Qudit hypergraph states and their properties

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Hypergraph states, a generalization of graph states, constitute a large class of quantum states with intriguing non-local properties, and they have promising applications in quantum information science and technology. In this paper, we study some features of an independently proposed generalization of hypergraph states to qudit hypergraph states, i.e., each vertex in the generalized hypergraph (multi-hypergraph) represents a d-level system instead of a two-level one. It is shown that multi-hypergraphs and d-level hypergraph states have a one-to-one correspondence, and the structure of a multi-hypergraph exhibits the entanglement property of the corresponding quantum state. We discuss their relationship with some well-known state classes, e.g., real equally weighted states and stabilizer states. The Bell nonlocality, an important resource in fulfilling many quantum information tasks, is also investigated.

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I. INTRODUCTION

In quantum information science and technology, graph states constitute an almost unique family of states for their appealing properties and applications [1–11]. They can be used to implement one-way quantum computation [1] and construct quantum codes [5–7]. Moreover, they can be used to characterize many kinds of widely used entangled states, such as cluster states [12], the Greenberger-Horne-Zeilinger (GHZ) states [13], and more generally, stabilizer states [14,15]. To make quantum states of suitable physical systems describable in the framework as that of graph states, Ref. [16] introduced an axiomatic method. Later, Refs. [17,18] generalized this approach and introduced a new class of quantum states named hypergraph states.

Like graph states, given a hypergraph, one can define an associated qubit hypergraph state, i.e., hypergraphs can be encoded into quantum states [17,18]. Besides this feature, every qubit hypergraph state corresponds to a stabilizer group [14,15]. However, generally speaking, the stabilizers are no longer products of local operators [18]. As a new class of quantum states, they possess many new properties, e.g., local unitary symmetries [19–22], entanglement properties [22–25], and nonlocal properties [22,26–28]. Besides these fundamental properties, these states also have many applications. Qubit hypergraph states are *real equally weighted states* [29,30], which have important applications in Grover [31] and Deutsch-Jozsa [32] algorithms. Recently, Ref. [33] has shown that, if one has a black box that can tell whether an input qubit hypergraph state is a product state, he or she can solve the NP-complete SAT problem efficiently [34]. Fully connected k-uniform qubit hypergraph states, a generalization of GHZ states, are applicable in Heisenberg-limited quantum metrology with more robustness to noise and particle losses [18,28]. Superior to one-way quantum computation based on graph states, measurement-based quantum computation with qubit hypergraph states is nonadaptive, making the measurement scheme simpler [35].

In this paper, by employing the concept of multi-hypergraph [36], we encode the multi-hypergraphs into multiqudit quantum states. (The so-called qudit hypergraph states were recently introduced in Ref. [37] in a way different from ours. The paper focuses on the stochastic local operations and classical communication (SLOCC) classification of these quantum states.) By investigating the encoding map, we discuss the relationship between the connectivity of the multi-hypergraph and the entanglement of the corresponding quantum state. We study the relationship between these quantum states and some well-known state classes, and we show the similarities and differences from the qubit case. The Bell nonlocality, a useful resource in quantum computation and quantum high-precision measurement, is also studied. Furthermore, a systematic approach for experimental detection is provided.

The paper is organized as follows: In Sec. II, we give some preliminary knowledge of hypergraph and qubit hypergraph states, and we explain related terminologies. We then generalize these concepts to represent a larger class of quantum states, which we call qudit hypergraph states, using a similar formalism. We show how the notion of a hypergraph should be modified when each vertex represents a qudit instead of a qubit. In Sec. III, we discuss the relation between multi-hypergraphs and qudit hypergraph states, mainly about the characteristics of the encoding map, and the relationship between the connectivity of a multi-hypergraph and the entanglement property of its corresponding quantum state. In Sec. IV, we discuss the relationship among qudit hypergraph states and some well-known state classes, such as real equally weighted states, qudit graph states, and stabilizer states. In Sec. V, we investigate the Bell nonlocality [38,39] of N-uniform qudit hypergraph states, and we propose a general

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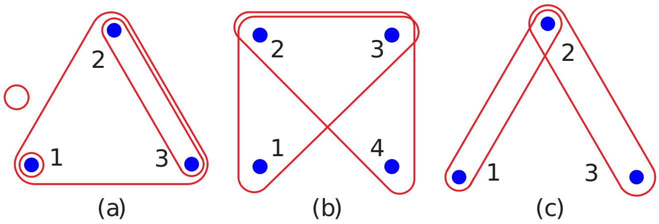

FIG. 1. Examples of hypergraphs. (a) A common hypergraph with $E = \{\emptyset,\{1\},\{2,3\},\{1,2,3\}\}$ (the red circle in the left represents an empty hyperedge). The corresponding qubit hypergraph state is $|\psi\rangle = (-|000\rangle - |001\rangle - |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle)/\sqrt{8}$, whose stabilizer group is generated by $X_1 C_\emptyset C_{\{2,3\}}$, $X_2 C_{\{3\}} C_{\{1,3\}}$, and $X_3 C_{\{2\}} C_{\{1,2\}}$. (b) Hypergraph with $E = \{\{1,2,3\},\{2,3,4\}\}$. As the cardinalities of the hyperedges are both 3, this hypergraph is a three-uniform hypergraph, a generalization of a conventional graph. (c) A hypergraph with $E = \{\{1,2\},\{2,3\}\}$. Because the hyperedges are both of cardinality 2, this hypergraph reduces to a conventional graph and the corresponding qubit hypergraph state becomes a conventional graph state.

detection scheme for illustrating the Bell nonlocality of general qudit hypergraph states. Conclusions are drawn in Sec. VI.

II. MULTI-HYPERGRAPHS AND QUDIT HYPERGRAPH STATES

In this section, we will introduce some preliminary knowledge of hypergraphs and qubit hypergraph states, and propose our main generalization of these concepts. Some important properties of qubit hypergraph states and qudit hypergraph states will be discussed.

A. Preliminary: Hypergraphs and qubit hypergraph states

A hypergraph H is composed of a set of vertices V and a set of hyperedges E [17,18,28], i.e., $H = (V,E)$. (For simplification, in this subsection, H represents such a hypergraph.) Suppose that the vertices are labeled as $1,2,\ldots,N$. Then $V = \{1,2,\ldots,N\}$. Unlike the edges defined in standard graphs, hyperedges in hypergraphs may connect more (or less) than two vertices, i.e., elements in E have a form $e = \{k_1,k_2,\ldots,k_{|e|}\}$, where $k_1,k_2,\ldots,k_{|e|}$ are the vertices connected by e, and $|e|$, the cardinality of e, ranges from 0 to N. If all the hyperedges in H are of the same cardinality k, then H is called k-uniform [18]. Standard graphs are in fact two-uniform hypergraphs. Some examples of hypergraphs are shown in Fig. 1.

Hypergraphs can be encoded into a class of quantum states named qubit hypergraph states, in which every vertex represents a two-level quantum system whose computational basis is $\{|0\rangle,|1\rangle\}$. The operator corresponding to the hyperedge $e = \{k_1,\ldots,k_{|e|}\}$ is defined as

$$C_e = \begin{cases} -1, & |e| = 0, \\ Z, & |e| = 1, \\ \sum_{i_{k_1},\ldots,i_{k_{|e|}}=0}^{1} (-1)^{i_{k_1}\cdots i_{k_{|e|}}} \hat{\Pi}_{i_{k_1}\cdots i_{k_{|e|}}}, & |e| \geqslant 2, \end{cases} \tag{1}$$

where $\hat{\Pi}_{i_{k_1}\cdots i_{k_{|e|}}} = |i_{k_1}\cdots i_{k_{|e|}}\rangle\langle i_{k_1}\cdots i_{k_{|e|}}|$ and $i_{k_1},\ldots,i_{k_{|e|}}$ denotes the value of the vertices $k_1,\ldots,k_{|e|}$, respectively. The qubit hypergraph state corresponding to H is

$$|H\rangle = \prod_{e\in E} C_e |+\rangle^{\otimes N}, \tag{2}$$

where $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. The state $|H\rangle$ can be interpreted as applying a series of C_e operations to $|+\rangle^{\otimes N}$. As all the C_e's are commutative with respect to each other, the order of the operations makes no difference, and a hypergraph corresponds to a definite qubit hypergraph state (see Fig. 1 for the examples).

As with graph states, qubit hypergraph states can also be characterized within the framework of stabilizers. Define a set of operators

$$g_k = \left(\prod_{e\in E} C_e\right) X_k \left(\prod_{e'\in E} C_{e'}\right)^\dagger = X_k \prod_{\{e|k\in e, e\in E\}} C_{e\setminus\{k\}}, \tag{3}$$

where X_k is the Pauli-X operator of the kth vertex. Then (see Fig. 1 for the examples)

$$g_k|H\rangle = |H\rangle. \tag{4}$$

Because

$$[g_k,g_{k'}] = \left(\prod_{e\in E} C_e\right)[X_k,X_{k'}]\left(\prod_{e'\in E} C_{e'}\right)^\dagger = 0, \tag{5}$$

the set $\{g_k|k \in V\}$ can generate an Abelian cyclic group called the stabilizer group of $|H\rangle$. Either $\{g_k|k \in V\}$ or the stabilizer group can determine a qubit hypergraph state up to a phase factor [18,40].

Qubit hypergraph states have interesting properties and important applications. The formalism offers a systematically pictorial representation of the *real equally weighted states*, which is a vivid way of demonstrating entanglement [18]. The entanglement and Bell nonlocality cause this class of quantum states to have a broad range of applications in quantum computation and quantum metrology [22,28].

B. Multi-hypergraphs and qudit hypergraph states

A multi-hypergraph, whose hyperedge can have a multiplicity larger than 1, is a generalization of a hypergraph (see Fig. 2 for examples). A multi-hypergraph whose vertices represent d-level quantum systems can be denoted as $H_d = (V,E)$, where $V = \{1,2,\ldots,N\}$ is the set of vertices, and E is a multiset of the hyperedges. The times an element e occurs in E is called multiplicity of e and is denoted as m_e ($m_e \in \{1,2,\ldots,d-1\}$) [11,16]. For the e that satisfies $e \in 2^V$ (2^V denotes the power set of V, which constitutes all the subsets of V) and $e \notin E$, its multiplicity m_e is defined to be 0. With this generalization, every H_d is associated with a definite multiplicity function $e \to m_e$, here $e \in 2^V$ and $m_e \in \{0,1,\ldots,d-1\}$. In the following, if not particularly specified, H_d refers to such a multi-hypergraph, and the multiplicity of e is denoted as m_e.

Now we define qudit hypergraph states corresponding to H_d. Suppose the computational basis of each vertex is $\{|0\rangle,|1\rangle,\ldots,|d-1\rangle\}$. Then in this basis the generalized

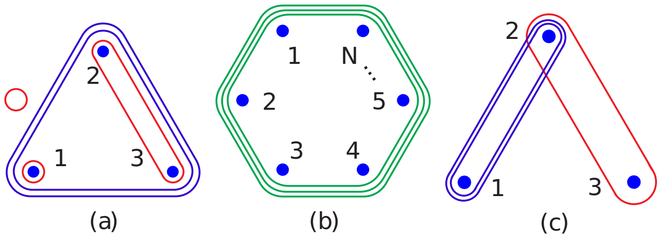

FIG. 2. Examples of multi-hypergraphs. Hyperedges with different multiplicities are drawn in different colors. (a) A common multi-hypergraph with $E = \{\emptyset,\{1\},\{2,3\},\{1,2,3\},\{1,2,3\}\}$, i.e., $m_\emptyset = 1, m_{\{1\}} = 1, m_{\{2,3\}} = 1, m_{\{1,2,3\}} = 2$, otherwise $m_e = 0$. (The red circle in the left represents an empty hyperedge.) Suppose each vertex represents a qutrit, then the corresponding quantum state is $|H_3\rangle = \sum_{i_1,i_2,i_3=0}^{2} \omega_3^{1+i_1+i_1i_2+2i_1i_2i_3} |i_1i_2i_3\rangle/\sqrt{27}$, where $\omega_3 = e^{i2\pi/3}$. The stabilizer group is generated by $X_1 C_\emptyset^\dagger (C_{\{2,3\}}^\dagger)^2$, $X_2 C_{\{3\}}^\dagger (C_{\{1,3\}}^\dagger)^2$, and $X_3 C_{\{2\}}^\dagger (C_{\{1,2\}}^\dagger)^2$. (b) An N-vertex multi-hypergraph with $m_{\{1,2,\dots,N\}} = 3$ (otherwise, $m_e = 0$). This multi-hypergraph is symmetric in the permutation of vertices. When encoding this multi-hypergraph into a quantum state, each vertex represents a quantum system whose dimension is larger than 3. (c) A multi-hypergraph with $E = \{\{1,2\},\{1,2\},\{2,3\}\}$. Because all the hyperedges in E are of cardinality 2, this multi-hypergraph is in fact a conventional multigraph that can be encoded into qudit graph states.

Pauli-X and Pauli-Z operators are [8–11]

$$X = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$Z = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \omega_d & 0 & \cdots & 0 \\ 0 & 0 & \omega_d^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \omega_d^{d-1} \end{pmatrix}, \tag{6}$$

in which $\omega_d = e^{i2\pi/d}$ and $XZ = \omega_d ZX$ (Manin's quantum plane algebra [41]). The operator corresponding to the hyperedge $e = \{k_1,k_2,\dots,k_{|e|}\}$ is defined as

$$C_e = \begin{cases} \omega_d, & |e| = 0, \\ Z, & |e| = 1, \\ \sum_{i_{k_1},\dots,i_{k_{|e|}}=0}^{d-1} \omega_d^{i_{k_1}\cdots i_{k_{|e|}}} \hat{\Pi}_{i_{k_1}\cdots i_{k_{|e|}}}, & |e| \geqslant 2, \end{cases} \tag{7}$$

where $\hat{\Pi}_{i_{k_1}\cdots i_{k_{|e|}}} = |i_{k_1}\cdots i_{k_{|e|}}\rangle\langle i_{k_1}\cdots i_{k_{|e|}}|$ and $i_{k_1},\dots,i_{k_{|e|}}$ denote the possible values of the vertices $k_1,\dots,k_{|e|}$ (in the computational basis), respectively. The unitary operators X, Z, and C_e satisfy

$$\begin{aligned} X^k &= \mathbb{I} \iff k = 0 \pmod d, \\ Z^k &= \mathbb{I} \iff k = 0 \pmod d, \\ C_e^k &= \mathbb{I} \iff k = 0 \pmod d. \end{aligned} \tag{8}$$

Denoting that $|+\rangle_d = \sum_{k=0}^{d-1} |k\rangle/\sqrt{d}$, then the d-level hypergraph state corresponding to H_d can be defined as

$$|H_d\rangle = \prod_{e\in 2^V} C_e^{m_e} |+\rangle_d^{\otimes N}. \tag{9}$$

Here the condition "$e \in 2^V$" is equivalent to "$e \in E$" because $C_e^0 = \mathbb{I}$ ($\forall\, e \in 2^V$). For simplicity, in the following we will not express it explicitly.

A qudit hypergraph state is also associated with a stabilizer group through which it can be determined up to a phase factor. For $H_d = (V,E)$, define

$$g_k = \Big(\prod C_e^{m_e}\Big) X_k \Big(\prod C_{e'}^{m_{e'}}\Big)^\dagger = X_k \prod_{e:k\in e} \Big(C_{e\setminus\{k\}}^\dagger\Big)^{m_e}. \tag{10}$$

Then

$$g_k |H_d\rangle = |H_d\rangle \tag{11}$$

and

$$[g_k, g_{k'}] = \Big(\prod C_e^{m_e}\Big)[X_k, X_{k'}]\Big(\prod C_{e'}^{m_{e'}}\Big)^\dagger = 0. \tag{12}$$

Note that the form of g_k in Eq. (10) is different from that in Eq. (3). The reason is that when $d = 2$, $\forall\, e$, C_e is Hermitian, while for general d this property cannot always hold. The set $\{g_k | k \in V\}$ generates a cyclic Abelian group named the stabilizer group of $|H_d\rangle$. Generally speaking, like those of qubit hypergraph states [22], the stabilizers of qudit hypergraph states are also nonlocal operators.

III. RELATION BETWEEN MULTI-HYPERGRAPHS AND QUDIT HYPERGRAPH STATES: CORRESPONDENCE AND ENTANGLEMENT PROPERTY

In this section, we will discuss the relation between multi-hypergraphs and qudit hypergraph states. Theorem 1 shows that the map from $\{H_d | H_d = (V,E)\}$ to $\{|H_d\rangle \,|H_d = (V,E)\}$, where H_d is mapped to $|H_d\rangle$, is a bijection. Theorem 2 demonstrates that the connectivity of a multi-hypergraph is closely related to the entanglement property of the corresponding quantum state. To prove these two theorems, we shall prove several lemmas first.

Lemma 1. Divide the hyperedge $e = \{1,2,\dots,n\}$ into the control part $e_C = \{1,2,\dots,m\}$ and the target part $e_T = \{m+1,m+2,\dots,n\}$. Then

$$C_e = \sum_{i_1,\dots,i_m=0}^{d-1} |i_1\cdots i_m\rangle\langle i_1\cdots i_m| C_{e_T}^{i_1\cdots i_m}. \tag{13}$$

Proof. From the definition in Eq. (7),

$$\begin{aligned} C_e &= \sum_{i_1,\dots,i_n=0}^{d-1} \omega_d^{i_1\cdots i_n} \hat{\Pi}_{i_1\cdots i_n}, \\ C_{e_C} &= \sum_{i_1,\dots,i_m=0}^{d-1} \omega_d^{i_1\cdots i_m} \hat{\Pi}_{i_1\cdots i_m}, \\ C_{e_T} &= \sum_{i_{m+1},\dots,i_n=0}^{d-1} \omega_d^{i_{m+1}\cdots i_n} \hat{\Pi}_{i_{m+1}\cdots i_n}. \end{aligned} \tag{14}$$

Because

$$\begin{aligned}&\sum_{i_{m+1},\ldots,i_n=0}^{d-1}\omega_d^{i_1\cdots i_n}\hat{\Pi}_{i_1\cdots i_n}\\&=\sum_{i_{m+1},\ldots,i_n=0}^{d-1}\omega_d^{i_1\cdots i_n}\hat{\Pi}_{i_1\cdots i_m}\hat{\Pi}_{i_{m+1}\cdots i_n}\\&=\hat{\Pi}_{i_1\cdots i_m}\sum_{i_{m+1},\ldots,i_n=0}^{d-1}\left(\omega_d^{i_{m+1}\cdots i_n}\right)^{i_1\cdots i_m}\hat{\Pi}_{i_{m+1}\cdots i_n}\\&=\hat{\Pi}_{i_1\cdots i_m}C_{e_T}^{i_1\cdots i_m},\end{aligned}\tag{15}$$

$$\begin{aligned}C_e&=\sum_{i_1,\ldots,i_n=0}^{d-1}\omega_d^{i_1\cdots i_n}\hat{\Pi}_{i_1\cdots i_n}\\&=\sum_{i_1,\ldots,i_m=0}^{d-1}\sum_{i_{m+1},\ldots,i_n=0}^{d-1}\omega_d^{i_1\cdots i_n}\hat{\Pi}_{i_1\cdots i_n}\\&=\sum_{i_1,\ldots,i_m=0}^{d-1}\hat{\Pi}_{i_1\cdots i_m}C_{e_T}^{i_1\cdots i_m},\end{aligned}\tag{16}$$

which is exactly the conclusion in Lemma 1. ■

Lemma 1 demonstrates that a hyperedge operation can be interpreted as a controlled operation: the products of the vertices in C determine the operations imposed on the target part T. In fact, one can choose an arbitrary subset of e as the control part, and the remaining part as the target, which originates from the symmetry of C_e.

Lemma 2. Consider a system composed of A and B, whose associated Hilbert spaces are $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Suppose $\{|1\rangle,|2\rangle,\ldots,|n\rangle\}$ is an orthonormal basis of $\mathcal{H}_A$ and $|\psi_1\rangle,|\psi_2\rangle,\ldots,|\psi_n\rangle$ are normalized vectors in $\mathcal{H}_B$. The vector

$$|1\rangle|\psi_1\rangle+|2\rangle|\psi_2\rangle+\cdots+|n\rangle|\psi_n\rangle\tag{17}$$

is a product state if and only if all the $|\psi_j\rangle$s ($1\leqslant j\leqslant n$) are parallel.

Proof. (i)"If." If all the $|\psi_j\rangle$s are parallel, then each $|\psi_j\rangle$ has a form $e^{i\phi_j}|\psi_0\rangle$. So

$$|1\rangle|\psi_1\rangle+|2\rangle|\psi_2\rangle+\cdots+|n\rangle|\psi_n\rangle=\left(\sum_{j=1}^{n}e^{i\phi_j}|j\rangle\right)|\psi_0\rangle,\tag{18}$$

which is a product state.

(ii)"Only if." Suppose the total system is in a product state. B remains the same physical state no matter what measurement is made to A and whatever the result is. By implementing the von Neumann measurement $\{M_j=|j\rangle\langle j||j\in\{1,2,\ldots,n\}\}$ to A, part B will collapse to one of the states in $\{|\psi_1\rangle,|\psi_2\rangle,\ldots,|\psi_n\rangle\}$. So all the $|\psi_j\rangle$'s are physically equivalent, i.e., they are parallel. ■

Lemma 3. Qudit hypergraph state $|H_d\rangle$ equals $|+\rangle_d^{\otimes N}$ if and only if $E=\emptyset$.

Proof. (i)"If." If $E=\emptyset$, by definition for all $e\in 2^V$, $m_e=0$, so $|H_d\rangle=|+\rangle_d^{\otimes N}$.

(ii)"Only if." The stabilizer group of $|H_d\rangle$ is generated by $\{X_k\prod_{e:k\in e}(C_{e\setminus\{k\}}^{\dagger})^{m_e}|k\in V\}$ while that of $|+\rangle_d^{\otimes N}$ is generated by $\{X_k|k\in V\}$.

If

$$\prod_e C_e^{m_e}|+\rangle_d^{\otimes N}=|+\rangle_d^{\otimes N},\tag{19}$$

the two-qudit hypergraph states will have the same stabilizer group, leading to

$$X_k\prod_{e:k\in e}(C_{e\setminus\{k\}}^{\dagger})^{m_e}=X_k\prod_{j\neq k}X_j^{p_j},\tag{20}$$

where $k\in V$ and $p_j\in\{0,1,\ldots,d-1\}$. The factor $\prod_{e:k\in e}(C_{e\setminus\{k\}}^{\dagger})^{m_e}$ is always diagonal in the computational basis, while $\prod_{j\neq k}X_j^{p_j}$ is diagonal only if $p_j=0$ ($\forall\ j\neq k$). That is to say, to make Eq. (20) hold,

$$\prod_{e:k\in e}(C_{e\setminus\{k\}}^{\dagger})^{m_e}=\prod_{j\neq k}X_j^0=\mathbb{I},\tag{21}$$

thus (notice that $C_{e\setminus\{k\}}$ is unitary)

$$\prod_{e:k\in e}C_{e\setminus\{k\}}^{m_e}|+\rangle_d^{\otimes N-1}=|+\rangle_d^{\otimes N-1}.\tag{22}$$

Implementing the above procedure several times, generally, one arrives at

$$\prod_{e:k_1,\ldots,k_n\in e}C_{e\setminus\{k_1,\ldots,k_n\}}^{m_e}|+\rangle_d^{\otimes N-n}=|+\rangle_d^{\otimes N-n}.\tag{23}$$

When $n=N-1$, Eq. (23) becomes

$$C_{\emptyset}^{m_{\{k_1,k_2,\ldots,k_{N-1}\}}}C_{\{k_N\}}^{m_{\{k_1,k_2,\ldots,k_N\}}}|+\rangle_d=|+\rangle_d,\tag{24}$$

indicating that

$$m_{\{k_1,k_2,\ldots,k_{N-1}\}}=m_{\{k_1,k_2,\ldots,k_N\}}=0,\tag{25}$$

because all the k_i ($i\in\{1,2,\ldots,N\}$) are arbitrarily arranged in order for all the e that satisfy $|e|=N$ or $N-1$, $m_e=0$.

When $n=N-2$, Eq. (23) becomes

$$\prod_{e:k_1,\ldots,k_{N-2}\in e}C_{e\setminus\{k_1,\ldots,k_{N-2}\}}^{m_e}|+\rangle_d^{\otimes 2}=|+\rangle_d^{\otimes 2}.\tag{26}$$

The product involves all the hyperedges containing $\{k_1,\ldots,k_{N-2}\}$, i.e., the cardinalities of these hyperedges are larger than or equal to $N-2$. As is shown in the previous paragraph, hyperedges whose cardinalities are larger than $N-2$ must have 0 multiplicity, thus contributing to identity factors. So Eq. (26) can be reduced to

$$C_{\emptyset}^{m_{\{k_1,k_2,\ldots,k_{N-2}\}}}|+\rangle_d^{\otimes 2}=|+\rangle_d^{\otimes 2},\tag{27}$$

indicating that $m_{\{k_1,k_2,\ldots,k_{N-2}\}}=0$. Generally, if $|e|=N-2$, $m_e=0$. Similarly, for all the e that satisfy $|e|=N-3,N-2,\ldots,0$, $m_e=0$. So if $|H_d\rangle=|+\rangle_d^{\otimes N}$, $m_e=0$ ($\forall\,e\in 2^V$), i.e., $E=\emptyset$. ■

With these lemmas, we can prove the following theorems.

Theorem 1. Suppose $H_d'=(V,E')$ and $H_d=(V,E)$. Then $|H_d'\rangle=|H_d\rangle$ if and only if $E'=E$.

Proof. (i)"If." By definition, in terms of representing d-level hypergraph states, a multi-hypergraph corresponds to a unique d-level hypergraph state.

(ii)"Only if." For $e \in 2^V$, denote its multiplicity corresponding to H_d' as m_e'. Then $|H_d'\rangle = \prod_e C_e^{m_e'}|+\rangle_d^{\otimes N}$. If $|H_d'\rangle = |H_d\rangle$,

$$|+\rangle_d^{\otimes N} = \prod_e C_e^{m_e'-m_e}|+\rangle_d^{\otimes N}. \tag{28}$$

According to Lemma 3, this equation holds if and only if for all e, $m_e' - m_e = 0$, i.e., $E' = E$. ■

Theorem 1 indicates that distinct multi-hypergraphs correspond to distinct quantum states, assuming that the systems are both N-qudit systems. An important entanglement property of qudit hypergraph states is revealed in the following theorem.

Theorem 2. If one part of a multi-hypergraph is connected with the other part, then these two corresponding subsystems are entangled.

Proof. Suppose $H_d = (V,E)$. Divide V into two parts, where one is called the control part ($C = \{c_1,c_2,\ldots,c_{|C|}\}$) and the other is called the target ($T = \{t_1,t_2,\ldots,t_{|T|}\}$), satisfying $C \cup T = V$ and $C \cap T = \emptyset$. Accordingly, we can define three sub-multisets of E, i.e., E_C, E_T, and Λ. E_C (E_T) constitutes all the elements in E that are subsets of C (T); Λ consists of all the elements in E that contains vertices in C and T simultaneously. If $\Lambda \neq \emptyset$, C and T are connected through hyperedges in Λ.

Define the multi-hypergraphs $H_d^C = (C,E_C)$ and $H_d^T = (T,E_T)$. Then

$$|H_d\rangle = C_\emptyset^{-m_\emptyset}\prod_{\epsilon\in\Lambda} C_\epsilon^{m_\epsilon}|H_d^C\rangle|H_d^T\rangle, \tag{29}$$

where $|H_d^C\rangle = \prod_{e'\in E_C} C_{e'}^{m_{e'}}|+\rangle_d^{\otimes|C|}$ and $|H_d^T\rangle = \prod_{e''\in E_T} C_{e''}^{m_{e''}}|+\rangle_d^{\otimes|T|}$ (notice that the multiplicity of each hyperedge in E_C, E_T, and Λ is the same as the one in E). Expanding $|H_d^C\rangle$ in the computational basis explicitly, one has

$$|H_d^C\rangle = \frac{1}{\sqrt{d}^{|C|}}\sum_{i_{c_1},\ldots,i_{c_{|C|}}=0}^{d-1} e^{i\phi(i_{c_1},\ldots,i_{c_{|C|}})}|i_{c_1}\cdots i_{c_{|C|}}\rangle. \tag{30}$$

According to Lemma 1, all the C_e's in Eq. (29) can be expressed in a form like Eq. (13), so

$$|H_d\rangle = \frac{C_\emptyset^{m_\emptyset}}{\sqrt{d}^{|C|}}\sum_{i_{c_1},\ldots,i_{c_{|C|}}=0}^{d-1} |i_{c_1}\cdots i_{c_{|C|}}\rangle'\hat{f}(i_{c_1},\ldots,i_{c_{|C|}})\big|H_d^T\big\rangle, \tag{31}$$

where $|i_{c_1}\cdots i_{c_{|C|}}\rangle' = e^{i\phi(i_{c_1},\ldots,i_{c_{|C|}})}|i_{c_1}\cdots i_{c_{|C|}}\rangle$ and $\hat{f}(i_{c_1},\ldots,i_{c_{|C|}})$ is some composite hyperedge transformation.

If $|H_d\rangle$ is a product state, all $\hat{f}(i_{c_1},i_{c_2},\ldots,i_{c_{|C|}})|H_d^T\rangle$ ($\forall i_{c_1},\ldots,i_{c_{|C|}} \in \{0,1,\ldots,d-1\}$) must be parallel (Lemma 2), i.e.,

$$\begin{aligned}\hat{f}(i_{c_1},\ldots,i_{c_{|C|}})|H_d^T\rangle &= e^{i\delta(i_{c_1},\ldots,i_{c_{|C|}})}\hat{f}(0,\ldots,0)|H_d^T\rangle\\ &= e^{i\delta(i_{c_1},\ldots,i_{c_{|C|}})}\big|H_d^T\big\rangle.\end{aligned} \tag{32}$$

Divide every ϵ in Λ into c_ϵ and t_ϵ, where $c_\epsilon = \epsilon \cap C$ and $t_\epsilon = \epsilon \cap T$. Then (Lemma 1)

$$\hat{f}(1,\ldots,1)\big|H_d^T\big\rangle = \prod_{\epsilon\in\Lambda} C_{t_\epsilon}^{m_\epsilon}\big|H_d^T\big\rangle. \tag{33}$$

So

$$\prod_{\epsilon\in\Lambda} C_{t_\epsilon}^{m_\epsilon}\big|H_d^T\big\rangle = e^{i\delta(1,\ldots,1)}\big|H_d^T\big\rangle = C_\emptyset^z\big|H_d^T\big\rangle, \tag{34}$$

where $z \in \{0,1,\ldots,d-1\}$, thus

$$C_\emptyset^{d-z}\prod_{\epsilon\in\Lambda} C_{t_\epsilon}^{m_\epsilon}|+\rangle_d^{\otimes|T|} = |+\rangle_d^{\otimes|T|}. \tag{35}$$

This equation cannot be true because of Lemma 3. So $|H_d\rangle$ cannot be a product state in a form like $|\psi\rangle_C|\phi\rangle_T$, i.e., the two parts are entangled. ■

Theorem 2 offers us an ability to knowing the entanglement structure of a qudit hypergraph state by reading the connectivity property of the multi-hypergraph. With the result in this theorem, we have the following two corollaries.

Corollary 1. If a multi-hypergraph H_d is connected, then $|H_d\rangle$ is genuinely entangled.

Proof. If H_d is connected, divide it into two arbitrary parts. Then the two parts are connected through some hyperedges. According to Theorem 2, these two parts are entangled. As the division is arbitrary, $|H_d\rangle$ is non-biseparable, i.e., it is genuinely entangled. ■

Corollary 2. Suppose an unconnected multi-hypergraph H_d is composed of several blocks ($H_d^{(i)}$) that are not connected to each other, and each one is a connected multi-hypergraph or possesses only one vertex. Then each $|H_d^{(i)}\rangle$ that possesses more than one vertex is a genuinely entangled state, and different blocks are not entangled with each other.

Proof. Different blocks are not connected to each other, so they are not entangled [see the definition in Eq. (9)]. For connected $H_d^{(i)}$, because $|H_d^{(i)}\rangle$ is also a qudit hypergraph state, it is genuinely entangled (Corollary 1). ■

Corollary 1 and Corollary 2 enable the multi-hypergraph to be a useful tool for visualizing the entanglement of its corresponding qudit hypergraph state.

IV. RELATIONSHIP AMONG QUDIT HYPERGRAPH STATES AND SOME WELL-KNOWN STATE CLASSES

In this section, we will discuss the relationships among qudit hypergraph states and some well-known state classes, i.e., generalized real equally weighted states, qudit graph states, and stabilizer states.

A. Qudit hypergraph states and generalized real equally weighted states

The *real equally weighted states* are the quantum states in which all the coefficients in the computational basis are real and with equal absolute value. For example, *real equally weighted states* describing N-qubit systems can all be represented in the form

$$|\psi(f,N)\rangle = \frac{1}{2^{N/2}}\sum_{i_1,\ldots,i_N=0}^{1}(-1)^{f(i_1,\ldots,i_N)}\,|i_1\cdots i_N\rangle\,, \tag{36}$$

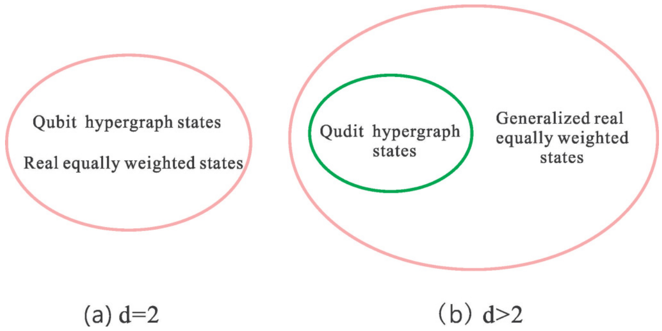

FIG. 3. Relationship between "qudit hypergraph states" and "generalized real equally weighted states." (a) When $d = 2$, "qudit hypergraph states" reduce to qubit hypergraph states, "generalized real equally weighted states" reduce to real equally weighted states, and the two sets are equivalent. (b) When $d > 2$, qudit hypergraph states form a proper subset of generalized real equally weighted states.

where $f(i_1,\ldots,i_N) \in \mathbb{Z}_2$. By interpreting -1 as ω_2, the generalized real equally weighted states (GREWSs) can be expressed as

$$|\psi(f,N)\rangle_d = \frac{1}{d^{N/2}} \sum_{i_1,\ldots,i_N=0}^{d-1} \omega_d^{f(i_1,\ldots,i_N)} |i_1 \cdots i_N\rangle, \tag{37}$$

in which $f(i_1,\ldots,i_N) \in \mathbb{Z}_d$.

It has been demonstrated in the literature that qubit hypergraph states are equivalent to *real equally weighted states* [17,18]. For the qudit case, it would be interesting to investigate whether a similar relationship exists. From the definition of qudit hypergraph states, we can see that every N-qudit hypergraph state can be expressed in the form of Eq. (37), i.e., all qudit hypergraph states are GREWSs. For specific N and d, the total number of GREWSs is d^{d^N}, while in total there are only d^{2^N} qudit hypergraph states (there are d^{2^N} such multi-hypergraphs in total, and Theorem 1 shows that the states and multi-hypergraphs have a one-to-one correspondence). Only if $d = 2$ is $d^{d^N} = d^{2^N}$, otherwise $d^{d^N} > d^{2^N}$. This indicates that if $d > 2$, the set of qudit hypergraph states is a proper subset of GREWSs. This relationship is different from the qubit case (see Fig. 3).

B. Relationship among qudit hypergraph states, qudit graph states, and stabilizer states

A qudit hypergraph state is a generalization of a qudit graph state, so qudit graph states form a subclass of qudit hypergraph states. According to Theorem 1, two qudit hypergraph states are equal only if their corresponding multi-hypergraphs are the same. Generally speaking, a multi-hypergraph can have hyperedges with cardinalities larger than 2, which is different from that of multigraphs. Therefore, in general, a qudit hypergraph state is not a qudit graph state.

Stabilizer states of N-qudit systems are the common eigenstates with eigenvalue 1 of N independent elements in the Pauli group $\mathcal{G}_N^{(d)}$ [42–44], where $\mathcal{G}_N^{(d)}$ is the N-fold product of $\mathcal{G}^{(d)}$, and $\mathcal{G}^{(d)} = \{\omega_d^a X^b Z^c | a,b,c \in \mathbb{Z}_d\}$ [X and Z are the qudit Pauli operators defined by Eq. (6)]. According to this definition, qudit graph states are all stabilizer states because there are N independent stabilizers that can be expressed in the form $g_k = X_k \prod_{n:\{k,n\}\in E} Z_n^{d-m_{\{k,n\}}}$, i.e., $g_k \in \mathcal{G}_N^{(d)}$ $(k \in \{1,2,\ldots,N\})$. As for the relationship between qudit hypergraph states and stabilizer states, we illustrate the result in the following proposition.

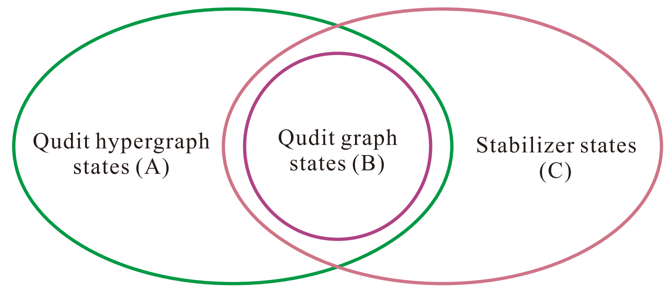

FIG. 4. Relationship among qudit hypergraph states (A), qudit graph states (B), and stabilizer states (C). B is a proper subset of $A \cap C$, because there are qudit hypergraph states that are qudit graph states acted upon by single-vertex hyperedge operations and zero-vertex hyperedge operations, i.e., they are stabilizer states but not qudit graph states.

Proposition 1. A qudit hypergraph state is a stabilizer state if and only if the cardinalities of the hyperedges are all no more than 2.

Proof. The stabilizer group of $|H_d\rangle$ is generated by $\{g_k = X_k \prod_{e:k\in e} C_{e\backslash\{k\}}^{d-m_e} | k = 1,2,\ldots,N\}$. If the cardinalities of the hyperedges are all no more than 2, then $\forall e,k$, $C_{e\backslash\{k\}}$ is ω_d or a Z operator. Thus in this case, $|H_d\rangle$ must be a stabilizer state. If some hyperedge in H_d has cardinality larger than 2 (suppose the vertex k is included by such a hyperedge), then $g_k \notin \mathcal{G}_N^{(d)}$. The reason is as follows. If $g_k \in \mathcal{G}_N$, then $X_k^{-1} g_k \in \mathcal{G}_N$. Define a new qudit hypergraph state $|H_d(k)\rangle = \prod_{e:k\in e} C_{e\backslash\{k\}}^{d-m_e} |+\rangle_d^{\otimes N}$. Then it must be a product state. If a hyperedge e satisfies $|e| > 2$, $H_d(k)$ possesses a hyperedge $e\backslash\{k\}$ satisfying $|e\backslash\{k\}| \geqslant 2$, which means that some vertices in $H_d(k)$ are connected by $e\backslash\{k\}$. According to Theorem 2, such a qudit hypergraph state cannot be a product state, which is contrary to $|H_d(k)\rangle$ being a product state. So only if the cardinalities of all the hyperedges are no more than 2 can $|H_d\rangle$ be a stabilizer state. ■

According to Proposition 1, a qudit hypergraph state that is also a stabilizer state at the same time may not be a qudit graph state (see Fig. 4). It may also be a qudit graph state operated by some generalized local Pauli operations.

To summarize, the relationship among qudit hypergraph states, qudit graph states, and stabilizer states can be expressed in Fig. 4, which is very similar to the qubit case studied in Ref. [17].

V. BELL NONLOCALITY OF QUDIT HYPERGRAPH STATES AND THE EXPERIMENTAL DETECTION

The exhibition of nonlocality by graph states and qubit hypergraph states is very important and even necessary in many quantum information tasks. Behind such an investigation is the challenging problem of the nonlocality of multipartite entangled states in quantum information theory. It has been proven that all entangled pure states are nonlocal, no matter how many particles there are and how many dimensions each particle contains [45,46]. In particular, a scheme of nonlocality exhibition was provided in an operational manner in Ref. [45]. Suppose there are N particles. The idea is that by projecting arbitrary $N-2$ particles to a product state, the remaining two particles can be measured to violate the Clauser-Horne-Shimony-Holt (CHSH) inequality [47]. Below, we discuss how it works in the scenario of qudit-hypergraph states.

A. Nonlocality exhibition by the CHSH inequality

For simplicity, we examine a multi-hypergraph $H_{N,d,m} = (V,E)$, in which $V = \{1,2,\ldots,N\}$ and $E = \{V,V,\ldots,V\}$ with $|E| = m$. The corresponding quantum state is

$$|H_{N,d,m}\rangle = C_V^m|+\rangle_d^{\otimes N} = \frac{1}{\sqrt{d^N}}\sum_{i_1,\ldots,i_N=0}^{d-1}\omega_d^{mi_1\cdots i_N}|i_1\cdots i_N\rangle. \quad (38)$$

Without losing generality, consider the case in which the Bell nonlocality of $|H_{N,d,m}\rangle$ can be exhibited by vertices 1 and 2 with the assistance of vertices $3,4,\ldots,N$ [48]. The assistance can be done by projecting the vertices to their respective $|+\rangle_d$. After this operation, the state of the remaining system (composed of vertices 1 and 2) becomes

$$|H_{N,d,m}^{(2)}\rangle = \frac{\mathcal{N}}{d^{N-1}}\sum_{i_1,i_2=0}^{d-1}\Omega_{i_1i_2}|i_1i_2\rangle, \quad (39)$$

where $\mathcal{N}$ is the normalization factor, $\Omega_{i_1i_2} = \sum_{i_3,\ldots,i_N=0}^{d-1}\omega_d^{mi_1i_2\cdots i_N}$ forming the $d\times d$ matrix Ω. We state that the remaining two vertices are entangled. The proof can be done through analyzing the rank of Ω. If $|H_{N,d,m}^{(2)}\rangle$ is separable, then the rank of Ω would be 1. However, the upper-left 2×2 submatrix $(i_1,i_2 \in \{0,1\})$ of Ω

$$\tilde{\Omega} = \begin{pmatrix} d^{N-2} & d^{N-2} \\ d^{N-2} & \sum_{i_3,\ldots,i_N=0}^{d-1}\omega^{mi_3\cdots i_N} \end{pmatrix} \quad (40)$$

has a nonzero determinant, therefore rank $(\Omega) \geqslant 2$ [49], indicating that the remaining two vertices are entangled.

To analyze the entanglement property and Bell nonlocality, it is convenient to transform $|H_{N,d,m}^{(2)}\rangle$ to its Schmidt form,

$$|H_{N,d,m}^{(2)}\rangle = \sum_{\mu=0}^{d-1}c_\mu|\mu\rangle_1|\mu\rangle_2, \quad (41)$$

where c_μ are the Schmidt coefficients, and $|\mu\rangle_1$ and $|\mu\rangle_2$ are the Schmidt bases for vertex 1 and 2, respectively. The entanglement of $|H_{N,d,m}^{(2)}\rangle$ implies that there is more than 1 nontrivial term on the right-hand side of Eq. (41). Thus, we can measure vertex 1 on the settings $S_1 = \sigma_z$ and $T_1 = \sigma_x$, and vertex 2 on the settings $S_2 = \sigma_z\cos 2t + \sigma_x \sin 2t$ and $T_2 = \sigma_z\cos 2t - \sigma_x\sin 2t$, where $\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$, $\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|$ on respective basis $|\mu\rangle_1$ and $|\mu\rangle_2$, and $\tan 2t = 2c_0c_1$. The measurement results will disclose the nonlocality by violating the following CHSH inequality [45]:

$$C = |E(S_1S_2||+\rangle_d^{\otimes N-2}) + E(S_1T_2||+\rangle_d^{\otimes N-2}) + E(T_1S_2||+\rangle_d^{\otimes N-2}) - E(T_1T_2||+\rangle_d^{\otimes N-2})| \leqslant 2. \quad (42)$$

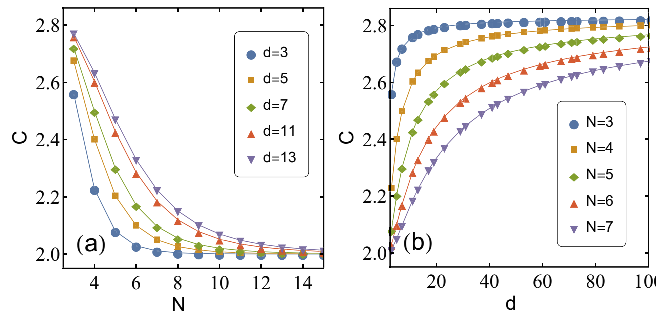

FIG. 5. Violation of the CHSH inequality for various combinations of d and N in our measurement scheme, where d is the dimension of each vertex (notice that d is assumed to be prime) and N is the number of vertices in the multi-hypergraph. The data points are connected for revealing the monotonicity of C with respect to N (d).

More precisely, the left-hand side of the above inequality can achieve $2\sqrt{1+4c_0^2c_1^2/(c_0^2+c_1^2)^2}$, such that the bound 2 is violated.

B. The prime-dimensional case

When the dimension of the qudits is prime ($d \in \mathbb{P}$), $\Omega_{i_1i_2}$ has a simple analytic form

$$\Omega_{i_1i_2} = \begin{cases} d^{N-2}, & i_1 = 0 \vee i_2 = 0, \\ d^{N-2} - d(d-1)^{N-3}, & i_1 \neq 0 \wedge i_2 \neq 0. \end{cases} \quad (43)$$

In this case, the Schmidt form of $|H_{N,d,m}^{(2)}\rangle$ is

$$|H_{N,d,m}^{(2)}\rangle = \frac{x_+|0\rangle_1|0\rangle_2 + x_-|1\rangle_1|1\rangle_2}{\sqrt{x_+^2+x_-^2}}, \quad (44)$$

where

$$x_\pm = \frac{\lambda \pm \sqrt{\lambda^2+4(d-\lambda)}}{2} \quad (45)$$

and

$$|0\rangle_k = \frac{1}{N_+}\left((x_+-\lambda+1)|0\rangle + \sum_{i=1}^{d-1}|i\rangle\right),$$
$$|1\rangle_k = \frac{1}{N_-}\left((x_--\lambda+1)|0\rangle + \sum_{i=1}^{d-1}|i\rangle\right), \quad (46)$$

with $N_\pm = \sqrt{(x_\pm-\lambda+1)^2+d-1}$, $k \in \{1,2\}$, and $\lambda = d - (d-1)^{N-2}/d^{N-3}$. The Schmidt number of $|H_{N,d,m}^{(2)}\rangle$ is 2, which indicates that the entanglement of vertices 1 and 2 is equivalent to the entanglement of two qubits. The results in the previous paragraph can be applied here directly except that here $\tan 2t = 2x_+x_-/(x_+^2+x_-^2)$. Explicitly, in this case the left-hand side of Eq. (42) can violate the CHSH inequality by an amount of $2\sqrt{1+4x_+^2x_-^2/(x_+^2+x_-^2)^2}$.

Figure 5 reveals the violation of the CHSH inequality for various combinations of d ($d \in \mathbb{P}$) and N in this measurement scheme. Here, C is always greater than 2, indicating that this measurement scheme can reveal the nonclassical correlation between the vertices. When d is fixed and N is large [see Fig. 5(a)], the matrix elements of the normalized Ω are nearly equal, i.e., the normalized quantum state of the remaining vertices is approximately $|+\rangle_d^{\otimes 2}$, thus C approaches 2 when N goes to infinity. When N is fixed and d increases [see Fig. 5(b)], $|H_{N,d,m}^{(2)}\rangle$ approaches $(|0\rangle \sum_{i=1}^{d-1} |i\rangle + \sum_{i=1}^{d-1} |i\rangle\, |0\rangle)/\sqrt{2(d-1)}$, which is equivalent to a two-qubit maximally entangled state, thus C approaches $2\sqrt{2}$ when d goes to infinity.

C. Discussion

Remarkably, the above scheme of exhibiting the nonlocality of multipartite quantum systems is potentially applicable in the current use of qudit hypergraph states and conventional qubit graph states. In fact, it can be, and in some cases has been, used in practice with current technology. In the case of entanglement verification, it involves only two measurement settings at each side, where the measurement settings of assistant qudits never change. Besides, the CHSH inequality can always reveal the "strong" nonlocality in the sense that the entanglement between two arbitrary faraway qudits can be revealed, as long as the two are connected by other vertices and edges. All these features make the above scheme rather experimentally friendly.

For example, the entanglement verification of cluster states (a special class of graph states) generated by cold-atom lattices is necessary work for future use in quantum computing. However, the detection of entanglement in large-scale cluster states is always a challenging problem [50]. From a practical perspective, the CHSH scheme discussed in this section can also be used as an entanglement witness for cluster states, especially for the *long-distance* entanglement. That is, one can always choose two interested particles (connected by other particles with C-phase operations), and test the entanglement correlation between them, no matter how far the two particles are.

Another example is its application in quantum networks [51,52], in which thousands of users complete a quantum-information task via a multipartite entangled state. A typical task is the so-called third-man quantum cryptography in which generation of a cryptographic key is controlled by a third operator who decides whether to activate the key generation [53]. Therefore, the scheme we discussed offers exactly an operational way to analyze the security of the third-man quantum cryptography.

An important problem in qudit hypergraph states we did not discuss is genuine multipartite entanglement. In particular, the relationship between the classification of multipartite entanglement and the property of hypergraphs deserves to be studied in depth, and the triple entanglement case has been discussed in [37]. As an analog, the concept of genuine multipartite nonlocality was also put forward in [37]. However, despite its significance in the theoretical study, its applications in quantum information processing need further study.

VI. CONCLUSIONS

In this work, we have proposed a large class of quantum states, called qudit hypergraph states, in which every vertex of the multi-hypergraph represents a d-level quantum system. We have investigated the operational definition of these states and studied their stabilizers, which possess potential applications in quantum codes and quantum computation.

The multi-hypergraphs and qudit hypergraph states have a one-to-one correspondence, and the entanglement of the qudit hypergraph states can be directly illustrated by the structure of their corresponding multi-hypergraphs. If a multi-hypergraph (or part of it) is connected, the corresponding quantum system (the quantum system corresponding to the connected part) is genuinely entangled. Such entanglement leads to potential exhibition of Bell nonlocality. As an example, we showed how to obtain the violation of Bell inequality in N-uniform qudit hypergraph states. The method is also applicable to other qudit hypergraph states and general N-qudit quantum states.

We also study the relationship among qudit hypergraph states and some important state classes. As for the real equally weighted states, we generalize them to the qudit case. It is shown that only in the two-level case are the two state classes ("generalized real equally weighted states" and "qudit hypergraph states") the same; otherwise, qudit hypergraph states are a subclass of "generalized real equally weighted states." The relationship among qudit hypergraph states, qudit graph states, and stabilizer states is discussed. Our results demonstrate that qudit graph states are a common subclass of qudit hypergraph states and stabilizer states. What is more, the union of these two state classes contains more than qudit graph states, which is very similar to the qubit case.

Nevertheless, much work is still needed to be done for the potential properties and applications of qudit hypergraph states. It is known that the set of qubit hypergraph states is the same as the set of *real equally weighted state*s, which is a class of quantum states having important applications in quantum algorithms. Qudit hypergraph states form a subclass of generalized real equally weighted states. In this sense, it is highly probable that qudit hypergraph states also have important applications in quantum algorithms. It has been shown in the literature that the unique entanglement form and Bell nonlocality of qubit hypergraph states have important applications in quantum metrology and novel quantum computation schemes. It is worthy of further study to see whether the qudit hypergraph states have similar applications. In this paper, we have focused on the simplest definition of entanglement (a quantum state is entangled if it cannot be written as a tensor product of two state vectors), while in fact there is much more comprehensive content in the study of multipartite entanglement, for example equivalent classes of multipartite entanglement. The discussion of such issues in the context of qudit hypergraph states is not only interesting by itself but also essential for future applications.

Note added. Recently, we became aware of a paper that proposed qudit hypergraph states in a different manner and discussed their SLOCC and LU classification [37].

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APPENDIX: DERIVATION OF EQ. (10)

If $k \notin e$, $C_e X_k C_e^\dagger = X_k$.

If $k \in e$, for simplicity of discussion, assume that $k = 1$ and $e = \{1,\dots,n\}$, then $C_e = \sum_{i_1=0}^{d-1} \hat{\Pi}_{i_1} C_{e\setminus\{1\}}^{i_1}$ (Lemma 1), thus

$$
\begin{aligned}
C_e X_1 C_e^\dagger &= \sum_{i_1,j_1=0}^{d-1} \hat{\Pi}_{i_1}(|0\rangle\langle 1| + |1\rangle\langle 2| + \cdots + |d-1\rangle\langle 0|)\hat{\Pi}_{j_1} C_{e\setminus\{1\}}^{i_1-j_1} \\
&= |0\rangle\langle 1|C_{e\setminus\{1\}}^{-1} + |1\rangle\langle 2|C_{e\setminus\{1\}}^{-1} + \cdots + |d-1\rangle\langle 0|C_{e\setminus\{1\}}^{-1} \\
&= X_1 C_{e\setminus\{1\}}^{-1} = X_1 C_{e\setminus\{1\}}^{\dagger}.
\end{aligned} \tag{A1}
$$

Generally, $C_e X_k C_e^\dagger = X_k C_{e\setminus\{k\}}^\dagger$.

Let X_k pass over all C_e. Then we have

$$
\left(\prod_{e\in E} C_e^{m_e}\right) X_k \left(\prod_{e'\in E} C_{e'}^{m_{e'}}\right)^\dagger = X_k \prod_{e:k\in e} (C_{e\setminus\{k\}}^\dagger)^{m_e}, \tag{A2}
$$

which is exactly what is demonstrated in Eq. (10).

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