Ultimate entanglement robustness of two-qubit states against general local noises

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We study the problem of optimal preparation of a bipartite entangled state, which remains entangled the longest time under action of local qubit noises. We show that for unital noises, such a state is always maximally entangled, whereas for nonunital noises, it is not. We develop a decomposition technique relating nonunital and unital qubit channels, based on which we find the explicit form of the ultimately robust state for general local noises. We illustrate our findings by amplitude damping processes at finite temperature, for which the ultimately robust state remains entangled up to two times longer than conventional maximally entangled states.

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I. INTRODUCTION

Quantum communication is one of the most developed subfields in the practical realization of quantum information protocols [1]. Dense coding [2], quantum teleportation [3], remote state preparation [4], and some cryptographic schemes [5–7] are based on the phenomenon of entanglement. Entanglement is also widely used in other quantum information applications [8]. When two laboratories A and B are taken into account, by entangled state we understand a density operator ϱ^{AB} (unit trace positive-semidefinite operator acting on some Hilbert space $\mathcal{H}$), which does not belong to a closure of separable states of the form $\varrho^{AB} = \sum_k p_k \varrho_k^A \otimes \varrho_k^B$, $p_k \geqslant 0$, $\sum_k p_k = 1$ [9]. Entangled states cannot be created by local operations and classical communication from factorized states [10], so entanglement between noninteracting laboratories A and B can only be created via sending parts of a locally prepared initial entangled state $\varrho_{\rm in}^{AB}$ to A and B, respectively (transmission of an entangled state can also be a stage in a more involved process such as entanglement swapping [11]). Since A and B are supposed to be far apart, transmission of the entangled state is carried out by means of local quantum channels $\Phi^{AB} = \Phi_1^A \otimes \Phi_2^B$; see Fig. 1. Quantum channel $\Phi : \mathcal{B}(\mathcal{H}) \mapsto \mathcal{B}(\mathcal{H})$ is a completely positive trace-preserving map that describes the result of quantum system transformation due to unavoidable interaction with environment (quantum noise) [12–14]. The longer the quantum channels between the entanglement source and laboratories A, B, the noisier and less entangled becomes the output state, $\varrho_{\rm out}^{AB} = (\Phi_1^A \otimes \Phi_2^B)[\varrho_{\rm in}^{AB}]$ [15–20]. The length of the quantum channels can be included in the above description by time t quantifying the duration of the system-environment interaction: $\varrho^{AB}(t) = \Phi_1^A(t) \otimes \Phi_2^B(t)[\varrho_{\rm in}^{AB}]$, with $\Phi_1^A(0)$ and $\Phi_2^B(0)$ being identity transformations (Id). Preservation of entanglement of the state $\varrho^{AB}(t)$ is the primary goal for implementing entanglement-based protocols. In fact, if A and B are both qubit systems and $\varrho^{AB}(t)$ is entangled, then by sending the same state $\varrho_{\rm in}^{AB}$ through a quantum channel $\Phi_1^A(t) \otimes \Phi_2^B(t)$ many times, one can distill maximally entangled states $\varrho_+ = |\psi_+\rangle\langle\psi_+|$, $|\psi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ that are useful in entanglement-based applications [21]. Given quantum noises $\Phi_1^A(t)$ and $\Phi_2^B(t)$, the entanglement lifetime of the state $\varrho_{\rm in}^{AB}$ is defined as the minimal time τ such that $\varrho^{AB}(t)$ is separable for all $t \geqslant \tau$. In other words, the entanglement lifetime (also referred to as disentangling time) is the time of entanglement sudden death [15]. The maximal possible entanglement lifetime $\widetilde{\tau} = \max_{\varrho_{\rm in}^{AB}} \tau$ provides the fundamental restriction on the length of quantum channels to A and B. The state $\widetilde{\varrho}_{\rm in}^{AB}$, which maximizes entanglement lifetime, exhibits the ultimate entanglement robustness to local noises, $\Phi_1^A(t) \otimes \Phi_2^B(t)$. If $\widetilde{\varrho}_{\rm in}^{AB}$ is the most robust to the loss of entanglement with respect to the dynamical map $\Phi_1^A(t) \otimes \Phi_2^B(t)$, then separability of $\Phi_1^A(t) \otimes \Phi_2^B(t)[\widetilde{\varrho}_{\rm in}^{AB}]$ implies separability of $\Phi_1^A(t) \otimes \Phi_2^B(t)[\varrho_{\rm in}^{AB}]$ for all input states $\varrho_{\rm in}^{AB}$. Note that the output of the channel $\Phi_1^A(\widetilde{\tau}) \otimes \Phi_2^B(\widetilde{\tau})$ is separable for all possible input states, i.e., such a channel is entanglement annihilating [22–27].

Despite the fact that entanglement of a two-qubit system can be readily and precisely verified via the Peres-Horodecki criterion [28,29] or concurrence [30,31], it is not that easy to resolve the maximin problem of entanglement lifetime $\widetilde{\tau}$ even for a simple semigroup dynamics $\Phi_1^A(t) \otimes \Phi_2^B(t) = e^{\mathcal{L}_1^A t} \otimes e^{\mathcal{L}_2^B t}$ describing generalized amplitude damping processes [32,33]. It is also not known how to find the optimal state $\widetilde{\varrho}_{\rm in}^{AB}$ analytically. There are three distinguished exceptions, however. The first one is the case of one-sided noiseless evolution, when $\Phi_1^A(t) \equiv {\rm Id}$, i.e., one part of the entangled system is perfectly preserved; then the maximally entangled state ϱ_+ has ultimate robustness [34–36]. The second exception is the case of local depolarizing noises, with ϱ_+ being ultimately robust [37]. The third exception is the case of local unital [38] two-qubit dynamical maps $\Upsilon(t) \otimes \Upsilon(t)$, for which the maximally entangled state ϱ_+ is the most robust to the loss of entanglement too [23]. In this paper, we extend these results to the case of general local unital channels $\Upsilon_1^A(t) \otimes \Upsilon_2^B(t)$ and prove that the maximally entangled state ϱ_+ is optimal for the transmission of entanglement through such channels. It is tempting to conclude that the maximally entangled state ϱ_+ exhibits ultimate robustness to general local two-qubit noises $\Phi_1^A(t) \otimes \Phi_2^B(t)$; however, this is not true [39–41] and we show

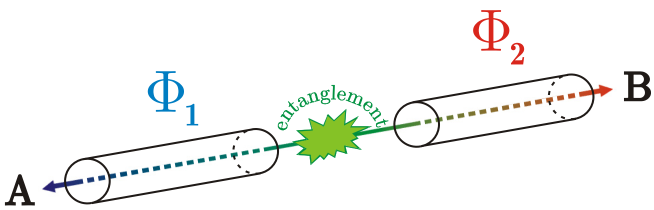

FIG. 1. Transmission of entangled state through local quantum channels.

that explicitly in this paper. Moreover, we analytically find the initial two-qubit state $\widetilde{\varrho}$, which is the most robust to a given nonunital local two-qubit dynamical map $\Phi_1(t) \otimes \Phi_2(t)$. The use of the optimal initial state for entanglement distribution enables essential extension of the length of communication lines, which we demonstrate by examples of generalized amplitude damping processes.

The paper is organized as follows. In Sec. II, we consider two-qubit local unital dynamical maps $\Upsilon(t) \otimes \Upsilon'(t)$ and prove that the ultimately robust state is necessarily maximally entangled. We also find a criterion to check if the map $\Upsilon(t) \otimes \Upsilon'(t)$ is entanglement annihilating, based on which one can straightforwardly calculate the maximal entanglement lifetime. In Sec. III A, we show how the results for unital dynamical maps are related with those for nonunital ones, provided a special decomposition is known. We find the explicit form of such a decomposition of nonunital channels in Sec. III B. In Sec. III C, we apply the developed theory to nonunital channels describing the process of amplitude damping due to qubit interaction with the environment of finite temperature. In Sec. IV, brief conclusions are given.

II. UNITAL CHANNELS

A unital qubit channel Υ is necessarily random unitary [42] and, with a suitable choice of input and output bases, can be represented in the form [43]

$$\Upsilon[X] = \frac{1}{2}\mathrm{tr}[X]I + \frac{1}{2}\sum_{i=1}^{3}\lambda_i \mathrm{tr}[\sigma_i X]\sigma_i, \tag{1}$$

where $\sigma_1,\sigma_2,\sigma_3$ is a conventional set of Pauli operators such that $\sigma_3|0\rangle = |0\rangle$ and $\sigma_3|1\rangle = -|1\rangle$. The map (1) is known to be positive if $-1 \leqslant \lambda_1,\lambda_2,\lambda_3 \leqslant 1$, completely positive if $1 \pm \lambda_3 \geqslant |\lambda_1 \pm \lambda_2|$, and entanglement breaking if $|\lambda_1| + |\lambda_2| + |\lambda_3| \leqslant 1$ [44]. We will associate every map Υ with the corresponding vector $\boldsymbol{\lambda} = (\lambda_1,\lambda_2,\lambda_3)^\top$.

Matrix representation $M_{ij}(\Upsilon) = \frac{1}{2}\mathrm{tr}[\sigma_i \Upsilon[\sigma_j]]$, $i,j = 0,\ldots,3$, $\sigma_0 = I$, of the map (1) reads

$$M(\Upsilon) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix} = \mathrm{diag}(1,\boldsymbol{\lambda}^\top). \tag{2}$$

A local unital two-qubit map $\Upsilon \otimes \Upsilon$ composed of identical unital maps Υ is known to be entanglement annihilating if $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 \leqslant 1$ [23], with the maximally entangled state ϱ_+ having the longest entanglement lifetime. Some sufficient and (separately) necessary conditions for entanglement annihilation of the general local unital two-qubit map $\Upsilon \otimes \Upsilon'$ are listed in Ref. [23]. We fill the gap in analysis of such maps and provide a criterion of entanglement annihilation.

Proposition 1. Suppose Υ and Υ' are positive qubit maps. Then the map $\Upsilon \otimes \Upsilon'$ is positive and entanglement annihilating if and only if $|\lambda_i|,|\lambda_i'| \leqslant 1$, $i = 1,2,3$, and $\boldsymbol{\lambda} P \boldsymbol{\lambda}' \leqslant 1$ for all signed permutation matrices P.

Proof. Sufficiency. Due to a convex structure of separable states, a map $\Upsilon \otimes \Upsilon'$ is entanglement annihilating if and only if $\Upsilon \otimes \Upsilon'[|\psi\rangle\langle\psi|]$ is separable for all pure states $|\psi\rangle$. On the other hand, any pure two-qubit state $|\psi\rangle$ can be represented as a linear combination of Bell-like states $|\varphi_i\rangle = \sigma_i \otimes I|\psi_+\rangle$, $i = 0,\ldots,3$,

$$|\psi\rangle = \sum_{i=0}^{3} c_i|\varphi_i\rangle = C \otimes I|\psi_+\rangle, \tag{3}$$

where $C = \sum_{i=0}^{3} c_i\sigma_i$. Denote $\Phi_C[X] = CXC^\dagger$; then the density operator of any two-qubit pure state takes the form

$$|\psi\rangle\langle\psi| = \Phi_C \otimes \mathrm{Id}[|\psi_+\rangle\langle\psi_+|]. \tag{4}$$

Kraus representation of the map Υ' is well known [43] and reads $\Upsilon'[X] = \sum_{j=0}^{3} q_j'\sigma_j X\sigma_j$, where real parameters $\{q_j'\}$ are uniquely expressed through parameters $\{\lambda_j'\}$. Since $I \otimes \sigma_j'|\psi_+\rangle = (\sigma_j')^\top \otimes I|\psi_+\rangle$, we get

$$\begin{aligned} \mathrm{Id} \otimes \Upsilon'[|\psi_+\rangle\langle\psi_+|] &= \sum_{j=0}^{3} q_j'(\sigma_j')^\top \otimes I|\psi_+\rangle\langle\psi_+|(\sigma_j')^\top \otimes I \\ &= \Upsilon' \otimes \mathrm{Id}[|\psi_+\rangle\langle\psi_+|], \end{aligned} \tag{5}$$

where we have taken into account that $(\sigma_2')^\top = -\sigma_2'$ and $(\sigma_j')^\top = \sigma_j'$ if $j = 0,1,3$. Combining (4) and (5), we can express the action of the map $\Upsilon \otimes \Upsilon'$ on any pure state as follows:

$$\begin{aligned} \Upsilon \otimes \Upsilon'[|\psi\rangle\langle\psi|] &= (\Upsilon \otimes \Upsilon') \circ (\Phi_C \otimes \mathrm{Id})[|\psi_+\rangle\langle\psi_+|] \\ &= (\Upsilon \circ \Phi_C \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \Upsilon')[|\psi_+\rangle\langle\psi_+|] \\ &= \Upsilon \circ \Phi_C \circ \Upsilon' \otimes \mathrm{Id}[|\psi_+\rangle\langle\psi_+|]. \end{aligned} \tag{6}$$

Therefore, the map $\Upsilon \otimes \Upsilon'$ is entanglement annihilating if and only if the output state (6) is separable for all matrices C. The necessary and sufficient criterion of separability of two-qubit states provides the reduction criterion [45], which states that the two-qubit state ϱ is separable if and only if $\mathcal{R} \otimes \mathrm{Id}[\varrho] \geqslant 0$, where the action of qubit map $\mathcal{R}$ reads $\mathcal{R}[X] = \mathrm{tr}[X]I - X$. Thus, the state (6) is separable if and only if $\mathcal{R} \circ \Upsilon \circ \Phi_C \circ \Upsilon' \otimes \mathrm{Id}[|\psi_+\rangle\langle\psi_+|] \geqslant 0$ or, equivalently,

$$\langle\chi|(\mathcal{R} \circ \Upsilon \circ \Phi_C \circ \Upsilon' \otimes \mathrm{Id}[|\psi_+\rangle\langle\psi_+|])|\chi\rangle \geqslant 0 \tag{7}$$

for all two-qubit states $|\chi\rangle$. Similarly to Eq. (4), we represent $|\chi\rangle = D \otimes I|\psi_+\rangle$ and conclude that $\Upsilon \otimes \Upsilon'$ is entanglement annihilating if and only if

$$\langle\psi_+|(\Phi_{D^\dagger} \circ \mathcal{R} \circ \Upsilon \circ \Phi_C \circ \Upsilon' \otimes \mathrm{Id}[|\psi_+\rangle\langle\psi_+|])|\psi_+\rangle \geqslant 0 \tag{8}$$

for all matrices C and D. Recalling $|\psi_+\rangle = \frac{1}{\sqrt{2}}\sum_{k=0}^{1}|k\rangle\otimes|k\rangle$, Eq. (8) is equivalent to

$$\sum_{k,l=0}^{1}\langle k|(\Phi_{D^\dagger}\circ\mathcal{R}\circ\Upsilon\circ\Phi_C\circ\Upsilon'[|k\rangle\langle l|])|l\rangle \geqslant 0. \tag{9}$$

The basis of matrix units $E_{kl}=|k\rangle\langle l|$ is orthonormal in the sense of Hilbert-Schmidt inner product $(X,Y)=\mathrm{tr}[X^\dagger Y]$. So is the basis of operators $\{\frac{1}{\sqrt{2}}\sigma_j\}_{j=0}^{3}$, and hence $E_{kl}=\sum_{j=0}^{3}W_{kl,j}\frac{1}{\sqrt{2}}\sigma_j$ and $\sum_{k,l=0}^{1}W_{kl,i}^{*}W_{kl,j}=\delta_{ij}$. Equation (9) takes the form

$$\begin{aligned}
0 &\leqslant \sum_{k,l=0}^{1}\mathrm{tr}[E_{kl}^{\dagger}\Phi_{D^\dagger}\circ\mathcal{R}\circ\Upsilon\circ\Phi_C\circ\Upsilon'[E_{kl}]]\\
&=\frac{1}{2}\sum_{i,j=0}^{3}\sum_{k,l=0}^{1}W_{kl,i}^{*}W_{kl,j}\mathrm{tr}[\sigma_i^{\dagger}\Phi_{D^\dagger}\circ\mathcal{R}\circ\Upsilon\circ\Phi_C\circ\Upsilon'[\sigma_j]]\\
&=\frac{1}{2}\sum_{i=0}^{3}\mathrm{tr}[\sigma_i\Phi_{D^\dagger}\circ\mathcal{R}\circ\Upsilon\circ\Phi_C\circ\Upsilon'[\sigma_i]]\\
&=\mathrm{tr}[M(\Phi_{D^\dagger}\circ\mathcal{R}\circ\Upsilon\circ\Phi_C\circ\Upsilon')]\\
&=\mathrm{tr}[M(\Phi_{D^\dagger})M(\mathcal{R})M(\Upsilon)M(\Phi_C)M(\Upsilon')]\\
&=\mathrm{tr}[M(\Phi_{D^\dagger})\mathrm{diag}(1,-\boldsymbol{\lambda}^{\top})M(\Phi_C)\mathrm{diag}(1,\boldsymbol{\lambda}'^{\top})]\\
&=(1,-\boldsymbol{\lambda}^{\top})\,M(\Phi_{D^\dagger})^{\top}*M(\Phi_C)\begin{pmatrix}1\\ \boldsymbol{\lambda}'\end{pmatrix}\\
&=(1,-\boldsymbol{\lambda}^{\top})\,M(\Phi_C)*M(\Phi_D)\begin{pmatrix}1\\ \boldsymbol{\lambda}'\end{pmatrix},
\end{aligned} \tag{10}$$

where $*$ denotes the Hadamard pointwise product, i.e., $(M*N)_{ij}=M_{ij}N_{ij}$.

To know matrix representations of maps Φ_C and Φ_D, we use singular-value decompositions,

$$C=U_C\begin{pmatrix}\sqrt{1+\sin\alpha_C} & 0\\ 0 & \sqrt{1-\sin\alpha_C}\end{pmatrix}V_C, \tag{11}$$

$$D=U_D\begin{pmatrix}\sqrt{1+\sin\alpha_D} & 0\\ 0 & \sqrt{1-\sin\alpha_D}\end{pmatrix}V_D, \tag{12}$$

which explicitly take into account that the states $|\psi\rangle$ and $|\chi\rangle$ are normalized, i.e., $\mathrm{tr}[C^\dagger C]=\mathrm{tr}[D^\dagger D]=2$. Here, U_C, V_C, U_D, V_D are unitary operators and $0\leqslant\alpha_C,\alpha_D\leqslant\frac{\pi}{2}$. Therefore,

$$M(\Phi_C)=\left(\begin{array}{c|c}1 & \mathbf{0}^{\top}\\ \hline \mathbf{0} & Q_{U_C}\end{array}\right)\left(\begin{array}{c|c}1 & \mathbf{t}_C^{\top}\\ \hline \mathbf{t}_C & T_C\end{array}\right)\left(\begin{array}{c|c}1 & \mathbf{0}^{\top}\\ \hline \mathbf{0} & Q_{V_C}\end{array}\right), \tag{13}$$

$$M(\Phi_D)=\left(\begin{array}{c|c}1 & \mathbf{0}^{\top}\\ \hline \mathbf{0} & Q_{U_D}\end{array}\right)\left(\begin{array}{c|c}1 & \mathbf{t}_D^{\top}\\ \hline \mathbf{t}_D & T_D\end{array}\right)\left(\begin{array}{c|c}1 & \mathbf{0}^{\top}\\ \hline \mathbf{0} & Q_{V_D}\end{array}\right), \tag{14}$$

where Q_U is a 3×3 orthogonal matrix corresponding to channel Φ_U, $\mathbf{0}=(0,0,0)^{\top}$, $\mathbf{t}_{C(D)}=(0,0,\sin\alpha_{C(D)})^{\top}$, and $T_{C(D)}=\mathrm{diag}(\cos\alpha_{C(D)},\cos\alpha_{C(D)},1)$.

By $\mathbf{u}_{C(D)1},\mathbf{u}_{C(D)2},\mathbf{u}_{C(D)3}$, denote three orthonormal columns of the matrix $Q_{U_{C(D)}}$ and, by $\mathbf{v}_{C(D)1}^{\top},\mathbf{v}_{C(D)2}^{\top},\mathbf{v}_{C(D)3}^{\top}$, denote three orthonormal rows of the matrix $Q_{V_{C(D)}}$. Introduce the vectors

$$\mathbf{u}_{kl}=\mathbf{u}_{Ck}*\mathbf{u}_{Dl},\quad \mathbf{v}_{kl}=\mathbf{v}_{Ck}*\mathbf{v}_{Dl}. \tag{15}$$

Then the direct calculation of the Hadamard product $M(\Phi_C)*M(\Phi_D)$ yields

$$M(\Phi_C)*M(\Phi_D)=\left(\begin{array}{c|c}1 & \sin\alpha_C\sin\alpha_D\mathbf{v}_{33}^{\top}\\ \hline \sin\alpha_C\sin\alpha_D\mathbf{u}_{33} & S\end{array}\right), \tag{16}$$

where

$$\begin{aligned}
S=\ &\mathbf{u}_{33}\mathbf{v}_{33}^{\top}+\cos\alpha_C[\mathbf{u}_{13}\mathbf{v}_{13}^{\top}+\mathbf{u}_{23}\mathbf{v}_{23}^{\top}]\\
&+\cos\alpha_D[\mathbf{u}_{31}\mathbf{v}_{31}^{\top}+\mathbf{u}_{32}\mathbf{v}_{32}^{\top}]\\
&+\cos\alpha_C\cos\alpha_D[\mathbf{u}_{11}\mathbf{v}_{11}^{\top}+\mathbf{u}_{12}*\mathbf{v}_{12}^{\top}+\mathbf{u}_{21}\mathbf{v}_{21}^{\top}+\mathbf{u}_{22}\mathbf{v}_{22}^{\top}].
\end{aligned} \tag{17}$$

By the Cauchy-Bunyakovsky-Schwarz inequality, $|(u_{kl})_x|+|(u_{kl})_y|+|(u_{kl})_z|\leqslant|\mathbf{u}_{Ck}|\cdot|\mathbf{u}_{Dl}|=1$ and $|(v_{kl})_x|+|(v_{kl})_y|+|(v_{kl})_z|\leqslant|\mathbf{v}_{Ck}|\cdot|\mathbf{v}_{Dl}|=1$. Thus, all the vectors $\mathbf{u}_{kl}$ and $\mathbf{v}_{kl}$ belong to the *octahedron* with vertices $(\pm1,0,0)$, $(0,\pm1,0)$, and $(0,0,\pm1)$. Moreover, since vectors $\mathbf{u}_{C(D)1},\mathbf{u}_{C(D)2},\mathbf{u}_{C(D)3}$ are mutually orthogonal, vectors $\mathbf{u}_{kl}$ and $\mathbf{u}_{k'l}$ ($\mathbf{u}_{kl'}$) cannot belong to the same octant or opposite octants if $k\neq k'$ ($l\neq l'$). Since vectors $\mathbf{u}_{kl}$ and $\mathbf{v}_{kl}$ linearly contribute to the expression

$$\begin{aligned}
&(1,-\boldsymbol{\lambda}^{\top})\,M(\Phi_C)*M(\Phi_D)\begin{pmatrix}1\\ \boldsymbol{\lambda}'\end{pmatrix}\\
&\quad=1-\sin\alpha_C\sin\alpha_D\boldsymbol{\lambda}^{\top}\mathbf{u}_{33}+\sin\alpha_C\sin\alpha_D\mathbf{v}_{33}^{\top}\boldsymbol{\lambda}'-\boldsymbol{\lambda}^{\top}S\boldsymbol{\lambda}',
\end{aligned} \tag{18}$$

the minimal value of (18) is achieved if some vectors $\mathbf{u}_{kl}$ and $\mathbf{v}_{kl}$ correspond to the extreme points of the octahedron, i.e., to vectors $(\pm1,0,0)$, $(0,\pm1,0)$, and $(0,0,\pm1)$. Without loss of generality, it can be assumed that $\mathbf{u}_{33}=\mathbf{v}_{33}=(0,0,1)$, which implies $\mathbf{u}_{k3}=\mathbf{u}_{3k}=\mathbf{0}$ and $\mathbf{v}_{k3}=\mathbf{v}_{3k}=\mathbf{0}$, $k=1,2$. Then either $S=\mathrm{diag}(\pm\cos\alpha_C\cos\alpha_D,\pm\cos\alpha_C\cos\alpha_D,1)$ or $S=\begin{pmatrix}0 & \pm\cos\alpha_C\cos\alpha_D & 0\\ \pm\cos\alpha_C\cos\alpha_D & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$, where signs $\pm$ are not correlated. Inequality (10) reduces to

$$\begin{aligned}
0\leqslant\ &1-\sin\alpha_C\sin\alpha_D(\lambda_3-\lambda_3')-\lambda_3\lambda_3'\\
&-\cos\alpha_C\cos\alpha_D\begin{cases}\pm\lambda_1\lambda_1'\pm\lambda_2\lambda_2',\\ \pm\lambda_1\lambda_2'\pm\lambda_2\lambda_1',\end{cases}
\end{aligned} \tag{19}$$

which is fulfilled for all $0\leqslant\alpha_C,\alpha_D\leqslant\frac{\pi}{2}$ if $\boldsymbol{\lambda}\begin{pmatrix}\pm1 & 0 & 0\\ 0 & \pm1 & 0\\ 0 & 0 & 1\end{pmatrix}\boldsymbol{\lambda}'\leqslant 1$, $\boldsymbol{\lambda}\begin{pmatrix}0 & \pm1 & 0\\ \pm1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\boldsymbol{\lambda}'\leqslant 1$, and $|\lambda_i|,|\lambda_i'|\leqslant 1$, $i=1,2,3$.

It can easily be checked numerically that in the general case of arbitrary vectors $\mathbf{u}_{kl}$ and $\mathbf{v}_{kl}$, inequality (18) is fulfilled whenever $|\lambda_i|,|\lambda_i'|\leqslant 1$, $i=1,2,3$, and $\boldsymbol{\lambda}P\boldsymbol{\lambda}'\leqslant 1$ for all signed permutation matrices P.

Necessity. Let the input state $|\psi\rangle=|\psi_+\rangle$; then the output state $\Upsilon\otimes\Upsilon'[|\psi_+\rangle\langle\psi_+|]$ is separable by Peres-Horodecki criterion if and only if $1+\lambda_3\lambda_3'\pm(\lambda_1\lambda_1'-\lambda_2\lambda_2')\geqslant 0$ and $1-\lambda_3\lambda_3'\pm(\lambda_1\lambda_1'+\lambda_2\lambda_2')\geqslant 0$. Also, the state $\mathcal{R}\circ\Upsilon\otimes\Upsilon'[|\psi_+\rangle\langle\psi_+|]$ must be separable, which corresponds to the change $\lambda_i\to-\lambda_i$. By permuting indices (1,2,3) of the second qubit, we obtain that the condition $\boldsymbol{\lambda}P\boldsymbol{\lambda}'\leqslant 1$ must be fulfilled for all signed permutation matrices P. Permutation of indices corresponds to the change of input state to the form $\frac{1}{\sqrt{2}}(|\varphi\rangle \otimes |\chi\rangle + |\varphi_\perp\rangle \otimes |\chi_\perp\rangle)$, where $\{|\varphi\rangle,|\varphi_\perp\rangle\}$ and $\{|\chi\rangle,|\chi_\perp\rangle\}$ are bases of eigenvectors of some Pauli operators. ■

Corollary 1. Suppose $1 \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant 0$ and $1 \geqslant \lambda_1' \geqslant \lambda_2' \geqslant \lambda_3' \geqslant 0$; then the local two-qubit unital map $\Upsilon \otimes \Upsilon'$ is entanglement annihilating if and only if $\boldsymbol{\lambda}^\top \boldsymbol{\lambda}' = \lambda_1\lambda_1' + \lambda_2\lambda_2' + \lambda_3\lambda_3' \leqslant 1$.

Proof. It is not hard to see that $\boldsymbol{\lambda}^\top P \boldsymbol{\lambda}'$ achieves maximum among signed permutation matrices P if $P = I$. Then the statement of Corollary 1 follows directly from Proposition 1. ■

In the necessity part of Proposition 1, we have noticed that ultimate robust states to local noises $\Upsilon(t) \otimes \Upsilon'(t)$ are the states of the form $\frac{1}{\sqrt{2}}(|\varphi\rangle \otimes |\chi\rangle + |\varphi_\perp\rangle \otimes |\chi_\perp\rangle)$, where $\{|\varphi\rangle,|\varphi_\perp\rangle\}$ and $\{|\chi\rangle,|\chi_\perp\rangle\}$ are bases of eigenvectors of some Pauli operators.

Proposition 2. Suppose a local two-qubit unital noise $\Upsilon(t) \otimes \Upsilon'(t)$, with matrix representations of $\Upsilon(t)$, $\Upsilon'(t)$ being diagonal in the basis of Pauli operators $\sigma_1,\sigma_2,\sigma_3$. Then the state with ultimate entanglement robustness is the maximally entangled state $|\psi_{\Upsilon\otimes\Upsilon'}\rangle = \frac{1}{\sqrt{2}}(|\varphi\rangle \otimes |\chi\rangle + |\varphi_\perp\rangle \otimes |\chi_\perp\rangle)$, where $\{|\varphi\rangle,|\varphi_\perp\rangle\}$ and $\{|\chi\rangle,|\chi_\perp\rangle\}$ are orthogonal eigenvectors of some Pauli operators (σ_1, σ_2, or σ_3).

Example 1. Consider an amplitude damping process of a two-level system (see, e.g., Ref. [12], sec. 8.3.5), when the temperature of the environment is so high (thermal energy $kT \gg \Delta E$, energy-level separation) that the rate of spontaneous emission equals the rate of spontaneous absorbtion. If this is the case, then Markov approximation leads to the following master equation in the interaction picture ([13], sec. 10.1):

$$\frac{d\varrho}{dt} = \gamma\big(\sigma_+\varrho\sigma_- - \tfrac{1}{2}\{\sigma_-\sigma_+,\varrho\}\big) + \gamma\big(\sigma_-\varrho\sigma_+ - \tfrac{1}{2}\{\sigma_+\sigma_-,\varrho\}\big), \tag{20}$$

where $\{\cdot,\cdot\}$ denotes anticommutator, $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm \sigma_2)$, and $\gamma > 0$ is the damping rate. Solution of this master equation results in a unital map (1) with $\lambda_1(t) = \lambda_2(t) = e^{-\gamma t}$ and $\lambda_3(t) = e^{-2\gamma t}$.

Suppose two qubits, each experiencing amplitude damping in a high-temperature environment with damping rates γ and γ', respectively. Then the maximally entangled state with one excitation $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle)$ exhibits the maximal entanglement robustness and the entanglement lifetime is determined by the equation $\lambda_1(t)\lambda_1'(t) + \lambda_2(t)\lambda_2'(t) + \lambda_3(t)\lambda_3'(t) = 1$, i.e., $2e^{-(\gamma+\gamma')t} + e^{-2(\gamma+\gamma')t} = 1$. The maximal entanglement lifetime equals $\widetilde{\tau} = \frac{\ln(\sqrt{2}+1)}{\gamma+\gamma'} \approx \frac{0.88}{\gamma+\gamma'}$.

Example 2. Suppose a pair of entangled qubits is prepared in laboratory A; one qubit is kept in the quantum memory cell of laboratory A and the other is sent to laboratory B. The qubit in laboratory A is subjected to amplitude damping in a high-temperature environment with damping rates γ, and the itinerant qubit experiences depolarization with dissipator $\mathcal{L} = \gamma' \sum_{j=1}^{3}(\sigma_j\varrho\sigma_j - \varrho)$. Then, $\lambda_1(t) = \lambda_2(t) = e^{-\gamma t}$, $\lambda_3(t) = e^{-2\gamma t}$, and $\lambda_1'(t) = \lambda_2'(t) = \lambda_3'(t) = e^{-\gamma' t}$. From Corollary 1, it follows that the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle)$ is ultimately robust to entanglement loss. The maximal entanglement lifetime $\widetilde{\tau}$ is a solution of $(1 + e^{-\gamma t})^2 = 1 + e^{\gamma' t}$ and approximately equals $\widetilde{\tau} \approx \frac{3\ln 3}{4\gamma+3\gamma'}$. This shows that entanglement is more sensitive to the decoherence rate in the memory cell (rate of the amplitude damping process).

III. NONUNITAL CHANNELS

A. Ultimate robustness

We continue using notation Φ_A for a completely positive map with a single Kraus operator A, i.e., $\Phi_A[X] = AXA^\dagger$. The recent result of Ref. [46] suggests that if Φ is a qubit map belonging to the interior of the cone of positivity-preserving maps, then there exist positive-definite operators A and B acting on $\mathcal{H}_2$ such that the map

$$\Upsilon = \Phi_A \circ \Phi \circ \Phi_B \tag{21}$$

is unital. This result can be viewed as a quantum analogue of Sinkhorn's theorem [47]. One can always treat map Υ as diagonal in the basis of Pauli operators because appropriate unitary rotations of input and output bases can be attributed to operators B and A, respectively. Alternatively, $\Phi = \Phi_{A^{-1}} \circ \Upsilon \circ \Phi_{B^{-1}}$. The latter equation is simply a decomposition of a nonunital positive qubit map Φ through some unital map Υ. The time-dependent version of this relation for quantum dynamical maps takes the form

$$\Phi(t) = \Phi_{A^{-1}(t)} \circ \Upsilon(t) \circ \Phi_{B^{-1}(t)}. \tag{22}$$

Proposition 3. Suppose a local two-qubit noise $\Phi(t) \otimes \Phi'(t)$, where both $\Phi(t)$ and $\Phi'(t)$ adopt decompositions (22) with nondegenerate operators $A(t)$, $B(t)$, $A'(t)$, $B'(t)$ and unital diagonal maps $\Upsilon(t)$ and $\Upsilon'(t)$. Then, $\Phi(t) \otimes \Phi'(t)$ is entanglement annihilating if and only if $\Upsilon(t) \otimes \Upsilon'(t)$ is entanglement annihilating. Ultimate robustness to loss of entanglement exhibits the state of the form

$$|\psi_{\Phi\otimes\Phi'}\rangle = \frac{B(\widetilde{\tau}) \otimes B'(\widetilde{\tau})|\psi_{\Upsilon\otimes\Upsilon'}\rangle}{\sqrt{\langle\psi_{\Upsilon\otimes\Upsilon'}|B^\dagger(\widetilde{\tau})B(\widetilde{\tau}) \otimes B'(\widetilde{\tau})^\dagger B'(\widetilde{\tau})|\psi_{\Upsilon\otimes\Upsilon'}\rangle}}, \tag{23}$$

where $|\psi_{\Upsilon\otimes\Upsilon'}\rangle$ is given by Proposition 2 and $\widetilde{\tau}$ is the maximal entanglement lifetime under noise $\Upsilon(t) \otimes \Upsilon'(t)$.

Proof. Since $\Phi(t) \otimes \Phi'(t)[|\psi\rangle\langle\psi|] = A^{-1}(t) \otimes A'^{-1}(t)[\Upsilon(t) \otimes \Upsilon'(t)][B^{-1}(t) \otimes B'^{-1}(t)|\psi\rangle\langle\psi|B^{\dagger-1}(t) \otimes B'^{\dagger-1}(t)]A^{\dagger-1}(t) \otimes A'^{\dagger-1}(t)$ and both $A(t)$ and $A'(t)$ are nondegenerate, then $\Phi(t) \otimes \Phi'(t)[|\psi\rangle\langle\psi|]$ is separable if and only if $[\Upsilon(t) \otimes \Upsilon'(t)][B^{-1}(t) \otimes B'^{-1}(t)|\psi\rangle\langle\psi|B^{\dagger-1}(t) \otimes B'^{\dagger-1}(t)]$ belongs to a cone of separable operators. Thus, $\Phi(t) \otimes \Phi'(t)[|\psi\rangle\langle\psi|]$ is separable for all $|\psi\rangle$ if and only if $[\Upsilon(t) \otimes \Upsilon'(t)][B^{-1}(t) \otimes B'^{-1}(t)|\psi\rangle\langle\psi|B^{\dagger-1}(t) \otimes B'^{\dagger-1}(t)]$ is a separable operator for all $|\psi\rangle$. As both $B(t)$ and $B'(t)$ are nondegenerate, the linear span of operators $B^{-1}(t) \otimes B'^{-1}(t)|\psi\rangle\langle\psi|B^{\dagger-1}(t) \otimes B'^{\dagger-1}(t)$ for all $|\psi\rangle$ is a cone of positive operators. Thus, $\Phi(t) \otimes \Phi'(t)[|\psi\rangle\langle\psi|]$ is separable for all $|\psi\rangle$ if and only if $[\Upsilon(t) \otimes \Upsilon'(t)][\varrho]$ is separable for all density operators ϱ, i.e., $\Upsilon(t) \otimes \Upsilon'(t)$ is entanglement annihilating. Since $\Phi(t) \otimes \Phi'(t)[|\psi_{\Phi\otimes\Phi'}\rangle\langle\psi_{\Phi\otimes\Phi'}|] \propto A^{-1}(t) \otimes A'^{-1}(t)[\Upsilon(t) \otimes \Upsilon'(t)][|\psi_{\Upsilon\otimes\Upsilon'}\rangle\langle\psi_{\Upsilon\otimes\Upsilon'}|]A^{\dagger-1}(t) \otimes A'^{\dagger-1}(t)$, then $\Phi(t) \otimes \Phi'(t)[|\psi_{\Phi\otimes\Phi'}\rangle\langle\psi_{\Phi\otimes\Phi'}|]$ is entangled if and only if the state $\Upsilon(t)\otimes\Upsilon'(t)[|\psi_{\Upsilon\otimes\Upsilon'}\rangle\langle\psi_{\Upsilon\otimes\Upsilon'}|]$ is entangled. Therefore, (23) exhibits ultimate robustness to loss of entanglement if $|\psi_{\Upsilon\otimes\Upsilon'}\rangle$ is ultimately robust to loss of entanglement due to unital noises $\Upsilon(t)\otimes\Upsilon'(t)$. ∎

B. Explicit decomposition of nonunital qubit maps

To utilize Proposition 3 for particular physical systems, one needs to know explicitly the operators A and B as well as the unital map Υ in formula (22) for a given qubit channel Φ. In what follows, we develop ideas of Ref. [46] to find such explicit expressions.

By a suitable choice of input and output bases, one can reduce the matrix representation of any nonunital qubit channel Φ to the following form [43]:

$$M(\Phi)=\begin{pmatrix}1&0&0&0\\ t_1&\lambda_1&0&0\\ t_2&0&\lambda_2&0\\ t_3&0&0&\lambda_3\end{pmatrix}. \tag{24}$$

Formula $\varrho=\frac{1}{2}(I+\sum_{j=1}^{3}r_j\sigma_j)$ establishes a one-to-one correspondence between qubit density operators ϱ and real Bloch vectors $\mathbf{r}=(r_1,r_2,r_3)^\top$ satisfying $|\mathbf{r}|=\sqrt{\sum_{j=1}^{3}r_j^2}\leqslant 1$. The Bloch vector of the density operator $\Phi[\varrho]$ is $(\lambda_1 r_1+t_1,\lambda_2 r_2+t_2,\lambda_3 r_3+t_3)^\top$. From this geometrical picture, it is not hard to see that positivity of the map Φ implies

$$\sum_{j=1}^{3}\frac{t_j^2}{(1-|\lambda_j|)^2}\leqslant 1. \tag{25}$$

This necessary condition for positivity of Φ means that the vector $\mathbf{t}$ has to belong to an ellipsoid with principal axes of length $2(1-|\lambda_j|)$, $j=1,2,3$. If $|\lambda_j|=1$, then $t_j=0$, and the ratio $\frac{t_j^2}{(1-|\lambda_j|)^2}$ should be treated as zero.

Following Ref. [46], we introduce operators $\widetilde{A}=\sqrt{S}$ and $\widetilde{B}=(\Phi^\dagger[S])^{-1/2}$, where the positive Hermitian operator S is a fixed point of the map $F[S]=(\Phi\{(\Phi^\dagger[S])^{-1}\})^{-1}$ and $\Phi^\dagger$ is a dual linear map such that $\mathrm{tr}[\Phi^\dagger[X]Y]=\mathrm{tr}[X\Phi[Y]]$ for all X,Y. Reference [46] shows that the map $\Phi_{\widetilde{A}}\circ\Phi\circ\Phi_{\widetilde{B}}$ is unital and positive if Φ belongs to the interior of the cone of positivity-preserving maps, although matrix representation of the map $\Phi_{\widetilde{A}}\circ\Phi\circ\Phi_{\widetilde{B}}$ is not necessarily diagonal.

We develop results of Ref. [46] and find S explicitly. We fix $\mathrm{tr}[S]=2$, denote $S=I+\sum_{j=1}^{3}x_j\sigma_j$, and introduce a new variable $y=1+\sum_{j=1}^{3}t_jx_j$. Then, $S=F[S]$ reduces to

$$y-1=y\sum_{j=1}^{3}\frac{t_j^2}{\lambda_j^2-y}, \tag{26}$$

$$x_j=\frac{yt_j}{\lambda_j^2-y},\quad j=1,2,3. \tag{27}$$

Equation (26) is simply a quartic equation,

$$y^4+by^3+cy^2+dy+e=0, \tag{28}$$

with coefficients

$$b=t_1^2+t_2^2+t_3^2-\lambda_1^2-\lambda_2^2-\lambda_3^2-1, \tag{29}$$

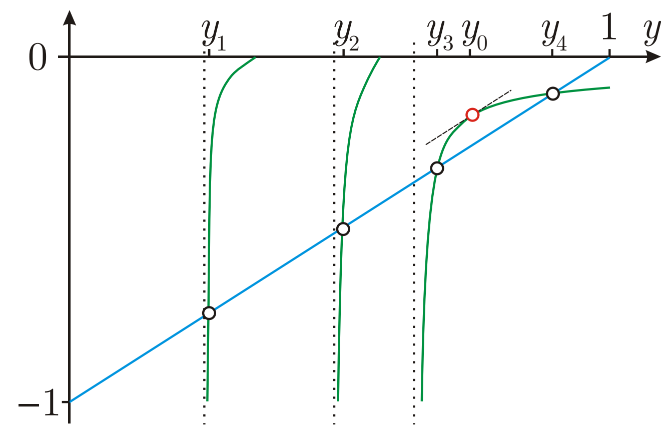

FIG. 2. Graphical solution of Eq. (26). All quantities are dimensionless.

$$c=\lambda_1^2(1-t_2^2-t_3^2)+\lambda_2^2(1-t_1^2-t_3^2)+\lambda_3^2(1-t_1^2-t_2^2)+\lambda_1^2\lambda_2^2+\lambda_2^2\lambda_3^2+\lambda_3^2\lambda_1^2, \tag{30}$$

$$d=t_1^2\lambda_2^2\lambda_3^2+\lambda_1^2t_2^2\lambda_3^2+\lambda_1^2\lambda_2^2t_3^2-\lambda_1^2\lambda_2^2\lambda_3^2-\lambda_1^2\lambda_2^2-\lambda_2^2\lambda_3^2-\lambda_3^2\lambda_1^2, \tag{31}$$

$$e=\lambda_1^2\lambda_2^2\lambda_3^2, \tag{32}$$

so it can be readily solved analytically, e.g., by Ferrari's method [48]. Let us demonstrate that if Φ is a positive map, then the obtained equation has four real non-negative roots (possibly degenerate), with the greatest one guaranteeing positivity of operator S.

In fact, a graph of the left-hand side of Eq. (26) is a line, and a graph of the right-hand side of Eq. (26) has (in general) three vertical asymptotes at points $y=\lambda_j^2\leqslant 1$; see Fig. 2. Thus, Eq. (26) definitely has two real roots, $y_{1,2}\in[0,\max\lambda_j^2)$. The derivative of the right-hand side of Eq. (26) equals 1 at point $y_0>\max\lambda_j^2$ satisfying $\sum_{j=1}^{3}\frac{t_j^2\lambda_j^2}{(\lambda_j^2-y_0)^2}=1$. From this follows that $y_0\leqslant|\lambda_j|$ for all $j=1,2,3$ because otherwise we encounter a contradiction $1=\sum_{j=1}^{3}\frac{t_j^2\lambda_j^2}{(\lambda_j^2-y_0)^2}<\sum_{j=1}^{3}\frac{t_j^2}{(1-|\lambda_j|)^2}\leqslant 1$; cf. Eq. (25). Thus, $y_0\leqslant|\lambda_j|$ and the right-hand side of Eq. (26) equals $y_0-\sum_{j=1}^{3}\frac{t_j^2y_0^2}{(\lambda_j^2-y_0)^2}\geqslant y_0-\sum_{j=1}^{3}\frac{t_j^2\lambda_j^2}{(\lambda_j^2-y_0)^2}=y_0-1$. Therefore, at point $y=y_0$, the right-hand side of Eq. (26) is larger than or equal to the left-hand side of Eq. (26), so Eq. (26) has two more real roots, $y_{3,4}\in(\max\lambda_j^2,1]$; see Fig. 2. Moreover, the derivative of the right-hand side of Eq. (26) at the largest root y_4 is less than or equal to 1, which readily implies that values x_1,x_2,x_3 corresponding to this root satisfy $\sum_{j=1}^{3}x_j^2\leqslant 1$, i.e., the operator S is positive semidefinite. If Φ belongs to the interior of positive maps, then S is positive.

Calculating $\widetilde{A}$, $\widetilde{B}$ and simplifying unitary map $\Phi_{\widetilde{A}}\circ\Phi\circ\Phi_{\widetilde{B}}$ as much as possible, we obtain the following result.

Proposition 4. Suppose a nonunital qubit map Φ, which belongs to the interior of the cone of positivity-preserving maps and is defined by matrix representation (24). Let the largest real root y of quartic equation (28) define coefficients x_j, $j = 1,2,3$, by Eq. (27). Let $x = \sqrt{\sum_{j=1}^3 x_j^2}$ and $\xi = \sqrt{\sum_{j=1}^3 \lambda_j^2 x_j^2}$; then operators

$$\widetilde{A} = \frac{\sqrt{1+x}+\sqrt{1-x}}{2} I + \frac{\sqrt{1+x}-\sqrt{1-x}}{2x}\sum_{j=1}^3 x_j\sigma_j, \tag{33}$$

$$\widetilde{B} = \frac{\sqrt{y+\xi}+\sqrt{y-\xi}}{2\sqrt{y^2-\xi^2}} I - \frac{\sqrt{y+\xi}-\sqrt{y-\xi}}{2\xi\sqrt{y^2-\xi^2}}\sum_{j=1}^3 \lambda_j x_j\sigma_j \tag{34}$$

are Hermitian and positive; the map $\Phi_{\widetilde{A}} \circ \Phi \circ \Phi_{\widetilde{B}}$ is unital, positive, trace preserving, and its matrix representation reads $M_{00}(\Phi_{\widetilde{A}} \circ \Phi \circ \Phi_{\widetilde{B}}) = 1$, $M_{0i}(\Phi_{\widetilde{A}} \circ \Phi \circ \Phi_{\widetilde{B}}) = M_{i0}(\Phi_{\widetilde{A}} \circ \Phi \circ \Phi_{\widetilde{B}}) = 0$,

$$M_{ij}(\Phi_{\widetilde{A}} \circ \Phi \circ \Phi_{\widetilde{B}}) = \frac{1-x^2}{\sqrt{y^2-\xi^2}}\left\{\frac{\lambda_i\delta_{ij}}{\sqrt{1-x^2}} + \left[\frac{1-\sqrt{1-x^2}}{x^2\sqrt{y^2-\xi^2}} - \frac{(y-\sqrt{y^2-\xi^2})\lambda_i^2}{\xi^2\sqrt{1-x^2}\,y}\right]x_i\lambda_j x_j\right\}, \tag{35}$$

where $i,j = 1,2,3.$ and δ_{ij} is the Kronecker delta. Conventional decomposition of matrix (35),

$$\left[M_{ij}(\Phi_{\widetilde{A}} \circ \Phi \circ \Phi_{\widetilde{B}})\right]_{i,j=1,2,3} = Q_{\widetilde{U}}\,\mathrm{diag}(\widetilde{\lambda}_1,\widetilde{\lambda}_2,\widetilde{\lambda}_3)Q_{\widetilde{V}}, \tag{36}$$

with orthogonal matrices $Q_{\widetilde{U}}$ and $Q_{\widetilde{V}}$, $\det Q_{\widetilde{U}} = \det Q_{\widetilde{V}} = 1$, leads to the unital map $\Upsilon = \Phi_{\widetilde{U}^\dagger\widetilde{A}} \circ \Phi \circ \Phi_{\widetilde{B}\widetilde{V}^\dagger}$ with diagonal matrix representation $M(\Upsilon) = \mathrm{diag}(1,\widetilde{\lambda}_1,\widetilde{\lambda}_2,\widetilde{\lambda}_3)$. Operators $A = \widetilde{U}^\dagger\widetilde{A}$ and $B = \widetilde{B}\widetilde{V}^\dagger$.

Proposition 4 allows one to reduce any nonboundary qubit channel Φ to a unital map Υ with diagonal matrix representation.

The obtained result becomes particularly simple in the case $t_1 = t_2 = 0$ because, in this case, Eq. (26) is readily solved and matrix (35) is automatically diagonal. Thus, no diagonalization (36) is needed, $A = \widetilde{A}$ and $B = \widetilde{B}$.

Corollary 2. Suppose a nonboundary qubit channel Φ given by matrix representation (24) with $t_1 = t_2 = 0$. If

$$A = \frac{2}{\sqrt{(1+t_3)^2-\lambda_3^2}+\sqrt{(1-t_3)^2-\lambda_3^2}} \times \begin{pmatrix}\sqrt{(1+|t_3|)^2-\lambda_3^2} & 0\\ 0 & \sqrt{(1-|t_3|)^2-\lambda_3^2}\end{pmatrix}, \tag{37}$$

$$B = \begin{pmatrix}\frac{1}{\sqrt{1+t_3x_3+|\lambda_3x_3|}} & 0\\ 0 & \frac{1}{\sqrt{1+t_3x_3-|\lambda_3x_3|}}\end{pmatrix},$$

$$x_3 = -t_3\frac{1-t_3^2+\lambda_3^2+\sqrt{[(1+t_3)^2-\lambda_3^2][(1-t_3)^2-\lambda_3^2]}}{1-t_3^2-\lambda_3^2+\sqrt{[(1+t_3)^2-\lambda_3^2][(1-t_3)^2-\lambda_3^2]}}, \tag{38}$$

then $\Upsilon = \Phi_A \circ \Phi \circ \Phi_B$ is a unital qubit channel with eigenvalues

$$\widetilde{\lambda}_1 = \frac{2\lambda_1}{\sqrt{(1+\lambda_3)^2-t_3^2}+\sqrt{(1-\lambda_3)^2-t_3^2}}, \tag{39}$$

$$\widetilde{\lambda}_2 = \frac{2\lambda_2}{\sqrt{(1+\lambda_3)^2-t_3^2}+\sqrt{(1-\lambda_3)^2-t_3^2}}, \tag{40}$$

$$\widetilde{\lambda}_3 = \frac{4\lambda_3}{\left[\sqrt{(1+\lambda_3)^2-t_3^2}+\sqrt{(1-\lambda_3)^2-t_3^2}\right]^2}. \tag{41}$$

C. Generalized amplitude damping processes at finite temperature

A two-level system with energy-level separation ΔE is coupled with a reservoir of finite temperature T, which results in a generalized amplitude damping process,

$$\frac{d\varrho}{dt} = \gamma w(2\sigma_+\varrho\sigma_- - \{\sigma_-\sigma_+,\varrho\}) + \gamma(1-w)(2\sigma_-\varrho\sigma_+ - \{\sigma_+\sigma_-,\varrho\}), \tag{42}$$

where $w, 1-w$ are the populations of ground and excited levels in thermal equilibrium, i.e., $\frac{1-w}{w} = \exp(-\frac{\Delta E}{kT})$. The resulting dynamical map $\Phi(t)$ is nonunital, and its matrix representation is

$$M(\Phi(t)) = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & e^{-\gamma t} & 0 & 0\\ 0 & 0 & e^{-\gamma t} & 0\\ (2w-1)(1-e^{-2\gamma t}) & 0 & 0 & e^{-2\gamma t}\end{pmatrix}. \tag{43}$$

Using Corollary 2, we find the corresponding unital dynamical map $\Upsilon(t)$ with eigenvalues

$$\widetilde{\lambda}_1(t) = \widetilde{\lambda}_2(t) = e^{-\gamma t}\{\sqrt{w(1-w)}(1-e^{-2\gamma t}) + \sqrt{[1-w(1-e^{-2\gamma t})][w+e^{-2\gamma t}(1-w)]}\}^{-1}, \tag{44}$$

and $\widetilde{\lambda}_3(t) = \widetilde{\lambda}_1^2(t) = \widetilde{\lambda}_2^2(t)$. The latter relation means that $\Upsilon(t)$ is simply an amplitude damping process with the infinite temperature of the environment considered in Examples 1 and 2, although the generator of $\Upsilon(t)$ is time dependent due to a time deformation. Exploiting Eq. (38), we also find

$$B(t) \propto \sqrt[4]{(1-w)[1-(1-w)(1-e^{-2\gamma t})]}\,\sigma_+\sigma_- + \sqrt[4]{w[1-w(1-e^{-2\gamma t})]}\,\sigma_-\sigma_+. \tag{45}$$

Example 3. Suppose two identical qubits, each experiencing amplitude damping in a reservoir with a finite temperature T such that $w, 1-w$ are the populations of ground and excited levels in thermal equilibrium (the case of two memory qubits [49]). What is the optimal preparation of the initial entangled state, whose entanglement lifetime is the longest? Surprisingly,

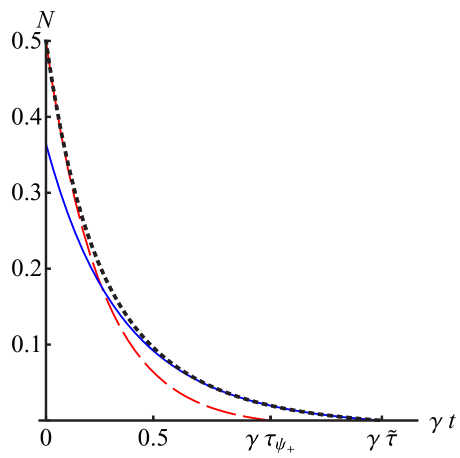

FIG. 3. Evolution of negativity under local generalized amplitude damping noise $\Phi(t)\otimes\Phi(t)$ with $w=0.01$ for the following initial states: the maximally entangled state (red dashed curve) and the ultimately robust state (blue solid curve). γt is dimensionless time. The dotted line represents a collection of negativities for states $\Phi(t)\otimes\Phi(t)[|\psi_t\rangle\langle\psi_t|]$, where the interpolating initial state $|\psi_t\rangle$ is given by Eq. (50).

it is not the maximally entangled state. Using Proposition 3 and Eq. (45), we conclude that ultimate robustness exhibits the state

$$|\psi_{\Phi\otimes\Phi}\rangle = \sqrt{\frac{(1-w)[1-(1-w)(1-e^{-2\gamma\tilde{\tau}})]}{1-(1-2w+2w^2)(1-e^{-2\gamma\tilde{\tau}})}}\,|0\rangle\otimes|1\rangle + \sqrt{\frac{w[1-w(1-e^{-2\gamma\tilde{\tau}})]}{1-(1-2w+2w^2)(1-e^{-2\gamma\tilde{\tau}})}}\,|1\rangle\otimes|0\rangle, \quad (46)$$

where $\tilde{\tau}$ is the maximal entanglement lifetime under unital noise $\Upsilon(t)\otimes\Upsilon(t)$. Using Corollary 1 and the explicit form of eigenvalues (44), we get

$$\tilde{\tau} = \frac{1}{2\gamma}\ln\frac{4(\sqrt{2}+1)w(1-w)}{1+4(\sqrt{2}+1)w(1-w)-\sqrt{1+8(\sqrt{2}+1)w(1-w)}}, \quad (47)$$

which is much greater than the entanglement lifetime of the maximally entangled state $|\psi_+\rangle$,

$$\tau_{\psi_+} = \frac{1}{2\gamma}\ln\frac{1+\sqrt{2w(1-w)}}{\sqrt{2w(1-w)}}. \quad (48)$$

If $w\to 0$, then $\tilde{\tau}/\tau_{\psi_+}\to 2$, i.e., the use of the ultimately robust state allows one to prolong the entanglement lifetime twice as compared with the entanglement lifetime of the maximally entangled state. A comparison of entanglement dynamics for initial states $|\psi_{\Phi\otimes\Phi}\rangle$ and $|\psi_+\rangle$ is depicted in Fig. 3. We use negativity $N(\varrho)=\frac{1}{2}(\|\varrho^\Gamma\|_1-1)$ as the entanglement measure of the state ϱ [50,51] (ϱ^Γ is the partial transpose of ϱ with respect to one of the qubits).

Finally, $\Phi(t)\otimes\Phi(t)$ is entanglement annihilating if and only if

$$1-e^{-2\gamma t}\geqslant\frac{\sqrt{1+8(\sqrt{2}+1)w(1-w)}-1}{4(\sqrt{2}+1)w(1-w)}. \quad (49)$$

This result solves the problem of characterizing entanglement annihilation by generalized amplitude damping noises raised in Ref. [23].

Although the state (46) is less entangled initially, it remains entangled longer than the maximally entangled state $|\psi_+\rangle$, whose entanglement is greater in the beginning of evolution; see Fig. 3. Thus, the state (46) is optimal for preserving entanglement as long as possible, whereas the maximally entangled state $|\psi_+\rangle$ is optimal for a short storage of entanglement. In practice, however, one may be interested in storing entanglement for some intermediate time t_0. An interpolation between $|\psi_+\rangle$ and the state (46) is the normalized state

$$|\psi_{t_0}\rangle\propto\left\{(1-w)[1-(1-w)(1-e^{-2\gamma\tilde{\tau}})]\right\}^{\frac{t_0}{2\tilde{\tau}}}|0\rangle\otimes|1\rangle + \left\{w[1-w(1-e^{-2\gamma\tilde{\tau}})]\right\}^{\frac{t_0}{2\tilde{\tau}}}|1\rangle\otimes|0\rangle. \quad (50)$$

One can see that the state $\Phi(t)\otimes\Phi(t)[|\psi_{t_0}\rangle\langle\psi_{t_0}|]$ has a high degree of entanglement at time moment t_0, which is illustrated by negativity in Fig. 3. Thus, using the state (50) as the initial state, one is able to reach a high degree of entanglement at time t_0.

In general, if a large degree of entanglement is desired at time t_0, then the interpolation for the optimally prepared state is a modification of Eq. (23),

$$|\psi_{\Phi\otimes\Phi'}(t_0)\rangle\propto[B(\tilde{\tau})\otimes B'(\tilde{\tau})]^{t_0/\tilde{\tau}}|\psi_{\Upsilon\otimes\Upsilon'}\rangle. \quad (51)$$

The state (51) always differs from the maximally entangled state $|\psi_+\rangle$ if at least one of the noises $\Phi(t)$ and $\Phi'(t)$ is nonunital and $t_0>0$.

Example 4. Suppose a pair of entangled qubits, with the first qubit experiencing generalized amplitude damping in a memory cell (parameters w, γ) and the second (itinerant) qubit being affected by a depolarizing noise with rate γ'. Suppose it takes time t_0 for the second qubit to reach another laboratory, after which an experiment with two apart qubits is performed. Maximal entanglement lifetime $\tilde{\tau}$ is a solution of $[1+\tilde{\lambda}_1(t)]^2=1+e^{\gamma' t}$, where $\tilde{\lambda}_1(t)$ is given by Eq. (44). Since operator B is defined by Eq. (45) and operator $B'=I$ in this case, then the optimal initial state guaranteeing a high degree of final entanglement for $t_0\in[0,\tilde{\tau})$ is

$$|\psi_{t_0}\rangle\propto\{(1-w)[1-(1-w)(1-e^{-2\gamma\tilde{\tau}})]\}^{\frac{t_0}{4\tilde{\tau}}}|0\rangle\otimes|1\rangle + \{w[1-w(1-e^{-2\gamma\tilde{\tau}})]\}^{\frac{t_0}{4\tilde{\tau}}}|1\rangle\otimes|0\rangle. \quad (52)$$

Note that this state is different from the state (50).

IV. CONCLUSIONS

We have analyzed entanglement dynamics of two-qubit entangled states subjected to local qubit noises of the most general form.

If the noise is unital, then the ultimately robust state to entanglement loss is maximally entangled. We have found a criterion (Proposition 1), which allows one to find the maximal entanglement lifetime in this case.

If the noise is nonunital, then we have reduced this problem to the previous one by developing a decomposition technique suggested in Ref. [46]. Hereby, we have solved the problem of full characterization of local two-qubit entanglement annihilating channels raised in Ref. [23]. Moreover, explicit decomposition of nonunital qubit maps (22) can find further applications in the analysis of n-tensor stable positive maps [52,53], absolutely separating quantum maps [54], and evaluation of channel capacities.

The ultimately robust state turns out to differ from the maximally entangled one for nonunital noises. By examples of generalized amplitude damping noises, we show that the ultimately robust state remains entangled about twice as long as compared with the maximally entangled one if environment temperature tends to zero. This fact shows that the use of an ultimately robust entangled state is beneficial for entanglement preservation. The communication length for entanglement-based protocols can be significantly increased by using optimal state preparation. Similarly, disentanglement time in physically implementable systems, e.g., electron spins, could be increased as compared to the disentanglement time for maximally entangled initial states [55,56].

Finally, we construct an interpolation initial state, which has a high degree of entanglement for a particular time moment t. This state is close to the maximally entangled state if t tends to zero and to the ultimately robust state if t approaches the maximal entanglement lifetime.

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[1] N. Gisin and R. Thew, Quantum communication, Nat. Photon. **1**, 165 (2007).
[2] C. Bennett and S. Wiesner, Communication Via One- and Two-Particle Operators on Einstein-Podolsky-Rosen States, Phys. Rev. Lett. **69**, 2881 (1992).
[3] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Teleporting an Unknown Quantum State via Dual Classical and Einstein-Podolsky-Rosen Channels, Phys. Rev. Lett. **70**, 1895 (1993).
[4] C. H. Bennett, P. Hayden, D. W. Leung, P. W. Shor, and A. Winter, Remote preparation of quantum states, IEEE Trans. Inf. Theory **51**, 56 (2005).
[5] A. K. Ekert, Quantum Cryptography Based on Bell's Theorem, Phys. Rev. Lett. **67**, 661 (1991).
[6] D. Deutsch, A. Ekert, R. Jozsa, C. Macchiavello, S. Popescu, and A. Sanpera, Quantum Privacy Amplification and the Security of Quantum Cryptography Over Noisy Channels, Phys. Rev. Lett. **77**, 2818 (1996).
[7] A. Acín, N. Brunner, N. Gisin, S. Massar, S. Pironio, and V. Scarani, Device-Independent Security of Quantum Cryptography against Collective Attacks, Phys. Rev. Lett. **98**, 230501 (2007).
[8] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, Rev. Mod. Phys. **81**, 865 (2009).
[9] R. F. Werner, Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model, Phys. Rev. A **40**, 4277 (1989).
[10] M. A. Nielsen, Conditions for a Class of Entanglement Transformations, Phys. Rev. Lett. **83**, 436 (1999).
[11] M. Żukowski, A. Zeilinger, M. A. Horne, and A. K. Ekert, "Event-Ready-Detectors" Bell Experiment via Entanglement Swapping, Phys. Rev. Lett. **71**, 4287 (1993).
[12] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
[13] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2002).
[14] A. S. Holevo, *Quantum Systems, Channels, Information* (Walter de Gruyter, Berlin, 2012).
[15] T. Yu and J. H. Eberly, Finite-Time Disentanglement Via Spontaneous Emission, Phys. Rev. Lett. **93**, 140404 (2004).
[16] B. Bellomo, R. Lo Franco, and G. Compagno, Entanglement dynamics of two independent qubits in environments with and without memory, Phys. Rev. A **77**, 032342 (2008).
[17] T. Yu and J. H. Eberly, Sudden Death of Entanglement, Science **323**, 598 (2009).
[18] C. Wang and Q.-H. Chen, Exact dynamics of quantum correlations of two qubits coupled to bosonic baths, New J. Phys. **15**, 103020 (2013).
[19] A. Shaham, A. Halevy, L. Dovrat, E. Megidish, and H. S. Eisenberga, Entanglement dynamics in the presence of controlled unital noise, Sci. Rep. **5**, 10796 (2015).
[20] L. Aolita, F. de Melo, and L. Davidovich, Open-system dynamics of entanglement: A key issues review, Rep. Prog. Phys. **78**, 042001 (2015).
[21] M. Horodecki, P. Horodecki, and R. Horodecki, Inseparable Two Spin-$\frac{1}{2}$ Density Matrices Can Be Distilled to a Singlet Form, Phys. Rev. Lett. **78**, 574 (1997).
[22] L. Moravčíková and M. Ziman, Entanglement-annihilating and entanglement-breaking channels, J. Phys. A: Math. Theor. **43**, 275306 (2010).
[23] S. N. Filippov, T. Rybár, and M. Ziman, Local two-qubit entanglement-annihilating channels, Phys. Rev. A **85**, 012303 (2012).
[24] S. N. Filippov and M. Ziman, Bipartite entanglement-annihilating maps: Necessary and sufficient conditions, Phys. Rev. A **88**, 032316 (2013).
[25] S. N. Filippov, A. A. Melnikov, and M. Ziman, Dissociation and annihilation of multipartite entanglement structure in dissipative quantum dynamics, Phys. Rev. A **88**, 062328 (2013).
[26] S. N. Filippov, PPT-Inducing, Distillation-Prohibiting, and Entanglement-Binding Quantum Channels, J. Russ. Laser Res. **35**, 484 (2014).

[27] S. N. Filippov, Influence of deterministic attenuation and amplification of optical signals on entanglement and distillation of gaussian and non-gaussian quantum states, EPJ Web Conf. **103**, 03003 (2015).
[28] A. Peres, Separability Criterion for Density Matrices, Phys. Rev. Lett. **77**, 1413 (1996).
[29] M. Horodecki, P. Horodecki, and R. Horodecki, Separability of mixed states: Necessary and sufficient conditions, Phys. Lett. A **223**, 1 (1996).
[30] S. Hill and W. K. Wootters, Entanglement of a Pair of Quantum Bits, Phys. Rev. Lett. **78**, 5022 (1997).
[31] W. K. Wootters, Entanglement of Formation of an Arbitrary State of Two Qubits, Phys. Rev. Lett. **80**, 2245 (1998).
[32] L. Jakóbczyk and A. Jamróz, Noise-induced finite-time disentanglement in two-atomic system, Phys. Lett. A **333**, 35 (2004).
[33] M. Ali, G. Alber, and A. R. P. Rau, Manipulating entanglement sudden death of two-qubit X-states in zero- and finite-temperature reservoirs, J. Phys. B: At. Mol. Opt. Phys. **42**, 025501 (2009).
[34] M. Horodecki, P. W. Shor, and M. B. Ruskai, Entanglement Breaking Channels, Rev. Math. Phys. **15**, 629 (2003).
[35] T. Konrad, F. de Melo, M. Tiersch, C. Kasztelan, A. Aragão, and A. Buchleitner, Evolution equation for quantum entanglement, Nat. Phys. **4**, 99 (2008).
[36] J.-S. Xu, C.-F. Li, X.-Y. Xu, C.-H. Shi, X.-B. Zou, and G.-C. Guo, Experimental Characterization of Entanglement Dynamics in Noisy Channels, Phys. Rev. Lett. **103**, 240502 (2009).
[37] M. Tiersch, Benchmarks and statistics of entanglement dynamics, Ph.D. thesis, University of Freiburg, 2009, http://www.freidok.uni-freiburg.de/volltexte/6878/.
[38] A linear map $\Upsilon : \mathcal{B}(\mathcal{H}) \mapsto \mathcal{B}(\mathcal{H})$ is called unital if it preserves the identity operator, i.e., $\Upsilon[I] = I$.
[39] X.-W. Wang, S.-Q. Tang, J.-B. Yuan, and L.-M. Kuang, Nonmaximally Entangled States can be Better for Quantum Correlation Distribution and Storage, Int. J. Theor. Phys. **54**, 1461 (2015).
[40] M. Ziman and V. Bužek, Entanglement-induced state ordering under local operations, Phys. Rev. A **73**, 012312 (2006).
[41] M. Ziman and V. Bužek, Entanglement Measures: State ordering vs. Local Operations, in *Quantum Communication and Security*, edited by M. Żukowski, S. Kilin, and J. Kowalik (IOS Press, Amsterdam, 2007), pp. 196–204.
[42] L. J. Landau and R. F. Streater, On Birkhoff's theorem for doubly stochastic completely positive maps of matrix algebras, Linear Algebra Appl. **193**, 107 (1993).
[43] M. B. Ruskai, S. Szarek, and E. Werner, An analysis of completely-positive trace-preserving maps on M_2, Linear Algebra Appl. **347**, 159 (2002).
[44] M. B. Ruskai, Qubit Entanglement Breaking Channels, Rev. Math. Phys. **15**, 643 (2003).
[45] M. Horodecki and P. Horodecki, Reduction criterion of separability and limits for a class of distillation protocols, Phys. Rev. A **59**, 4206 (1999).
[46] G. Aubrun and S. J. Szarek, Two proofs of Størmer's theorem, arXiv:1512.03293.
[47] L. Gurvits, Classical complexity and quantum entanglement, J. Comput. Syst. Sci. **69**, 448 (2004).
[48] A. Kurosh, *Higher Algebra* (Mir, Moscow, 1980).
[49] K. M. Fonseca-Romero, J. Martínez-Rincón, and C. Viviescas, Statistical portrait of the entanglement decay of two-qubit memories, Phys. Rev. A **86**, 042325 (2012).
[50] K. Życzkowski, P. Horodecki, A. Sanpera, and M. Lewenstein, Volume of the set of separable states, Phys. Rev. A **58**, 883 (1998).
[51] G. Vidal and R. F. Werner, Computable measure of entanglement, Phys. Rev. A **65**, 032314 (2002).
[52] A. Müller-Hermes, D. Reeb, and M. M. Wolf, Positivity of linear maps under tensor powers, J. Math. Phys. **57**, 015202 (2016).
[53] S. N. Filippov and K. Y. Magadov, Positive tensor products of maps and n-tensor-stable positive qubit maps, J. Phys. A: Math. Theor. **50**, 055301 (2017).
[54] S. N. Filippov, K. Y. Magadov, and M. A. Jivulescu, Absolutely separating quantum maps and channels, New J. Phys. **19**, 083010 (2017).
[55] P. Mazurek, K. Roszak, R. W. Chhajlany, and P. Horodecki, Sensitivity of entanglement decay of quantum-dot spin qubits to the external magnetic field, Phys. Rev. A **89**, 062318 (2014).
[56] I. Bragar and L. Cywiński, Dynamics of entanglement of two electron spins interacting with nuclear spin baths in quantum dots, Phys. Rev. B **91**, 155310 (2015).