Editors' Suggestion

Lamb shift and fine structure at $n = 2$ in a hydrogenlike muonic atom with the nuclear spin $I = 0$

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The paper is devoted to the Lamb shift and fine structure in a hydrogenlike muonic atom with a spinless nucleus up to the order $\alpha^5 m$ with all the recoil corrections included. Enhanced contributions of a higher order are also considered. We present the results on the pure QED contribution and on the finite-nuclear-size contribution, proportional to R_N^2, with the higher-order corrections included. We also consider the consistency of the pure QED theory and the evaluation of the nuclear-structure effects. Most of the QED theory is the same as the theory for the case of the nuclear spin 1/2. Additional nuclear-spin-dependent terms are considered in detail. The issue of the difference for the theories with a spinor nucleus and a scalar one is discussed for the recoil contributions in the order $(Z\alpha)^4 m$, $\alpha(Z\alpha)^4 m$, and $(Z\alpha)^5 m$. The numerical results are presented for the muonic atoms with two lightest scalar nuclei, helium-4 and beryllium-10. We compare the theory of those muonic atoms with theory for the muonic hydrogen. Some higher-order finite-nuclear-size corrections for the Lamb shift in muonic hydrogen are revisited.

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I. INTRODUCTION

Determination of the accurate values of the nuclear charge radii is one of the problems of nuclear physics. Studies of spectra of muonic atoms contribute very much in the field (see, e.g., [1,2]). The previous generation of such studies was based in the investigation of the emission spectra of the excited states, appearing through muon capture. There is, however, another possibility for muonic spectroscopy. A small, but observable portion of highly excited states, appearing after muon capture, decays through a cascade of fast transitions to the metastable $2s$ state. This state lives relatively long and can be excited by a laser radiation, and the splitting between the states of $2s$ and $2p$ can be determined with a high accuracy. The results, interpreted in the terms of the nuclear radii, have unprecedented accuracy.

Such an approach has not been realized until recently. There have been published results on muonic hydrogen [3], muonic deuterium [4], while ones on the muonic helium ions are expected soon [5]. Experiments with some heavier nuclei seem possible.

The higher experimental accuracy, actual and expected, requires a more accurate theory. One of the most interesting targets is the determination of the Lamb shift in such atoms, which is the interval between the $2s_{1/2}$ and $2p_{1/2}$ states. Another interval of interest is the fine-structure splitting between the states of $2p_{1/2}$ and $2p_{3/2}$. There have been a number of efforts to improve the related theory and to prepare the review and compilation on the results.

In particular, there have been a number of reviews and compilations on the Lamb shift in the muonic helium-4 ion [6–8]. The theory is mostly the quantum electrodynamics (QED) one, but there is an important correction which is proportional to mean square of the nuclear charge radius R_N^2. Its contribution is relatively small in comparison to the leading QED term, but it is sufficiently large to be determined with a high accuracy, which in turn allows us to determine the rms charge radius of the α particle.

Here we revisit quantum electrodynamics theory of muonic helium-4 ion. The paper follows our consideration of muonic hydrogen [9] and the two-body muonic atoms with the nuclear spin 1/2 [10]. The α particle has a nuclear spin zero and that makes a certain difference for the QED consideration. We also discuss here a hydrogenlike muonic atom with another scalar nuclei, namely, Be-10.

The purpose of a QED theory, which we pursue in this paper, is to present the result for the Lamb shift as a sum of three terms (cf. [9,10]) with sufficient accuracy.

(i) The largest contribution is a pure QED one for pointlike particles. It should be calculated with a high accuracy. We build a theory of such contributions up to the order $\alpha^5 m$ with the relevant recoil effects included. Some enhanced $\alpha^6 m$ are also considered.

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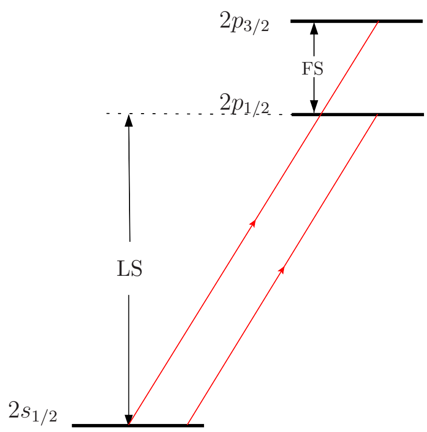

FIG. 1. Level structure for a muonic atom with a scalar nucleus. Not to scale.

(ii) The largest finite-nuclear-structure effect is presented with the term

$$\Delta E_{\text{fns:lead}}(nl) = \frac{2}{3}\frac{(Z\alpha)^4 m_r c^2}{n^3}\left(\frac{m_r R_N c}{\hbar}\right)^2 \delta_{l0}. \quad (1)$$

It is important that the term is a product of the mean square of the charge radius R_N^2 and a certain coefficient, which does not depend on the nuclear structure at all. The coefficient itself is a result of calculations for a pointlike nucleus.

Some QED corrections to the leading finite-size term are of a similar form. The ultimate purpose of the calculation of this type of contribution is to find the coefficient in front of R_N^2 with a sufficient accuracy. This term is the "signal" for the determination of the nuclear charge radius. We consider all $\alpha^5(mR_N)^2m$ contributions and some enhanced $\alpha^6(mR_N)^2m$ ones to the leading finite-size term (1).

(iii) A number of nuclear-structure corrections cannot be expressed in the terms of const $\times$ R_N^2. Many of them are not QED corrections to (1), but still some QED input is important. It is necessary to present those contributions in the way consistent with the QED theory.

The largest of them is the two-photon-exchange (TPE) term and the full TPE term contains a certain pointlike physics which should be subtracted in an appropriate way. The issue is not trivial since the two-photon diagrams in QED were studied in detail, but only for spinor particles (see, e.g., [11,12]). That makes the results applicable, e.g., for ordinary and muonic hydrogen, while the calculation for muonic helium may need an adjustment.

The level structure of a muonic atom with a spinless nucleus is depicted in Fig. 1. Two transitions which can be measured are shown there. The calculations correspond to the Lamb shift (LS) and fine-structure (FS) intervals.

Our evaluation of the QED theory follows our previous consideration of the muonic hydrogen atom [9] and its generalization for muonic atoms with other light nuclei in [10]. We start with unperturbed QED problem which is to be solved as the bound-state problem. We note that the theory for the case of different nuclear spins contains terms of order $(m/M)^2$ which are nuclear-spin dependent. For this reason we first consider the theory using the expressions for the nuclear-spin-1/2 and

TABLE I. Parameters of light muonic atoms: charge Z and mass M of a nucleus [13], the reduced muon mass m_r, the characteristic atomic momentum $\overline{p}$ for the $n=2$ energy levels, the values for the rms nuclear charge radii for proton and α particle are from scattering [14], while the result for the nucleus of ^{10}Be is from [1]. For the rough numerical estimation we apply the following numerical values (see below) $R_p = 0.84$ fm, $R_\alpha = 1.6$ fm, $R_{\text{Be}} = 2.4$ fm. [We use two similar notations: while R_N stands for the (dimensional) value of the rms charge radius, r_N is its numerical value in the fermis. Both types of values are widely applied and it is good to distinguish them.]

	μH	μ^4He	μ^{10}Be
Z	1	2	4
M (u)	1.007 276	4.001 506	10.011 340
m_r (u)	0.101 948 55	0.110 302 23	0.112 158 16
$\overline{p} = Z\alpha m_r/2$			
(MeV/c)	0.346 495	0.749 773	1.524 78
m_μ/M	0.112 61	0.028 347	0.011 33
r_N	0.895(18)	1.681(4)	2.43(5)

physical parameters, which are presented in Table I for two most light scalar nuclei. We compare all the results with the most advanced and most cross-checked theory, which is the one for muonic hydrogen, and the parameters of the latter are also given there.

The consideration of the unperturbed QED problem (see Sec. II) is with the expressions for the nuclear spin 1/2, however the terms due to the hyperfine effects are excluded from the theory of muonic helium-4 and muonic beryllium-10 ions. The other nuclear-spin-dependent terms are considered separately in Sec. V.

The most important corrections beyond the unperturbed problem are due to the contributions of the diagrams with closed electron loops. Those are muonic-atom specific terms. They are considered in Sec. III, that contains the theory up to the order $\alpha^5 m$ with all the recoil corrections included. The $(m/M)^2$ terms are taken there from the theory of the nuclear spin 1/2. The nuclear-spin-dependent terms of the perturbed problem as well as of the specific part of the theory are due to the Darwin term and electronic-vacuum-polarization correction to it. They are considered separately in Sec. V.

The other QED contributions, which do not contain the closed electron loops, are similar to the theory of ordinary hydrogenlike atoms. Some of such diagrams contain no closed fermion loops at all; those are the self-energy or leading recoil contributions. They could be easily "rescaled" by a substitution of m_e in the case of ordinary atoms for m_μ in muonic ones.

Some other "standard" diagrams contain the closed muon loops. The results for the contributions of the μ loops in muonic atoms and e loops in the case of ordinary atoms are also different only by the substitution of the mass of the fermion. The rescaled diagrams are considered in Sec. IV. Only one term there contains $(m/M)^2$ contributions and therefore may be nuclear-spin dependent. That is the so-called Salpeter term, the leading part of which is of order m/M, but it is known exactly in m/M. In Sec. IV we apply the standard theory of the two-photon-exchange contributions for the nuclear spin 1/2 and deal with the standard Salpeter term. The difference

TABLE II. Contributions to the unperturbed energy levels for the Lamb shift interval $\Delta E(2p_{1/2} - 2s_{1/2})$ in the muonic hydrogen atom and in the helium-4 and beryllium-10 ions. The corrections marked with an asterisk (*) are exact in m/M. The order shown is the leading order in m/M. Such a notation is used for all the tables through the paper. $(Z\alpha)^{4+}$ stands for $(Z\alpha)^4$ and all higher-order terms in $(Z\alpha)$. ⋆ Here we present the complete BP term for muonic hydrogen [9,10] (given with the bold italic font), not only its leading term [15,16] (see the Appendix in [9] for details). It is absent for the case of a scalar nucleus. The most important contributions are given in bold font. The notation follows [9].

No.	Designation	Order	Refs.	$\Delta E(\mu\text{H})$ (meV)	$\Delta E(\mu^4\text{He})$ (meV)	$\Delta E(\mu^{10}\text{Be})$ (meV)
0.1	Rel	$(Z\alpha)^{4+}m$		0	0	0
0.2	Rel-Rec*	$(Z\alpha)^4 m^2/M$		0	0	0
0.3	BG*	$(Z\alpha)^4(m/M)^2 m$	[17]	**0.057 47**	**0.073 795**	**0.198 31**
0.4	BP⋆	$(Z\alpha)^4(m/M)^2 m$	[9,10]	**−0.108** ***42***	0	0

in the two-photon-exchange effects due to the nuclear spin 0 is discussed in Sec. VII.

The consideration of the mentioned above contributions completes our consideration of "pure" QED terms. The leading finite-nuclear-size contribution (1) is considered in Sec. V together with the corrections which may effectively affect the definition of the rms charge radius. Those corrections include the hadronic vacuum polarization and nuclear-line QED corrections. The radiative corrections to (1), results for which maintain the form of const × R_N^2, are summarized in Sec. VI. The nuclear-structure contributions, which are not reduced to such a simple form, are in part discussed in Sec. VII. The concern of that section is not a discussion of the nuclear-physics calculations, but such a definition of the nuclear-structure correction which would be consistent with the pointlike QED theory presented in the QED sections of the paper.

In Sec. VIII, we summarize the results for the $2p_{1/2} - 2s_{1/2}$ interval (the Lamb shift) and compare them with the summary tables of other authors. The results for the $2p_{3/2} - 2p_{1/2}$ interval (the fine structure) are considered in detail in the Appendix.

II. UNPERTURBED QED TWO-BODY PROBLEM

The muonic atom is a two-body bound system and the first step in building an accurate theory is to formulate an "unperturbed" two-body problem, which can be solved, and to build a systematic perturbation series, which would allow us in principle to calculate contributions up to an arbitrary level of accuracy. Such an unperturbed QED problem and the related perturbative expansion are very similar to the case of the nuclear spin 1/2. Actually, the most important part of the unperturbed problem in the case of a two-body problem with the orbiting particle essentially lighter than the nuclei can be solved with the help of an effective Dirac equation (see, e.g., [11,12]). The latter allows us to incorporate many important recoil contributions in order m/M into the leading approximation. As far as the recoil effects are linear in m/M they do not depend on the nuclear spin. However, the effects in higher order in $(m/M)^2$ do.

There are two differences between theories with the nuclear spin 0 and 1/2. First, for spin 0 there is no nuclear spin and therefore no hyperfine effects. That simplifies the consideration and we take that into account in this section immediately. Second, the $(m/M)^2$ terms are somewhat different. We present here the relevant part of the theory for the nuclear spin 1/2 for the related contributions, while the difference between the cases of $I = 1/2$ and $I = 0$ is discussed in Sec. V.

The unperturbed nuclear-spin-1/2 QED theory, presented in Table II, includes the standard so-called Barker-Glover (BG) term for the nuclear spin 1/2. Exclusion of the hyperfine effects (from the 1/2 theory) leads to the absence of the Brodsky-Parsons (BP) term for the nuclear spin 0. The theory (without the BP term) is well covered in [11,12] (see also [9,10]).

The total results for the unperturbed problem (in this section) include the result for the Schödinger equation with the reduced mass, the relativistic corrections (from the Dirac equation), all the m/M corrections to the results from the Dirac equation, and a special consideration for the leading relativistic contribution of order $(Z\alpha)^4 m$ exactly in m/M for the nuclear spin 1/2.

III. SPECIFIC MUONIC-ATOM QED CONTRIBUTIONS

The QED theory has diagrams with closed fermion loops. In the case of ordinary atoms, the lightest particle in the loop is an electron, i.e., the particle identical to the orbiting one. In muonic atoms, the lightest particle in the loop is essentially lighter than the orbiting particle. The mass of the lightest particle in the closed loop is comparable to the characteristic atomic momentum $m_e \sim \overline{p}$ (see Table I), which makes specific a certain group of the contributions. (The units in which $\hbar = c = 1$ are used through the paper.) The diagrams with the loop particle lighter than the orbiting one are enhanced. They are not only specific, it is more important that their contributions larger than standard radiative contributions.

Specific muonic-atom QED contributions are those from the Feynman diagrams with the closed electron loops. Most of the loops are due to the electron vacuum polarization (eVP) and some are due to the electron-loop contribution to the light-by-light (LbL) scattering. The results obtained previously for the isotopes of muonic hydrogen and muonic helium are generalized here for beryllium-10. Specific muonic-atom results are summarized in Table III.

The largest contribution is the eVP1 one (eVP of the first order in α, related to the so-called Uehling potential) and we need to find it together with the recoil effects. The $(m/M)^2$ term is to be nuclear-spin dependent and in this section we present the result for $I = 1/2$ following [18–21]. The correction due to the actual value $I = 0$ for muonic helium-4 and beryllium-10 is considered in Sec. V (cf. [19]). The eVP2 contribution (eVP of the second order, with a potential of the two-loop eVP related to the so-called Källen-Sabry potential, and with

TABLE III. Specific muonic-atom contributions to the Lamb shift interval $\Delta E(2p_{1/2} - 2s_{1/2})$ in the muonic hydrogen atom and in the helium-4 and beryllium-10 ions due to closed e loops. ⋆ indicates that the LbL contribution is a combination of terms with a different Z dependence (cf. Table V). We follow the notation of [9].

No.	Designation	Order	$\Delta E(\mu\mathrm{H})$ (meV)	$\Delta E(\mu^4\mathrm{He})$ (meV)	$\Delta E(\mu^{10}\mathrm{Be})$ (meV)
1.1	eVP1 (NR)*	$\alpha(Z\alpha)^2m$	**205.007 37**	**1665.7730**	**9270.7116**
1.2	eVP1 (Rel)	$\alpha(Z\alpha)^4m$	0.020 84	0.532 04	10.0291
1.3	eVP1 (Rel-Rec)*	$\alpha(Z\alpha)^4\frac{m^2}{M}$	−0.002 08	−0.012 61	−0.090 7
2	eVP2 (NR)*	$\alpha^2(Z\alpha)^2m$	**1.658 85**	**13.2769**	**78.4768**
3	eVP3 (NR)*	$\alpha^3(Z\alpha)^2m$	0.007 52	0.074(3)	0.576(2)
4	LbL*⋆	α^5m	−0.000 89(2)	−0.0134(6)	−0.17 8(13)
5	eVP+SE	$\alpha^2(Z\alpha)^4m$	−0.002 54	−0.0646	−1.4(1)
6	SE (eVP)	$\alpha^2(Z\alpha)^4m$	−0.001 52	−0.0307	−0.5166

the double iteration of the Uehling potential) is found within a nonrelativistic approximation, since the relativistic corrections are too small, being of order $\alpha^2(Z\alpha)^4m$.

The most important new results for muonic beryllium are with eVP3 (see Fig. 2) and LbL (see Fig. 3) terms. The eVP3 diagrams (see Fig. 2) contain a triple iteration of the Uehling potential, a second combined iteration with the Uehling and Källen-Sabry potentials, and a three-loop eVP. The results on eVP3 contributions for various isotopes of muonic hydrogen and muonic helium are described in detail in [10,22–24]. Here, the results are generalized for muonic beryllium; they are summarized in Table IV.

The situation with the LbL contributions is somewhat similar. The characteristic diagrams are depicted in Fig. 3. The LbL results for muonic hydrogen and helium are known and can be found in [10,25,26]. To obtain a result for beryllium we use a somewhat different method, instead of a direct calculation. For the 1:3 and 3:1 contributions to the Lamb shift in muonic beryllium we apply the effective potential introduced in [27] [see Eq. (7.31) in [12]]. To find the 2:2 contribution for the muonic beryllium we use the expression from [25,26] [see Eq. (17) of [26]] which introduced an effective potential for the virtual Delbruck scattering contribution.

To find the effective potential we calculated it for a number of different values of r and next fitted it with a Padé approximation. The fit was subsequently utilized to calculate the correction to the Lamb shift. The results for the LbL contributions are presented in Table V. The details will be described elsewhere.

To check this approach we have obtained the results for muonic hydrogen and the muonic helium with the fits and compared them to the results of our direct calculations for the muonic helium ion [25,26]. They are consistent within their uncertainty.

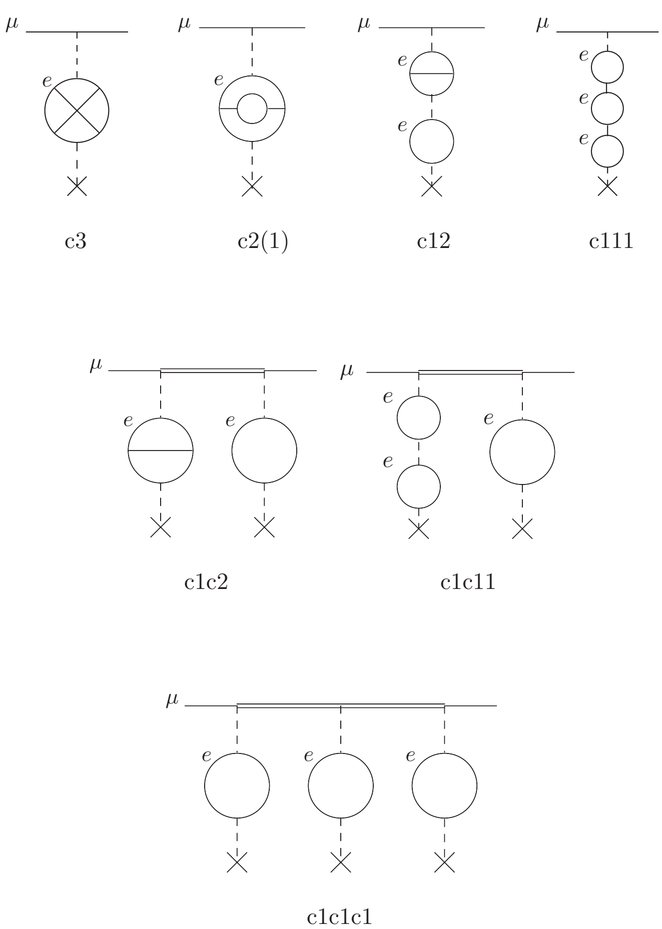

FIG. 2. Characteristic vacuum polarization contributions in order $\alpha^3(Z\alpha)^2m$ (eVP3). The first graph is for the complete irreducible eVP of the third order without any internal eVP loops, while the second is for the complete irreducible contribution with an internal eVP. The other types of contributions are either due to the reducible part of three-loop eVP or for iterations of the Uehling and Källen-Sabry potentials. The double muon line is for the Coulomb Green function.

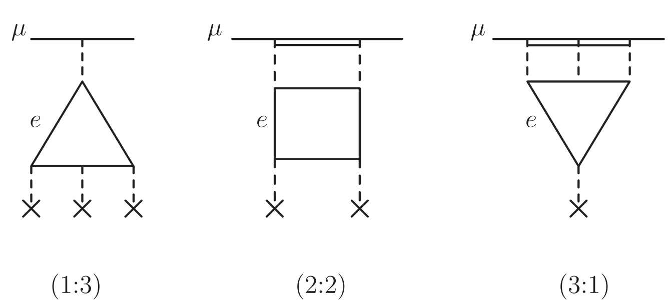

FIG. 3. Characteristic diagrams induced by the light-by-light scattering.

TABLE IV. The individual eVP3 contribution to the $\Delta E(2p_{1/2} - 2s_{1/2})$ interval in the muonic beryllium-10 (see Fig. 2). The notation follows [22].

No.	$E(2s)$ $\left[\frac{\alpha^3}{\pi^3}(Z\alpha)^2 m_r\right]$	$E(2p)$ $\left[\frac{\alpha^3}{\pi^3}(Z\alpha)^2 m_r\right]$	$E(2p-2s)$ $\left[\frac{\alpha^3}{\pi^3}(Z\alpha)^2 m_r\right]$
c3	−0.047 33	−0.078 75	−0.031 42
c111	−0.096 80	0.004 46	0.101 26
c2(1)			0.020(2)
c12	−0.272 55	−0.030 90	0.241 65
c1c2	−0.151 70	−0.072 79	0.078 91
c1c11	−0.110 05	−0.007 90	0.102 15
c1c1c1	−0.009 40	−0.005 01	0.004 38

IV. STANDARD QED CONTRIBUTIONS RESCALED

The standard QED theory of two-body systems is well covered in, e.g., [11,12]. They consider diagrams without closed fermion loops (such a fermion's self-energy or relativistic recoil effects) or with closed fermion loops, the fermion in which is of the same type as the orbiting particle (i.e., a muon in muonic atoms and an electron in ordinary ones). To rescale a standard theoretical contribution from an electronic hydrogenlike atom to a muonic one we have to use a substitution,

$$m_e \to m_\mu.$$

The ordinary theory contains many terms and only a very few of them are required for muonic atoms (see, e.g., [9]). That happens because of two reasons. First, the purpose of the study of muonic atoms is to determine the rms nuclear charge radius. Its contribution is strongly enhanced. While the standard part of theory scales as $\propto m$, the leading finite-size contribution [see (1)] scales as $\propto m^3$. Because of this enhancement, a relatively low accuracy of experiment and theory is required to accurately determine the radius. The other reason is that while the standard part of QED is a dominant contribution in the ordinary case, it is just a small correction to specific contributions (see the previous section), which dominate in the case of muonic atoms. Since the correction is small it does not require by itself a high accuracy for an eventually accurate complete theory.

The rescaled contributions of the standard theory are collected in Table VI. That is a theory with the nuclear spin 1/2. One of the contributions [due to the $(Z\alpha)^5 m^2/M$], the so-called Salpeter term, is known exactly in m/M and therefore may contain the nuclear-spin-dependent contribution in order $(m/M)^2$ or higher. The issue is discussed in Sec. VII.

V. DEFINITION OF THE NUCLEAR CHARGE RADIUS, THE DARWIN TERM, AND QED IN THE NUCLEAR LINE

The vertex of the interaction of a scalar particle (with charge Ze) with the electromagnetic field,

$$-iZe(p_\mu + p'_\mu)G(q^2), \tag{2}$$

is a product of a pointlike vertex and the electric form factor $G(q^2)$. That is a nuclear-structure form factor. The form factor is a standard one which is applied for the interpretation of the scattering results. As usual, the Fourier transform of the form factor is approximately equal to the charge distribution density (or, more accurately they are equal in the nonrecoil approximation, which is the most appropriate for larger distances and smaller momentum; the approximation is essentially better for few-nucleon nuclei than for a single proton).

The rms radius, applied in all the expressions derived from the first principle, is defined as

$$\left.\frac{\partial G(\mathbf{q}^2)}{\partial \mathbf{q}^2}\right|_{\mathbf{q}^2=0} = -\frac{1}{6}R_N^2. \tag{3}$$

The approximation of it is an average of the nuclear value of r^2 over the nuclear charge density. Such an average does not appear by itself in the base formulas, but is a result of their approximation.

There is a certain "natural" choice among the definitions of the form factor and the rms radius. There are also some certain "technical" problems in their practical extraction from the experimental data. We briefly overview both of them below.

In principle, there is an interplay between the definition of the charge radius and the so-called Darwin term (see, e.g., [28]). The definition of (2) and (3) means that $G(0) = 1$ and $R_N = 0$ in the case of a pointlike scalar nucleus. Such a definition means some additional (to the case of the nuclear spin 1/2) contributions in order $(Z\alpha)^4(m/M)^2 m$ and $\alpha(Z\alpha)^4(m/M)^2 m$. (It is possible to take another definition of the form factor which could absorb the Darwin term.) The first of them is due to the Darwin term itself,

$$\Delta E_{\rm Darw}(nl_j) = \langle nl_j|\frac{\pi Z\alpha}{2M^2}\delta^3(r)|nl_j\rangle,$$

which is present in the case of $I = 1/2$ [17] and absent in the case of $I = 0$ [29]. The other nuclear-spin dependent $(m/M)^2$ term mentioned is the Uehling correction to the Darwin term (cf. [17–19]). The results are summarized in Table VII. The "additional" corrections are supposed to cancel the Darwin-type contributions for the QED theory for the nuclear spin 1/2.

The definition of (2) and (3) is not a complete one in a practical sense. We have a number of corrections which we can in principle treat in a different way, and the difference in the treatment would effectively mean a certain shift in the

TABLE V. The individual light-by-light contributions to the $\Delta E(2p_{1/2} - 2s_{1/2})$ interval in the muonic hydrogen atom and in the helium-4 and beryllium-10 ions at order $\alpha^5 m$. The notation follows [26] (see Fig. 3).

No.	Designation	Order	$\Delta E(\mu$H) (meV)	$\Delta E(\mu^4$He) (meV)	$\Delta E(\mu^{10}$Be) (meV)
4.1	LbL (1:3)	$\alpha(Z\alpha)^4 m$	–0.001 018(4)	–0.01988(6)	–0.2414(7)
4.2	LbL (2:2)	$\alpha^2(Z\alpha)^3 m$	0.001 15(1)	0.0114(4)	0.0787(7)
4.3	LbL (3:1)	$\alpha^3(Z\alpha)^2 m$	–0.001 02(1)	–0.0050(2)	–0.0150(25)

TABLE VI. The rescaled QED terms, originating from a standard theory of ordinary hydrogen (cf. [9,12]). The results are presented for the Lamb shift $\Delta E(2p_{1/2} - 2s_{1/2})$ interval in the muonic hydrogen atom and the muonic helium-4 and beryllium-10 ions. Here, the designation is not unique, but together with order of the terms it is sufficient to distinguish the corrections. We follow the notation of [9].

No.	Designation	Order	$\Delta E(\mu$H) (meV)	$\Delta E(\mu^4$He) (meV)	$\Delta E(\mu^{10}$Be) (meV)
7.1	QED (Rad)*	$\alpha(Z\alpha)^4 m$	**−0.663 45**	**−10.926**	**−143.4535**
7.2	QED (Rad)	$\alpha(Z\alpha)^5 m$	−0.004 43	−0.1797	−6.0448
7.3.	QED (Rad-Rec)	$\alpha(Z\alpha)^5 m^2/M$	0.000 19	0.0022	0.0301
8	QED (Rec)*	$(Z\alpha)^5 m^2/M$	**−0.044 97**	**−0.4326**	**−5.3989**

definition of the form factor. Technically, the whole vertex involves the whole form factor. We mention above that $G(q^2)$ is a nuclear-structure form factor. There are also QED contributions to the form factor, i.e., the *total* form factor is a sum of the QED term and the nuclear-structure term. All the effects, which are not the nuclear-structure one, but come instead from QED, must be subtracted from the whole form factor and treated separately as QED corrections. If, by any reason or by chance, some of them are not treated separately and therefore their contribution is not subtracted, then technically that means that we put those QED contributions into $G(q^2)$.

Following [12], we introduce the leading nuclear-line QED correction as

$$\Delta E_{\rm N:QED}(nl) = \frac{4(Z^2\alpha)(Z\alpha)^4}{\pi n^3}\frac{m_r^3}{M^2} \times \left\{\left[\frac{1}{3}\ln\frac{M}{(Z\alpha)^2 m_r} + \frac{11}{72}\right]\delta_{l0} - \frac{1}{3}\ln k_0(nl)\right\}, \quad (4)$$

where $\ln k_0(nl)$ is the Bethe logarithm. Any variation in its definition by using (for any reason) another constant instead of 11/72 would mean a slightly different definition of the form factor $G(q^2)$ and the related change in the definition of the charge radius R_N.

The correction is due to the QED form factor of the nuclear particle. The electric QED form factor is infrared (IR) divergent. The IR divergence is treated differently in scattering and bound-state problems because of different physics behind the near-on-shell effects which make IR finite the total result. In the cases of scattering that is a soft-photon emission and in the case of the bound states that is a virtuality due to the binding. Because of such a specific treatment of the IR terms, it is important to deal with them in the both situations explicitly (and in a consistent way). The IR terms are related to a soft radiative photon, virtual or real, which effectively interact with a pointlike nucleus. As for the IR finite terms, they can be freely added and subtracted, because there is always a contribution of hard radiative photons, which is sensitive to the nuclear structure and cannot be found from the first principle. It is not important which IR finite term we use, but it is important to maintain the consistency in all the methods.

The related numerical results for the nuclear-line QED contribution, defined in (4), are for the muonic atoms of interest,

$$\Delta E_{\rm N:QED}(2p_{1/2} - 2s_{1/2}) = \begin{cases} -0.010\,41 \ \text{meV}, & \text{for } \mu\text{H}, \\ -0.0530 \ \text{meV}, & \text{for } \mu^4\text{He}, \\ -0.541 \ \text{meV}, & \text{for } \mu^{10}\text{Be}. \end{cases} \quad (5)$$

Another technical problem of the "practical definition" is due to a contribution of the hadronic vacuum polarization, which can be presented in the terms of a substitution for the photon propagators,

$$\frac{1}{q^2} \to [1 + \Pi_{\rm hVP}(q^2)]\frac{1}{q^2},$$

where $\Pi_{\rm hVP}(q^2)$ is the hadronic contribution to the vacuum-polarization operator.

That is a modification of the photon propagator due to hadronic intermediate states. Experimentally, when a lepton interacts with a compound nucleus, the form factor of the latter and the vacuum polarization (not necessary, the hadronic one) enter various expression always in a combination

$$G(q^2) \times [1 + \Pi_{\rm VP}(q^2)]. \quad (6)$$

If a certain contribution to $\Pi_{\rm VP}(q^2)$ is not included into the evaluation of the scattering explicitly, that means that it is hidden in the would-be extracted form factor. Instead of $G(q^2)$ we use the product from (6) with some contributions to $\Pi_{\rm VP}(q^2)$.

All the vacuum-polarization contributions (be it muonic or hadronic), which have not been accounted as QED corrections

TABLE VII. Additional Darwin corrections for system with nuclear spin equal to 0 for $2p_{1/2} - 2s_{1/2}$ for the Lamb shift (or for $2s$) in the muonic hydrogen atom and in the helium-4 and beryllium-10 ions.

No.	Designation	Order	$\Delta E(\mu$H) (meV)	$\Delta E(\mu^4$He) (meV)	$\Delta E(\mu^{10}$Be) (meV)
		Unperturbed quantum mechanics			
0.3ad	BG*	$(Z\alpha)^4(m/M)^2 m$	0	**0.221 385**	**0.5949**
		Specific QED			
1.3ad	eVP1 (Rel-Rec)	$\alpha(Z\alpha)^4(m/M)^2 m$	0	0.001 84	0.0068

TABLE VIII. Some nuclear-structure contributions to $\Delta E(2p_{1/2} - 2s_{1/2})$ in the muonic hydrogen atom and in the helium-4 and beryllium-10 ions. Here we summarize the finite-nuclear-size contribution (FNS) and the pointlike TPE κ term [9,34] (which is present in muonic hydrogen, but not in muonic helium-4 and beryllium-10). We present the numerical results with help of equations in the terms of r_N (which is the numerical value of R_N in the fermis). The results given in italics are not used directly and are given for reference purposes only. The numerical values are given for $R_p = 0.84$ fm, $R_\alpha = 1.6$ fm, $R_{Be} = 2.4$ fm to characterize the contributions. We follow the notation of [9].

No.	Designation	Order	ΔE for H (meV)		ΔE for ^{4}He (meV)		ΔE for ^{10}Be (meV)	
			Value	Estimation	Value	Estimation	Value	Estimation
10	FNS (NR)	$(Z\alpha)^4(mR_N)^2m$	$\mathbf{-5.1974}\ r_p^2$	***−3.7***	$\mathbf{-105.32}\ r_\alpha^2$	***−270***	$\mathbf{-1771.66}\ r_{Be}^2$	***−10200***
11	FNS (Rel)	$(Z\alpha)^6(mR_N)^2m$	$-0.0016\ r_p^2$		$-0.098\ r_\alpha^2$		$-5.0\ r_{Be}^2$	
			$+0.000\,24(r_p^2)^2$	*−0.0001*	$+0.018(r_\alpha^2)^2$	*−0.14*	$+0.93(r_{Be}^2)^2$	*2.1*
12	FNS (eVP)	$\alpha(Z\alpha)^4(mR_N)^2m$	$-0.0282\ r_p^2$	*−0.020*	$-0.878\ r_\alpha^2$	*−2.2*	$-20.4\ r_{Be}^2$	*−120*
13	FNS (SE+μVP)	$\alpha(Z\alpha)^5(mR_N)^2m$	0.000 36		0.058		3.87	
14.κ	TPE (κ)	$(Z\alpha)^5m^4/M^3$	−0.003 05		–		–	

to the scattering, are effectively included in the would-be $G(q^2)$ and would-be R_N.

The calculation of the hadronic vacuum-polarization contribution, using the dispersion relations and the experimental data for electron-positron annihilation, produces rather a marginal contribution (cf. [9,30]),

$$\Delta E_{\rm hVP}(2p_{1/2} - 2s_{1/2}) = \begin{cases} 0.0106(10) & \text{meV, for } \mu\text{H},\\ 0.22(2) & \text{meV, for } \mu^4\text{He},\\ 3.6(3) & \text{meV, for } \mu^{10}\text{Be}.\end{cases} \quad (7)$$

However, it is not clear whether we should include it into the final result. Our purpose is to extract the value of R_N from the muonic spectroscopy. We are to compare it to the values from spectroscopy of ordinary atoms and from the electron-nucleus scattering.

The hadronic vacuum-polarization contribution may be treated as either a separate contribution or an effective correction to the form factor. It is not that important how it is treated, but that it is treated in a consistent way. In the case of helium and beryllium there are no accurate data on an ordinary hydrogenlike ions. There are some data on the isotopic shift in the neutral helium-3 and helium-4 (e.g., [31]) and in beryllium isotopes [32] (for Li-like beryllium), however, the contribution of the hadronic VP polarization to isotopic shift cancels out. Therefore, until the hydrogenlike helium experiment will succeed (and such a program is on the way at MPQ [33]), we should only care that the electron-nucleus scattering and the muonic-atom Lamb shift have consistent definitions of the form factor. (Technically, that is the question which corrections are taken into account in the evaluation of the scattering data—muonic and hadronic vacuum polarization are not always taken in account explicitly and therefore they may be effectively included into the data values for the scattering form factors.)

VI. FINITE NUCLEAR SIZE CONTRIBUTIONS, PROPORTIONAL TO R_N^2

The leading finite-nuclear-size (FNS) term has the form of a product of a coefficient, determined by a pointlike physics, and a value of R_N^2 [see (1)]. There are a number of other contributions which have the same generic form. We can consider the related coefficients as corrections to the coefficient in the leading FNS term (1). In this section we consider relativistic and QED corrections to the coefficient in (1). The results are summarized in Table VIII. The results for the radiative corrections to the FNS effects are obtained within the nonrecoil limit.

Mostly we follow our procedure in [9]. Still two corrections to our original procedure in [9] are made. First, we fix an error in our tables for μH which comes from a misprint in [12] (see [10] for details). The relativistic correction is now presented in the form [item no. 11, FNS (Rel)],

$$\Delta E_{\rm fns:rel} = (Z\alpha)^2\left[1 - \frac{2}{3}(m_r R_N)^2\right]\Delta E_{\rm fns:lead}\ \ln\frac{1}{Z\alpha m R_N}. \quad (8)$$

The uncertainty of this logarithmic expression is estimated as 50%.

The other difference is a recalculation of the standard radiative correction to the leading term (term no. 13). Previously [9,10], we related that term to R_N^2 following the suggestion of [12]. Now we present a complete result without approximating the nuclear form factor by its first nontrivial term,

$$G(q^2) - 1 \approx -R_N^2 q^2/6.$$

For the muonic hydrogen we utilize a scope of realistic fits [35–40]. (For those we consider the fits which have a good value of χ^2 for data with spacelike q^2, are defined in all the area of integration, and have there a reasonable asymptotic behavior (in contrast to, e.g., polynomials).)

For muonic ions of helium-4 and beryllium we apply the homogeneous-sphere distribution with actual values of their charge radii taken from [1,14] and given above in Table I. Because of the smallness of the correction such a rough approximation seems appropriate. The results for this correction are the most different from the previous treatment for the heaviest of considered atoms. Another difference in the treatment of those radiative corrections is in the way of the presentation of the results. While applying [9,10] the approximation suggested in [12], the result appeared in the form of coefficient × R_N^2. Now we use a more complicated approach and the correlation with R_N^2 is not that simple. In contrast to our previous evaluations [9,10], we treat this contribution as independent from R_N^2.

TABLE IX. The QED summary table on the Lamb shift interval $\Delta E(2p_{1/2} - 2s_{1/2})$. We follow the notation in [9]. The uncertainty in the *total* values is due to the estimation of the higher-order contributions.

No.	$\Delta E(\mu\mathrm{H})$ (meV)	$\Delta E(\mu^4\mathrm{He})$ (meV)	$\Delta E(\mu^{10}\mathrm{Be})$ (meV)	Refs.
		Unperturbed quantum mechanics		
0	**−0.050 95**	**0.073 79**	**0.1983**	Table II
		Specific QED		
1	**205.026 13**	**1666.2925**	**9280.650**	Table III
2	**1.658 85**	**13.2769**	**78.4768**	Table III
3	0.007 52	0.074 (3)	0.576(2)	Table III
4	−0.000 89(2)	−0.0134(6)	−0.178(12)	Table III
5	−0.002 54	−0.0646	−1.4(1)	Table III
6	−0.001 52	−0.0307	−0.517	Table III
		Rescaled QED		
7	**– 0.667 69**	**–11.1035**	**–149.468**	Table VI
8	**– 0.044 97**	**–0.4326**	**–5.399**	Table VI
		Nuclear-line QED		
9	−0.010 41	−0.0530	−0.541	Eq. (5)
		Finite-nuclear size		
10	**−5.1974** r_p^2	**−105.322** r_α^2	**−1771.66** r_h^2	Table VIII
11	−0.0016 r_p^2 +0.000 24$(r_p^2)^2$	−0.098 r_α^2 +0.018$(r_\alpha^2)^2$	−5.0 r_{Be}^2 +0.93$(r_{Be}^2)^2$	Table VIII
12	−0.0282 r_p^2	−0.88 r_α^2	−20.39 r_{Be}^2	Table VIII
13	0.00036	0.058	3.87	Table VIII
14.κ	−0.003 05	0	0	Table VIII
		Hadronic VP		
16	0.010 6(10)	0.22(2)	3.6(3)	Eq. (7)
Total	205.9215(10) − 5.2271(8) r_p^2 +0.0003 $(r_p^2)^2$	1668.29(2) − 106.3(5) r_α^2 +0.02 $(r_\alpha^2)^2$	9209.9(4) − 1797.1(30) r_{Be}^2 +0.9(4) $(r_{Be}^2)^2$	

Our results are in part similar to those in [41], however we use more realistic charge distribution. The details will be discussed elsewhere.

VII. TWO-PHOTON EXCHANGE, ITS SUBTRACTION, AND THE (EFFECTIVE) SALPETER TERM

In Sec. IV we have already discussed the standard QED theory. The related table of the results (see Table VI) contains term no. 8 which is of order $(Z\alpha)^5 m^2/M$. That is the so-called Salpeter term and it is known exactly in (m/M) (for the nuclear spin 1/2). The "hard" part of the term comes from two-photon-exchange (TPE) QED diagram (see Fig. 4). "Hard" means that the momentum transfer through the loop ($\sim m_\mu$) is much above the atomic scale. (There is also a "soft" part of the contribution, which is not our concern in this section.)

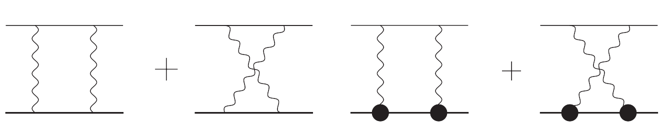

FIG. 4. The hard-photon part of the Salpeter contribution (left) and the elastic finite-nuclear-size TPE contribution (right). The closed circles are for the vertex of the extended nucleus. Appropriate subtractions for both contributions are assumed.

In general, TPE diagrams play an important role, because they are responsible for the next-to-leading finite-nuclear-size term [that is so-called Friar term of order $(Z\alpha)^5 m^4 R_N^3$] as well as for the nuclear-polarizability contribution. Both such contributions suggest a hard integration over the loop momentum. The hard integration of straightforwardly introduced integrals is IR divergent by itself. Therefore it is not possible to evaluate the integrals without a certain subtraction of some pointlike contributions, which would make them IR finite. The only valid reason to make a subtraction is that we are to take the subtracting terms into account elsewhere and, namely, as a part of the pointlike QED theory.

The TPE contributions (for the nuclear spin 0) with a subtraction are considered in [7] (cf. [42–44]). The total expression for the finite-size TPE contribution is

$$\Delta E(nl) = -\frac{16(Z\alpha)^5\, m_r^4}{\pi}\, I_{\mathrm{eTPE}}\,\frac{\delta_{l0}}{n^3},$$

$$I_{\mathrm{eTPE}} = I_3^{\mathrm{Fr}} + I_{\mathrm{rec}}^{\mathrm{E}},$$

$$I_3^{\mathrm{Fr}} = \int_0^\infty \frac{dq}{q^4}[(G(q^2))^2 - 1 - 2G'(0)\,q^2],$$

$$I_{\mathrm{rec}}^{\mathrm{E}} = \int_0^\infty \frac{dq}{q^4} f(m,M;q^2)[(G(q^2))^2 - 1], \qquad (9)$$

where

$$f(m,M;q^2) = f_E(m,M;q^2) + f_F(m,M;q^2),$$
$$f_E(m,M;q^2) = \frac{M\gamma_2(\tau_N) - m\gamma_2(\tau_\mu)}{M-m} - 1,$$
$$f_F(m,M;q^2) = \frac{M+m}{m}\,\tau_N\gamma_1(\tau_\mu), \quad (10)$$

and

$$\tau_N = \frac{q^2}{4M^2},$$
$$\tau_\mu = \frac{q^2}{4m^2},$$
$$\gamma_1(\tau) = (1-2\tau)(\sqrt{1+\tau} - \sqrt{\tau}) + \sqrt{\tau},$$
$$\gamma_2(\tau) = (1+\tau)^{3/2} - \tau^{3/2} - \frac{3}{2}\sqrt{\tau}. \quad (11)$$

With such a subtraction the q integration is IR finite, which means that all the IR divergent pointlike contributions are subtracted. But that does not mean that IR finite pointlike contributions are subtracted properly. It can be a mismatch between the subtractions and the calculation of the pointlike terms.

The problem of consistency of the pointlike subtraction in (9) and the pointlike QED calculations was considered in [45]. There are two questions to clarify. The practical one is to what "pure QED" contribution the subtracted pointlike physics is related. In general, the related pointlike two-photon exchange (which we refer to as an "effective Salpeter term") could deviate from the standard Salpeter term [46,47] (see also [11,12]). The nuclear-structure TPE contribution to the Lamb shift [see (9)] vanishes for all the states, but the ns ones, and therefore only their pointlike contribution in order $(Z\alpha)^5m$, may be affected by the choice of the subtraction terms. The standard Salpeter contribution (for the nuclear spin 1/2) for an ns state is of the form

$$\Delta E(ns) = \left\{\frac{2}{3}\ln\frac{1}{Z\alpha} - \frac{8}{3}\ln k_0(ns) - \frac{1}{9} - 2\ln\frac{m}{m_r} + \frac{14}{3}\left(\ln\frac{2}{n} + \psi(n+1) - \psi(1) + \frac{2n-1}{2n}\right) + \frac{2m^2}{M^2-m^2}\ln\frac{M}{m}\right\}\frac{(Z\alpha)^5}{\pi n^3}\frac{m^2}{M}\left(\frac{m_r}{m}\right)^3, \quad (12)$$

where $\psi(z)$ is the logarithmic derivative of the Γ function. The soft part of the correction comes from the area of the atomic momenta and the nucleus acts as a spinless pointlike charge. The hard part of the correction comes from higher momenta. The presence of $\ln(M/m)$ is a clear manifestation that the momenta comparable to the nuclear mass contribute. With such a high momentum transfer the nuclear spin will indeed affect the nuclear part of the diagrams. Possible nuclear-spin-dependent corrections should be considered.

The second question is how the effective Salpeter term, related to the subtraction, corresponds to a pointlike *ab initio* theory for a two-body bound system with a light fermion (as an orbiting particle) and a heavy scalar (as a nucleus). That is in part a question of interpretation. We need to combine a pointlike QED theory and a nuclear-structure contributions with subtracted integrals. A pointlike theory, consistent with the subtractions in TPE, is not necessarily a would-be *ab initio* QED theory for pointlike particles (see below).

The answer to the first question is that in principle the effective Salpeter term does depend on the spins of the orbiting particle and the nucleus. E.g., the result for two scalar particles differs from the result for a scalar and an 1/2-spinor [45]. The individual contributions for the scalar-spinor (1/2) effective Salpeter term differ from those for the standard Salpeter term for two 1/2 fermions. However, it happens occasionally that albeit the differences in the individual contributions for a scalar and an 1/2 fermion [it does not matter which of them is the (light) orbiting particle and which is the (heavy) nucleus], the total result is the same as for the standard case of two 1/2 fermions [45].

Responding to the second question, we find that in principle, the effective Salpeter term differs from the "true" *ab initio* Salpeter-type contribution for the two-body bound system of two pointlike particles with various spins. In particular, that was observed for the nuclear spin 1 [48]. However, in the case of the scalar-fermion subtraction, the subtraction exactly corresponds to the *ab initio* pointlike physics [45], and the effective Salpeter term is equal to the *ab initio* Salpeter-type contribution for the two-body bound system of two pointlike particles with spins 1/2 and 0.

Eventually, despite the fact that in general there may be additional terms (and in some situations there are such terms) and that there are additional partial contributions, occasionally there is no additional correction in total for the nuclear-spin-dependent terms in order $(Z\alpha)^5m$ [exactly in (m/M)].

Once the expression for the nuclear-structure two-photon-exchange elastic corrections is introduced, we have to add there the nuclear polarizability contribution appropriately. The elastic and inelastic terms could be calculated either directly using experimental data on the form factor and dispersion relations and the related experimental data on the nuclear polarizability, or using various nuclear models. In the case of muonic helium-4 the former approach has not fully been realized up to date in contrast to other light atoms (cf. [44,49]), while the nuclear-model calculations have been done in [50,51]. Since the purpose of our paper is a QED theory of muonic atoms, the detailed consideration of the nuclear structure effects is beyond the scope of this paper.

VIII. SUMMARY

Concluding, we have revisited QED theory of the $2s - 2p_{1/2}$ Lamb shift in hydrogenlike muonic atoms with scalar nuclei. The QED theory with pure QED contributions and with QED corrections to the leading FNS term is summarized in Table IX. Most of the terms come from the theory for the nuclear-spin 1/2, but terms of order $(Z\alpha)^4m$ (term no. 0) and $\alpha(Z\alpha)^4m$ (term no. 1) have been corrected because of the additional nuclear-spin-dependent recoil contributions. We have also looked for an additional recoil nuclear-spin-dependent contribution in order $(Z\alpha)^5m$. We have found that it occasionally vanishes for the nuclear spin 0 and therefore the term no. 8 does not need any correction.

Treatment of various QED corrections in our paper differs somewhat from those in the earlier calculations. Nevertheless, our results for the muonic helium-4 ion are in a reasonable

TABLE X. Theory of the $2p$ fine-structure interval $\Delta E(2p_{3/2} - 2p_{1/2})$. The radiative correction (item no. 7) is for the fine structure completely determined by the muon anomalous magnetic moment. The uncertainty in the *total* values is due to the estimation of the higher-order contributions. We follow the notation of [9]. We use here an experimental value of a_μ [13] and the related term includes higher order in α.

No.	Designation	Order	Refs.	$\Delta E(\mu\text{H})$ (meV)	$\Delta E(\mu^4\text{He})$ (meV)	$\Delta E(\mu^{10}\text{Be})$ (meV)
			Unperturbed quantum mechanics			
0.1	Rel	$(Z\alpha)^{4+}m$		**8.415 64**	**145.6980**	**2371.339**
0.2	Rel-Rec*	$(Z\alpha)^6 m^2/M$		$5.1\cdot10^{-6}$	0.0001	0.0028
0.3	BG*	$(Z\alpha)^4(m/M)^2m$	[17]	**−0.086 21**	**−0.1107**	**−0.297**
0.4	BP (tot)*	$(Z\alpha)^4(m/M)^2m$	[9]	***0.162 63(2)***	***0***	***0***
			Specific QED			
1	eVP1 Rel*	$\alpha(Z\alpha)^4m$		0.005 02	0.2753	10.116
			Rescaled QED			
7	$(g-2)_\mu$	$\alpha(Z\alpha)^4m$		0.017 64	0.3303	5.465
			Finite-nuclear size			
11	FNS (Rel)	$(Z\alpha)^6m$		$-0.000\,05\,r_p^2$	$-0.0042\,r_\alpha^2$	$-0.283\,r_{Be}^2$
Total				8.514 72(6) $-\,0.000\,05\,r_p^2$	146.193(5) $-\,0.0042\,r_\alpha^2$	2386.62(9) $-\,0.283\,r_{Be}^2$

agreement with those in [6–8], while the results on the muonic beryllium-10 ion are consistent with those in [52].

In the summary table (Table IX) the vacuum-polarization contribution (term no. 16) and the nuclear-line QED corrections (term no. 9) are included. They should be treated with caution. As we explain in the text (see Sec. V), it is crucial that they are taken into account consistently in various calculations (muonic atoms–ordinary atoms–scattering).

Study of $n = 2$ levels involves also the fine-structure splitting, which is considered in the Appendix.

We focus in this paper on pure QED contributions to the energy levels and on finite-nuclear-size contributions, proportional to R_N^2. The latter serve as a signal for the determination of the nuclear charge radius and the coefficient in front of R_N^2 is a value, determined by pointlike physics, and therefore it is a part of QED consideration. The QED part of the theory has a generic form and it is essentially the same for any low-Z hydrogenlike muonic atoms with the nuclear spin 0. The nuclear effects, which cannot be expressed in such a simplified form, depend on details of the nuclear structure and requires separately studies for each isotope (see, e.g., [50,51] for the results on muonic helium-4). It is important however to obtain QED expressions more accurate than a possible uncertainty for the nuclear-structure effects. We develop the theory of the Lamb shift and fine structure up to the order $\alpha^5 m$ including the recoil effects. We also consider some enhanced $\alpha^6 m$ contributions. The results on muonic helium-4 and beryllium-10 are presented in this paper. They can be generalized for other isotopes with scalar light nuclei.

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APPENDIX: FINE STRUCTURE AT $n = 2$

The consideration of the $n = 2$ fine structure (the splitting between the states of $2p_{3/2}$ and $2p_{1/2}$) is similar to the consideration of the Lamb shift (the splitting between the states of $2p_{1/2}$ and $2s_{1/2}$), which is done in the body of the paper. The calculations are simpler since many contributions vanishes for non-s states. The results are summarized in Table X.

The results on muonic-helium fine structure are consistent with those in [6] and [8].

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