the state component with $\nu-1$ photons (see text). The probabilities shown are $\mathcal{P}(\nu-2,\tau)$ (red, dashed line), $\mathcal{P}(\nu-1,\tau)$ (black, solid line), and $\mathcal{P}(\nu,\tau)$ (blue, dotted line). Here, $\Delta\tau=1/2$ and the total excitation number is $M=\nu=3$.

where $\phi_{M+q+r-2,qr}(\tau)$ is a time-dependent coefficient which is determined by considering the coupling interaction of the Hamiltonian and

$$\mathcal{E}_{M+q+r-2,q\,r}=\Omega M+(1-q)\Delta_{23}+(1-r)\Delta_{12} \tag{26}$$

is the energy given by the diagonal contribution of the Hamiltonian for each state $|\nu;\,1\,q\,r\rangle$, i.e., $\boldsymbol{H}_D|\nu;\,1\,q\,r\rangle=\mathcal{E}_{\nu qr}|\nu;\,1\,q\,r\rangle$. Notice that we are *not* necessarily considering the resonant case. For the resonant case $\Delta_{23}=\Delta_{12}=0$ one has that $\mathcal{E}_{M+q+r-2,q\,r}=\Omega M$. Since $|\psi(\tau)\rangle$ should be a solution of the time-dependent Schrödinger equation, the coefficients $\phi_{\nu qr}(\tau)$ must obey the following system of coupled differential equations:

$$\dot{\phi}_{M+q+r-2,\,q\,r}(\tau)=-i\sum_{q'=0}^{1}\sum_{r'=0}^{q'}e^{-i(\mathcal{E}_{M+q'+r'-2,q'r'}-\mathcal{E}_{M+q+r-2,qr})\tau}\langle M+q+r-2;\,1\,q\,r|\boldsymbol{H}_{\rm int}|M+q'+r'-2;\,1\,q'\,r'\rangle\phi_{M+q'+r'-2,\,q'\,r'}(\tau), \tag{27}$$

where the matrix elements of $\boldsymbol{H}_{\rm int}\equiv-\frac{1}{\sqrt{N_a}}\sum_{i<j=1}^{3}\mu_{ij}(\boldsymbol{a}^\dagger\boldsymbol{A}_{ij}+\boldsymbol{a}\boldsymbol{A}_{ji})$ are given by

$$\begin{aligned}\langle\nu';\,1\,q'\,r'|\boldsymbol{H}_{\rm int}|\nu;\,1\,q\,r\rangle=&-\mu_{12}(\tau)(\sqrt{(\nu+1)(q-r)(r+1)}\delta_{\nu'\nu+1}\delta_{r'r+1}+\sqrt{\nu(q-r+1)r}\delta_{\nu'\nu-1}\delta_{r'r-1})\delta_{q'q}\\&-\mu_{13}(\tau)(\sqrt{(\nu+1)(1-q)(r+1)}\delta_{\nu'\nu+1}\delta_{q'q+1}\delta_{r'r+1}+\sqrt{\nu(2-q)r}\delta_{\nu'\nu-1}\delta_{q'q-1}\delta_{r'r-1})\\&-\mu_{23}(\tau)(\sqrt{(\nu+1)(1-q)(q-r+1)}\delta_{\nu'\nu+1}\delta_{q'q+1}\delta_{r'r}+\sqrt{\nu(2-q)(q-r)}\delta_{\nu'\nu-1}\delta_{q'q-1}\delta_{r'r}).\end{aligned} \tag{28}$$

The system of coupled differential equations for $\phi_{\nu qr}(\tau)$ (27) can be rewritten in matrix form:

$$\frac{d}{dt}\boldsymbol{\phi}=\boldsymbol{W}(\tau)\boldsymbol{\phi}, \tag{29}$$

where $\boldsymbol{W}$ is a matrix with elements $W_{\kappa_1,\kappa_2}(\tau)$, for indices $\kappa_1,\kappa_2=1,2,3$, since we are interested in considering cases where $M\geqslant 2$.

We now review in detail the procedure to generate dynamically the $\mathcal{C}_n$ state of light:

First, we solve the system of differential equations (29) for the first atom in the ground state coupled with the cavity with a definite number of photons $|\psi_1(0)\rangle=|\nu_0\rangle\otimes|1\,1\,1\rangle$. Then we determine the properties of the system; for example, the atom-field correlations through the calculation of the von Neumann or linear entropy, the photon number probabilities, and the Husimi function for the electromagnetic field. The photon number probabilities during the time inside the cavity for the first atom are shown in Fig. 3. Two different exit times have been chosen which annihilate the state component with ν_0-1 photons: $\tau_1=t_s+\Delta\tau=\pi/\mathcal{E}_\Xi+\Delta\tau$ (left), and $\tau_2=3t_s+\Delta\tau=3\pi/\mathcal{E}_\Xi+\Delta\tau$ (right), where t_s is given by Eq. (21). Recall that t_s is the time at which the probability of finding $\nu=\nu_0-1$ photons vanishes, according to the analytic solution obtained in the previous section. When considering the coupling strengths dependent on time, there is however a delay equal to $\Delta\tau=1/2$ to get the same condition. The different probabilities to find ν_0, ν_0-1, and ν_0-2 photons, correspond to those shown in Fig. 2. The linear entropy is shown for the same two different exit times in Fig. 4.

Thus, at $t_s+\Delta\tau$, the first atom through the cavity leaves the electromagnetic field in the superposition

$$\cos\theta|\nu_0-2\rangle+e^{i\xi}\sin\theta|\nu_0\rangle.$$

When a second atom enters the cavity we need to solve the system of differential equations (29). Choosing $\nu_0=3$, at $\tau=0$ we have as the initial state the superposition given in Eq. (24), with $M_1=\nu_0-2$ and $M_2=\nu_0$. By calculating the photon number probabilities for $\nu=0,1,2,3$ we see that there is a tof time when the only surviving contributions are those for $\nu=0$ and $\nu=3$ photons [cf. Fig. 5 (right)]. The cavity is then left in a superposition with $\Delta M=3$. If we want to study the dynamics of matter inside a cavity in a superposition of

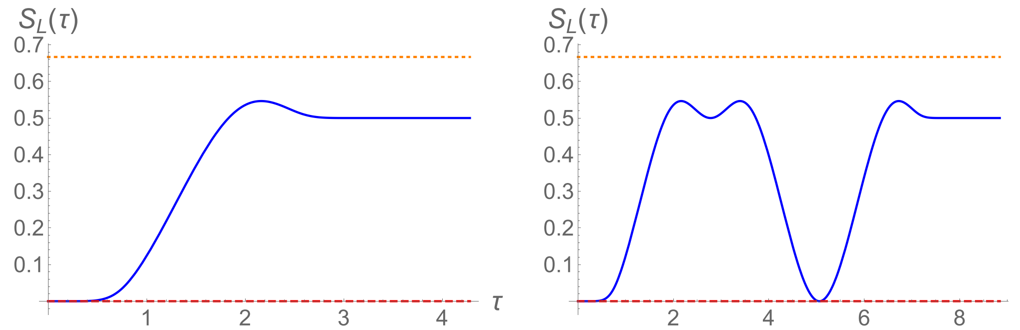

FIG. 4. Evolution of the linear entropy inside the cavity for the first atom. Two different times have been chosen which annihilate the state component with $\nu - 1$ photons: τ_1 (left), and τ_2 (right). For these figures we considered a total excitation number of $M = 3$. The horizontal dashed lines correspond to the maximum and minimum possible values of the matter-field entanglement.

such states, we may follow the same procedure by sending a third atom through the cavity.

More generally, one may prepare in this manner a $\mathcal{C}_n$ state of light with an arbitrary difference in the total excitation number. If we start with ν_0 photons inside the cavity, the first atom passing through will leave it in a superposition state of ν_0 and $\nu_0 - 2$ photons, at an exit time $t_s + \Delta\tau$. A second atom entering the cavity will see a $\mathcal{C}_2$ light state as depicted in Fig. 5 (left), and will leave the cavity in a photon superposition of two Fock states as shown in Fig. 5 (right). While passing through the cavity it will allow us to obtain light states with various $\Delta\nu$ in superposition; an example is shown in Table I, where $\mu_{12}(\tau) = \mu(\tau_{\rm tof},\tau)$, $\mu_{23}(\tau) = \sqrt{2}\,\mu(\tau_{\rm tof},\tau)$. Note that in our construction $\Delta M = \Delta\nu$.

For $\Delta M = n$, the Husimi function given in Eq. (19) presents a cyclic point symmetry given by the group $\mathcal{C}_n$. The particular case $\Delta M = 5$ is depicted in Fig. 6.

It is interesting to see the evolution of a light-superposition state in a cavity prepared as described above. An animation of the evolution, as given by the Husimi function, of a superposition of 2 and 7 photons with a Ξ-configuration atom in a cavity is presented in the supplemental material [40] for a time step of $\delta\tau = \pi/32$. The effect of the parameter ξ in the Husimi function itself, at a fixed time, is also shown as a rotation in the q-p quadrature parameter space.

V. EXTENSION TO AN ARBITRARY FINITE NUMBER OF ATOMS

The expressions (25)–(29) can be generalized to any number of atoms N_a. In this case the dimension of the matrix system is given by the degeneracy of a three-dimensional oscillator with N_a quanta, because we consider $M \geqslant 2N_a$. One may study the evolution of N_a atoms inside a cavity prepared in a light superposition state as described in the last section. Figure 7 shows this evolution for two three-level atoms in their ground state, in the Ξ configuration, having entered a cavity prepared in a $\mathcal{C}_4$ light state of 1 and 5 photons, at three different times: entering time $\tau = 0$, halfway through $\tau = t_{\rm tof}/2$, and leaving time $\tau = t_{\rm tof}$. Similar plots are obtained at other values of (μ_{12}, μ_{23}), for different numbers of atoms, and for all atomic configurations.

Next we prove the invariance under $\mathcal{C}_n$ transformations of the Husimi function of light. To that effect, we first establish the general solution of the Schrödinger equation as a time-dependent linear combination of the basis states introduced in Eq. (3),

$$|\Phi_M(\tau)\rangle = \sum_{q,r} C^{\rm M}_{qr}(t)|M - (q-r)\lambda_2 - (N_a - q)\lambda_3; N_a q r\rangle. \tag{30}$$

Since $M - (q-r)\lambda_2 - (N_a - q)\lambda_3$ denotes the number of photons, we write it as $\nu(M,q,r)$. The corresponding density matrix is constructed, then one takes the partial trace with respect to the matter components to get the reduced density matrix of the electromagnetic field and calculates its expectation value with respect to the coherent state of light to find the Husimi function

$$Q^{\rm M}_H(\alpha,t) = \sum_{q,r} \left|C^{\rm M}_{Mqr}(t)\right|^2 \frac{e^{-|\alpha|^2}}{\nu(M,q,r)!}|\alpha|^{2\nu(M,q,r)}. \tag{31}$$

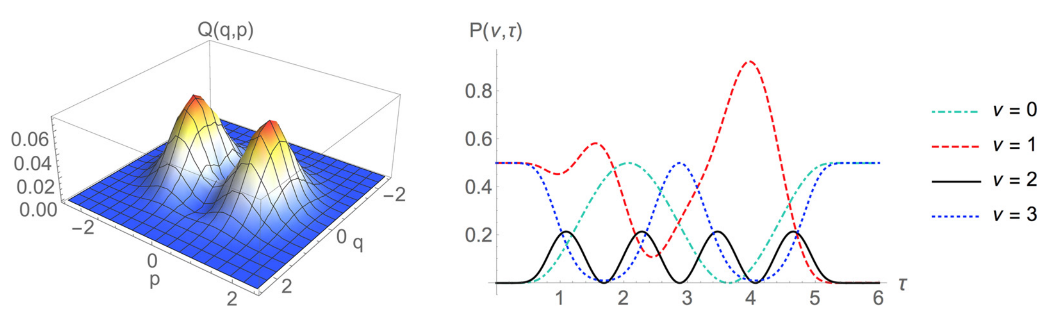

FIG. 5. (left) $\mathcal{C}_2$ state with ν and $\nu - 2$ photons, left by the passing of the first atom, and as seen by the second atom when it enters the cavity. In this case $\nu = 3$. (right) Evolution of the photon number probability while the second atom traverses the cavity.

TABLE I. Photon number probabilities $\mathcal{P}(\nu,\tau)$ for the given tof time τ_{tof}, for the second atom in a cavity with a superposition of a total excitation number given by M_1 and M_2. When the atom leaves the cavity, the latter has a light superposition state with the $\Delta\nu$ shown. The probabilities are in sequential order for $\nu = \max\{0, M_1 - 2\}$ to $\nu = M_2$ photons.

M_1	M_2	$\Delta\nu$	τ_{tof}	$\mathcal{P}(\nu,\tau)$
1	3	3	5.749	0.4993, 0.0007, 0.0000, 0.5000
3	5	4	4.510	0.4851, 0.0041, 0.0108, 0.0015, 0.4985
1	5	5	2.685	0.4935, 0.0065, 0, 0.0001, 0.0082, 0.4918

It is immediate that this Husimi function is invariant under rotations because it is a function of $\alpha = \rho e^{i\phi}$. We now wish to show that a linear superposition of two or more Fock states leads to a Husimi function with a discrete symmetry. To that end we consider the initial state

$$|\psi(0)\rangle = \big(\cos\theta|\nu_1\rangle + e^{i\xi}\sin\theta|\nu_2\rangle\big)\otimes|N_aN_aN_a\rangle,$$
$$= \cos\theta|\Phi_{\nu_1}(0)\rangle + e^{i\xi}\sin\theta|\Phi_{\nu_2}(0)\rangle. \quad (32)$$

This state has the symmetry of $\mathcal{C}_{\nu_2-\nu_1}$. As the cavity evolves in the presence of the N_a atoms a complete superposition of many Fock states of light will arise (together with their corresponding atomic excitations), all of which obey the $\mathcal{C}_{\nu_2-\nu_1}$ symmetry since the Hamiltonian preserves this symmetry at all times. By following the same procedure as for the one-particle case the Husimi function takes the form

$$Q_H(\nu_1,\nu_2,\alpha,\theta,\xi,\tau) = \cos^2\theta\, Q_H^{\nu_1}(\alpha,t) + \sin^2\theta\, Q_H^{\nu_2}(\alpha,t) + \frac{1}{2}\sin 2\theta[e^{-i\xi}\mathcal{F}_{\nu_1,\nu_2}(\alpha,t) + \text{c.c.}], \quad (33)$$

where we have defined

$$\mathcal{F}_{\nu_1,\nu_2}(\alpha,t) = \sum_{q,r} C_{qr}^{\nu_1}(t)C^{*\nu_2}_{qr}(t) \times \frac{e^{-|\alpha|^2}\rho^{\nu_1+\nu_2-2(q-r)\lambda_2-2(N_a-q)\lambda_3}e^{i\phi(\nu_1-\nu_2)}}{\sqrt{\nu(\nu_1,q,r)!\nu(\nu_2,q,r)!}},$$

so that Q_H is invariant under the transformation $\phi \to \phi + 2\pi/|\nu_1 - \nu_2|$, as we wanted to prove. We have thus left the cavity with an electromagnetic field described by a superposition of Fock states with a $\mathcal{C}_{|\nu_1-\nu_2|}$ symmetry. For the case $\nu_1 = 2$, $\nu_2 = 7$, it has contributions from $\nu = 0,1,\ldots,7$ components.

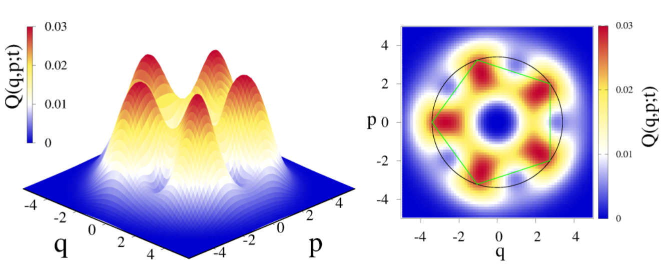

FIG. 6. The complete Husimi function plotted as a function of the field quadratures, $\alpha = (q + ip)/\sqrt{2} = \varrho e^{i\phi}$, for $\Delta M = \nu_2 - \nu_1 = 5$. We use $\xi = 0$, $\theta = \pi/4$, $\Delta_{12} = 0$, and $\tau = \pi$, and we evaluate at $(\mu_{12},\mu_{23}) = (1,\sqrt{2})$.

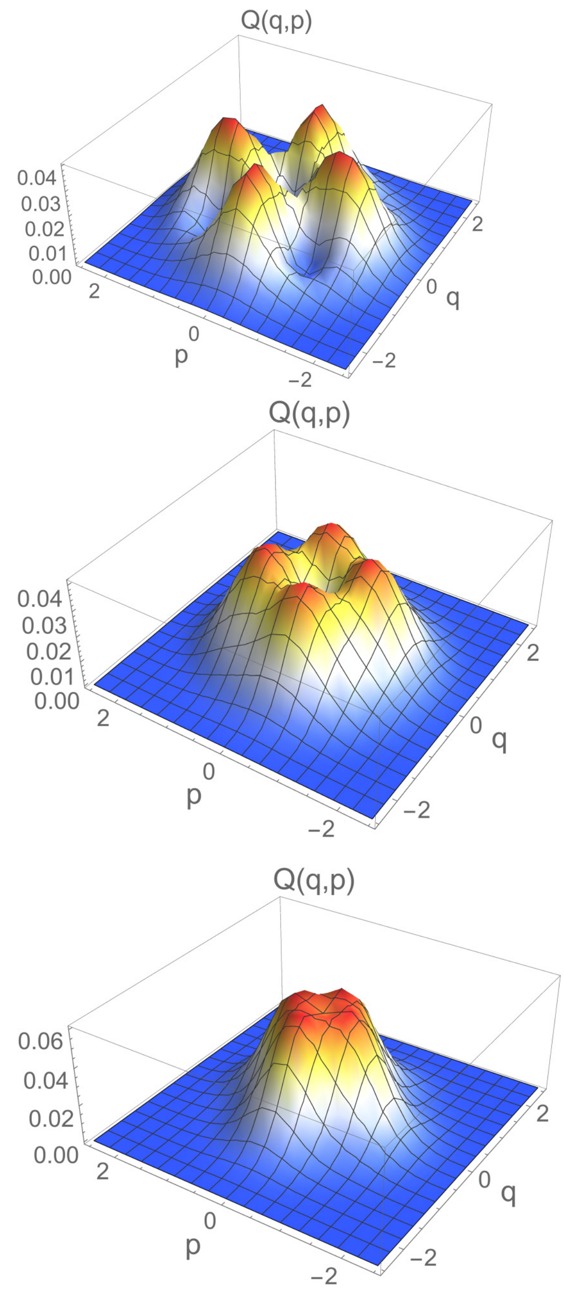

FIG. 7. The Husimi function plotted as a function of the field quadratures, for the evolution of two three-level atoms in their ground state having entered a cavity prepared in a $\mathcal{C}_4$ light state of 1 and 5 photons. We use $\xi = 0$, $\theta = \pi/4$, $\Delta_{12} = 0$, and $\tau = 0$ (top left), $\tau = \tau_{\text{tof}}/2$ (top right), and $\tau = \tau_{\text{tof}}$ (bottom). We evaluate at $(\mu_{12},\mu_{23}) = (1/\sqrt{2}, 1/\sqrt{2})$.

VI. CONCLUSIONS

In this work we have shown how to generate superpositions of photon number operator states within the generalized Tavis–Cummings model (GTC), independently of the atomic dipolar strengths and of the number of atoms. Here, by "atom" we mean throughout "artificial atom," in the sense that it can be a system consisting of real atoms, spin systems, quantum dots, or quantum circuits, with the Hamiltonian in Eq. (1) being an *effective* Hamiltonian.

As a *Gedankenexperiment*, the procedure for a possible experimental setup was presented to generate these state superpositions with a fixed difference of the total excitation number, $\Delta M = |M_1 - M_2|$, and it was shown that the field sector is invariant under point transformations of $\mathcal{C}_n$, the cyclic group in n dimensions, with $n = |M_1 - M_2|$.

The stationary states of the GTC model, for the one-particle case, were given in analytic form together with the corresponding evolution operator. This operator was used to study the dynamics of an arbitrary initial state. By appropriately selecting the time of flight of an atom through a resonant cavity, we show how to obtain crystallized states of light. In particular, we considered a superposition of two states with M_1 and M_2 total excitation numbers, and the crystallized states were given explicitly. The results for the one-atom case were extended to any number of atoms, both under resonant conditions and with detuning.

In the supplemental material [40] we exhibit the dynamics of the Husimi function associated with a single atom inside a cavity with $\nu_1 = 2$ and/or $\nu_2 = 7$ photons, the rotation effect of the parameter ξ of the initial state (24), and the corresponding symmetry associated with the cyclic group of dimension $\nu_2 - \nu_1$ (see Appendix C for details).

ACKNOWLEDGMENTS

This work was partially supported by CONACyT-México (under Project No. 238494), and DGAPA-UNAM (under Project No. IN101217).

APPENDIX A: DRESSED STATES FOR A THREE-LEVEL ATOM

The dressed states for a single atom can be obtained in analytic form. They are explicitly given for each atomic configuration in what follows:

For the Ξ configuration with the detuning conditions $\Delta_{12} + \Delta_{23} = 0$:

$$|\psi_0\rangle_\Xi = -\frac{\sqrt{M_\Xi}\mu_{12}}{\Omega_\Xi}|M_\Xi - 2; 1\,0\,0\rangle + \frac{\sqrt{M_\Xi - 1}\mu_{23}}{\Omega_\Xi}|M_\Xi; 1\,1\,1\rangle,$$

$$|\psi_\pm\rangle_\Xi = \frac{1}{\mathcal{E}_\Xi\left(2 \pm \frac{\Delta_{12}}{\mathcal{E}_\Xi}\right)^{1/2}}\{\sqrt{M_\Xi - 1}\mu_{23}|M_\Xi - 2; 1\,0\,0\rangle + \sqrt{M_\Xi}\mu_{12}|M_\Xi; 1\,1\,1\rangle - (\Delta_{12}/2 \pm \mathcal{E}_\Xi)|M_\Xi - 1; 1\,1\,0\rangle\}. \tag{A1}$$

For the Λ configuration with $\Delta_{13} - \Delta_{23} = 0$:

$$|\psi_0\rangle_\Lambda = -\frac{\mu_{13}}{\sqrt{\mu_{13}^2 + \mu_{23}^2}}|M_\Lambda; 1\,1\,0\rangle + \frac{\mu_{23}}{\sqrt{\mu_{13}^2 + \mu_{23}^2}}|M_\Lambda; 1\,1\,1\rangle,$$

$$|\psi_\pm\rangle_\Lambda = \frac{1}{\mathcal{E}_\Lambda\left(2 \pm \frac{\Delta_{13}}{\mathcal{E}_\Lambda}\right)^{1/2}}\left\{\sqrt{M_\Lambda}\mu_{13}|M_\Lambda; 1\,1\,1\rangle - \left(\pm\frac{\Delta_{13}}{2} + \mathcal{E}_\Lambda\right)|M_\Lambda - 1; 1\,0\,0\rangle + \sqrt{M_\Lambda}\mu_{23}|M_\Lambda; 1\,1\,0\rangle\right\}. \tag{A2}$$

For the V configuration with $\Delta_{12} - \Delta_{13} = 0$:

$$|\psi_0\rangle_V = -\frac{\mu_{12}}{\sqrt{\mu_{12}^2 + \mu_{13}^2}}|M_V - 1; 1\,0\,0\rangle + \frac{\mu_{13}}{\sqrt{\mu_{12}^2 + \mu_{13}^2}}|M_V - 1; 1\,1\,0\rangle,$$

$$|\psi_\pm\rangle_V = \frac{1}{\mathcal{E}_V\left(2 \mp \frac{\Delta_{12}}{\mathcal{E}_V}\right)^{1/2}}\left\{\mp\sqrt{M_V}\mu_{13}|M_V - 1; 1\,0\,0\rangle \mp \sqrt{M_V}\mu_{12}|M_V - 1; 1\,1\,0\rangle + \left(\mp\frac{\Delta_{12}}{2} + \mathcal{E}_V\right)|M_V; 1\,1\,1\rangle\right\}. \tag{A3}$$

APPENDIX B: EVOLUTION OPERATOR FOR A THREE-LEVEL ATOM

This operator in the interaction picture can be determined in analytic form for the detuning conditions indicated in Appendix A. They take the following expressions for each atomic configuration:

For the Ξ configuration:

$$U_I(\tau)_{11} = \frac{1}{\mathcal{E}_\Xi^2 - \Delta_{12}^2/4}\left\{M_\Xi\mu_{12}^2 + (M_\Xi - 1)\mu_{23}^2\left(\cos\mathcal{E}_\Xi\tau + i\Delta_{12}\frac{\sin\mathcal{E}_\Xi\tau}{2\mathcal{E}_\Xi}\right)e^{-i\Delta_{12}\tau/2}\right\},$$

$$U_I(\tau)_{12} = i\sqrt{M_\Xi - 1}\mu_{23}\frac{\sin\mathcal{E}_\Xi\tau}{\mathcal{E}_\Xi}e^{-i\Delta_{12}\tau/2},$$

$$U_I(\tau)_{13} = \frac{\sqrt{M_\Xi(M_\Xi - 1)}\mu_{12}\mu_{23}}{\mathcal{E}_\Xi^2 - \Delta_{12}^2/4}\left\{-1 + \left(\cos\mathcal{E}_\Xi\tau + i\Delta_{12}\frac{\sin\mathcal{E}_\Xi\tau}{2\mathcal{E}_\Xi}\right)e^{-i\Delta_{12}\tau/2}\right\},$$

$$U_I(\tau)_{22} = \left(\cos\mathcal{E}_\Xi\tau - i\Delta_{12}\frac{\sin\mathcal{E}_\Xi\tau}{2\mathcal{E}_\Xi}\right)e^{-i\Delta_{12}\tau/2}, \tag{B1}$$

$$U_I(\tau)_{23} = i\sqrt{M_\Xi}\,\mu_{12}\frac{\sin\mathcal{E}_\Xi\tau}{\mathcal{E}_\Xi}e^{-i\Delta_{12}\tau/2},$$

$$U_I(\tau)_{33} = \frac{1}{\mathcal{E}_\Xi^2-\Delta_{12}^2/4}\left\{(M_\Xi-1)\mu_{23}^2 + M_\Xi\mu_{12}^2\left(\cos\mathcal{E}_\Xi\tau + i\Delta_{12}\frac{\sin\mathcal{E}_\Xi\tau}{2\mathcal{E}_\Xi}\right)e^{-i\Delta_{12}\tau/2}\right\}.$$

For the Λ configuration:

$$\begin{aligned}
U_I(\tau)_{11} &= e^{-i\Delta_{13}\tau/2}\left\{\cos\mathcal{E}_\Lambda\tau - i\Delta_{13}\frac{\sin\mathcal{E}_\Lambda\tau}{2\mathcal{E}_\Lambda}\right\},\\
U_I(\tau)_{12} &= \frac{-i\sqrt{M_\Lambda}\mu_{23}e^{-i\Delta_{13}\tau/2}}{\mathcal{E}_\Lambda}\sin\mathcal{E}_\Lambda\tau,\\
U_I(\tau)_{13} &= \frac{-i\sqrt{M_\Lambda}\mu_{13}e^{-i\Delta_{13}\tau/2}}{\mathcal{E}_\Lambda}\sin\mathcal{E}_\Lambda\tau,\\
U_I(\tau)_{22} &= \frac{M_\Lambda}{\mathcal{E}_\Lambda^2-\Delta_{13}^2/4}\left\{\mu_{13}^2 + \mu_{23}^2e^{-i\Delta_{13}\tau/2}\left(\cos\mathcal{E}_\Lambda\tau + i\Delta_{13}\frac{\sin\mathcal{E}_\Lambda\tau}{2\mathcal{E}_\Lambda}\right)\right\},\\
U_I(\tau)_{23} &= -\frac{M_\Lambda\mu_{12}\mu_{23}}{\mathcal{E}_\Lambda^2-\Delta_{13}^2/4}\left\{1 - e^{-i\Delta_{13}\tau/2}\left(\cos\mathcal{E}_\Lambda\tau + i\Delta_{13}\frac{\sin\mathcal{E}_\Lambda\tau}{2\mathcal{E}_\Lambda}\right)\right\},\\
U_I(\tau)_{33} &= \frac{M_\Lambda}{\mathcal{E}_\Lambda^2-\Delta_{13}^2/4}\left\{\mu_{23}^2 + \mu_{13}^2e^{-i\Delta_{13}\tau/2}\left(\cos\mathcal{E}_\Lambda\tau + i\Delta_{13}\frac{\sin\mathcal{E}_\Lambda\tau}{2\mathcal{E}_\Lambda}\right)\right\}.
\end{aligned}\tag{B2}$$

For the V configuration:

$$\begin{aligned}
U_I(\tau)_{11} &= \frac{M_V}{\mathcal{E}_V^2-\Delta_{12}^2/4}\left\{\mu_{12}^2e^{-i\Delta_{12}\tau} + \mu_{13}^2e^{-i\Delta_{12}\tau/2}\left(\cos\mathcal{E}_V\tau - i\Delta_{12}\frac{\sin\mathcal{E}_V\tau}{2\mathcal{E}_V}\right)\right\},\\
U_I(\tau)_{12} &= -\frac{M_V\mu_{12}\mu_{13}}{\mathcal{E}_V^2-\Delta_{12}^2/4}\left\{e^{-i\Delta_{12}\tau} - e^{-i\Delta_{12}\tau/2}\left(\cos\mathcal{E}_V\tau - i\Delta_{12}\frac{\sin\mathcal{E}_V\tau}{2\mathcal{E}_V}\right)\right\},\\
U_I(\tau)_{13} &= \frac{i\sqrt{M_V}\mu_{13}e^{-i\Delta_{12}\tau/2}}{\mathcal{E}_V}\sin\mathcal{E}_V\tau,\\
U_I(\tau)_{22} &= \frac{M_V}{\mathcal{E}_V^2-\Delta_{12}^2/4}\left\{\mu_{13}^2e^{-i\Delta_{12}\tau} + \mu_{12}^2e^{-i\Delta_{12}\tau/2}\left(\cos\mathcal{E}_V\tau - i\Delta_{12}\frac{\sin\mathcal{E}_V\tau}{2\mathcal{E}_V}\right)\right\},\\
U_I(\tau)_{23} &= \frac{i\sqrt{M_V}\mu_{12}e^{-i\Delta_{12}\tau/2}}{\mathcal{E}_V}\sin\mathcal{E}_V\tau,\\
U_I(\tau)_{33} &= e^{-i\Delta_{12}\tau/2}\left\{\cos\mathcal{E}_V\tau + i\Delta_{12}\frac{\sin\mathcal{E}_V\tau}{2\mathcal{E}_V}\right\},
\end{aligned}\tag{B3}$$

and for each configuration the corresponding Hermitian conjugates.

Note that, for the Λ and V configurations, the dependence on the total number of excitations appears only in the argument of the trigonometric functions.

APPENDIX C: EVOLUTION OF THE HUSIMI FUNCTION OF LIGHT

The corresponding animations of the material presented here appear in the supplemental material [40].

1. QED Cavity with 2 or 7 photons

As an example we consider three-level atoms in the Ξ configuration with the following parameters: $\Omega = 1$ for the field frequency, $\omega_1 = 0$, $\omega_2 = 1$, $\omega_3 = 2$ for the atomic level frequencies ($\hbar = 1$), and dipolar strengths $\mu_{12} = 1$ and $\mu_{23} = \sqrt{2}$.

The most general superposition of two states with different values of the total number of excitations is given by

$$|\psi_0\rangle = \cos(\theta)|\nu_1; N_a\; q_1\, r_1\rangle + e^{i\xi}\sin(\theta)|\nu_2; N_a\; q_2\, r_2\rangle, \tag{C1}$$

where each state is given by the direct product of the photon (Fock basis) and matter (Gelfand basis) states $|\nu; N_a\, q\, r\rangle = |\nu\rangle\otimes|N_a\, q\, r\rangle$. In terms of these quantum numbers, the total number of excitations for N_a atoms in the Ξ configuration reads as $M = \nu + 2N_a - q - r$.

For the numerical calculation we consider the particular case of a single particle $N_a = 1$ in its ground state, i.e., $q_1 = q_2 = r_1 = r_2 = N_a$, and the cavity in a superposition of states with $\nu_1 = 2$ and $\nu_2 = 7$ photons. For this case the initial state is separable and takes the form

$$|\psi_0\rangle = [\cos(\theta)|\nu_1\rangle + e^{i\xi}\sin(\theta)|\nu_2\rangle] \otimes |N_a\, N_a\, N_a\rangle. \quad \text{(C2)}$$

In animations 1(a) and 1(b) in the supplemental material [40] the Husimi quasiprobability is shown as a function of dimensionless time $\tau = \Omega t$ for the case of an initial state with $\xi = 0$ and $\theta = 0$ (left), which has a single $M = \nu_1$ contribution; and for an initial state with $\xi = 0$ and $\theta = \pi/2$ (right), which corresponds to $M = 2$ and $M = 7$ excitations. In the animations one may observe that both present a radial symmetry and that their profiles evolves as volcano shapes. The radius of the volcano mouth oscillates in the range $\sqrt{M-2} \leqslant R \leqslant \sqrt{M}$ as a function of time. R is defined as the square root of the expectation value of the number of photons, $R^2 = (1/2)(q^2 + p^2)$. Thus for the state with $M = 2$ the volcano shape can evolve into a single peak, which corresponds to the vacuum state of the cavity.

2. $\mathcal{C}_5$ light state

Animation 2 in the supplemental material [40] describes the evolution of a three-level atom in the Ξ configuration inside a cavity, in a superposition of photon states as in Eq. (C2) with $\nu_1 = 2$ and $\nu_2 = 7$.

3. Effect of parameter ξ

The effect of the ξ parameter in the Husimi function is shown in animation 3 in the supplemental material [40]. The initial state is taken to be that considered above in the $\mathcal{C}_5$ case. We take $\theta = \pi/4$, $\tau = \pi/2$, and analyze the contour plot of the Husimi function as function of ξ.

[1] H. Walther, B. H. T. Varcoe, B. Englert, and T. Becker, Rep. Prog. Phys. **69**, 1325 (2006).
[2] H. J. Kimble, Phys. Scr. **T76**, 127 (1998).
[3] R. Miller, T. E. Northup, K. M. Birnbaum, A. Boca, A. D. Boozer, and H. J. Kimble, J. Phys. B: At., Mol. Opt. Phys. **38**, S551 (2005).
[4] R. H. Dicke, Phys. Rev. **93**, 99 (1954).
[5] E. Jaynes and F. Cummings, Proc. IEEE **51**, 89 (1963).
[6] Y. Zhang, G. Chen, L. Yu, Q. Liang, J.-Q. Liang, and S. Jia, Phys. Rev. A **83**, 065802 (2011).
[7] M. Tavis and F. W. Cummings, Phys. Rev. **170**, 379 (1968).
[8] I. Carusotto and C. Ciuti, Rev. Mod. Phys. **85**, 299 (2013).
[9] J. I. Cirac, P. Zoller, H. J. Kimble, and H. Mabuchi, Phys. Rev. Lett. **78**, 3221 (1997).
[10] R. López Peña, S. Cordero, E. Nahmad-Achar, and O. Castaños, Phys. Scr. **90**, 068016 (2015).
[11] E. Schrödinger, Naturwissenschaften **23**, 844 (1935).
[12] V. V. Dodonov, I. A. Malkin, and V. I. Man'ko, Physica **72**, 597 (1974).
[13] D. F. Walls, Nature (London) **306**, 141 (1983).
[14] J. N. Hollenhorst, Phys. Rev. D **19**, 1669 (1979).
[15] H. P. Yuen, Phys. Rev. A **13**, 2226 (1976).
[16] M. M. Nieto and D. R. Truax, Phys. Rev. Lett. **71**, 2843 (1993).
[17] R. J. Glauber, Phys. Rev. **130**, 2529 (1963).
[18] J. R. Anderson and A. V. Gold, Phys. Rev. Lett. **10**, 227 (1963).
[19] J. R. Klauder, J. Math. Phys. **4**, 1055 (1963).
[20] O. Castaños, R. López-Peña, and V. I. Man'ko, J. Russ. Laser Res. **16**, 477 (1995).
[21] O. Castaños and J. A. López-Saldívar, J. Phys.: Conf. Ser. **380**, 012017 (2012).
[22] B. Yurke and D. Stoler, Phys. Rev. Lett. **57**, 13 (1986).
[23] B. M. Garraway and P. L. Knight, Phys. Rev. A **49**, 1266 (1994).
[24] S. Song, C. M. Caves, and B. Yurke, Phys. Rev. A **41**, 5261 (1990).
[25] B. Yurke, W. Schleich, and D. F. Walls, Phys. Rev. A **42**, 1703 (1990).
[26] J. Gea-Banacloche, Phys. Rev. Lett. **65**, 3385 (1990).
[27] M. Brune, S. Haroche, J. M. Raimond, L. Davidovich, and N. Zagury, Phys. Rev. A **45**, 5193 (1992).
[28] V. Bužek, H. Moya-Cessa, P. L. Knight, and S. J. D. Phoenix, Phys. Rev. A **45**, 8190 (1992).
[29] G. Kirchmair, B. Vlastakis, Z. Leghtas, S. E. Nigg, H. Paik, E. Ginossar, M. Mirrahimi, L. Frunzio, S. M. Girvin, and R. J. Schoelkopf, Nature (London) **495**, 205 (2013).
[30] L. Bretheau, P. Campagne-Ibarcq, E. Flurin, F. Mallet, and B. Huard, Science **348**, 776 (2015).
[31] D. I. Schuster, A. A. Houck, J. A. Schreier, A. Wallraff, J. M. Gambetta, A. Blais, L. Frunzio, J. Majer, B. Johnson, M. H. Devoret, S. M. Girvin, and R. J. Schoelkopf, Nature (London) **445**, 515 (2002).
[32] U. Leonhardt and H. Paul, Phys. Rev. A **47**, R2460(R) (1993).
[33] N. Lambert, C. Emary, and T. Brandes, Phys. Rev. Lett. **92**, 073602 (2004).
[34] N. Lambert, C. Emary, and T. Brandes, Phys. Rev. A **71**, 053804 (2005).
[35] O. Castaños, R. López-Peña, E. Nahmad-Achar, J. G. Hirsch, E. López-Moreno, and J. E. Vitela, Phys. Scr. **79**, 065405 (2009).
[36] O. Castaños, E. Nahmad-Achar, R. López-Peña, and J. G. Hirsch, Phys. Scr. **80**, 055401 (2009).
[37] H. I. Yoo and J. H. Eberly, Phys. Rep. **118**, 239 (1985).
[38] S. Cordero, R. López-Peña, O. Castaños, and E. Nahmad-Achar, Phys. Rev. A **87**, 023805 (2013).
[39] O. Castaños, S. Cordero, R. López-Peña, and E. Nahmad-Achar, J. Phys.: Conf. Ser. **512**, 012006 (2014).
[40] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevA.97.013808 for an animation of the evolution of the Husimi function of the state of light inside the cavity, given for an atom in the Ξ configuration. This is done for two states of light $|\Phi_{\nu_0=2}(\tau)\rangle$ and $|\Phi_{\nu_0=7}(\tau)\rangle$.
[41] P. W. H. Pinkse, T. Fischer, P. Maunz, and G. Rempe, Nature (London) **404**, 365 (2000).
[42] C. J. Hood, T. W. Lynn, A. C. Doherty, A. S. Parkins, and H. J. Kimble, Science **287**, 1447 (2000).
[43] C. Guerlin, J. Bernu, S. Deleglise, C. Sayrin, S. Gleyzes, S. Kuhr, M. Brune, J.-M. Raimond, and S. Haroche, Nature (London) **448**, 889 (2007).
[44] J. Krause, M. O. Scully, and H. Walther, Phys. Rev. A **36**, 4547 (1987).