Equivalence of qubit-environment entanglement and discord generation via pure dephasing interactions and the resulting consequences

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We find that when a qubit initialized in a pure state experiences pure dephasing due to interaction with an environment, separable qubit-environment states generated during the evolution also have zero quantum discord with respect to the environment. What follows is that the set of separable states which can be reached during the evolution has zero volume, and hence, such effects as sudden death of qubit-environment entanglement are very unlikely. In the case of the discord with respect to the qubit, a vast majority of qubit-environment separable states is discordant, but in specific situations zero-discord states are possible. This is conceptually important since there is a connection between the discordance with respect to a given subsystem and the possibility of describing the evolution of this subsystem using completely positive maps. Finally, we use the formalism to find an exemplary evolution of an entangled state of two qubits that is completely positive, and occurs solely due to interaction of only one of the qubits with its environment (so one could guess that it corresponds to a local operation, since it is local in a physical sense), but which nevertheless causes the enhancement of entanglement between the qubits. While this simply means that the considered evolution is completely positive, but does not belong to local operations and classical communication, it shows how much caution has to be exercised when identifying evolution channels that belong to that class.

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I. INTRODUCTION

There is little or no ambiguity in the study of quantum correlations for pure states, as long as the potentially correlated parties are well defined and completely distinguishable. Such correlations can be fully described by entanglement, and none of the pure separable states (states with no entanglement) exhibit any type of behaviors which can be associated with quantum correlations. The two main characteristics of pure entangled states are that (a) it is not possible to prepare an entangled state via local operations and classical communication (criterion of preparation) and (b) it is not possible to fully determine the state of either of the entangled subsystems by local measurements on this subsystem and classical communication alone without disturbing it (criterion of measurement). It is this second characteristic which results in the property of entangled states that appropriately chosen measurements on one subsystem determine the state of the other subsystem, which underlie many applications of entangled states such as quantum algorithms [1,2] or quantum teleportation [3,4].

In the case of mixed states, the situation becomes more complicated. Mixed state entanglement [5–7] is defined using the above criterion of preparation, meaning that a state of two subsystems is entangled if and only if it cannot be written as a statistical mixture of product states of the two subsystems, and, equivalently, if it cannot be prepared by local operations and classical communication (LOCC). All entangled states satisfy the criterion of measurement as well, but there exist separable (not entangled) states which satisfy the criterion of measurement for one or both subsystems (while obviously not satisfying the criterion of preparation). A measure of quantum correlations which is based on the criterion of measurement is called the quantum discord [8–10]. The set of discordant states is larger than the set of entangled states, and in fact, it includes the set of entangled states, so although there do not exist entangled states with zero discord, there do exist separable discordant states [10]. It is important to note that there is an inherent asymmetry in the definition of the quantum discord with respect to the potentially correlated subsystems, since the criterion of measurement can be fulfilled for one of the subsystems while it is not fulfilled for the other.

Entanglement and the quantum discord differ significantly when it comes to the qualitative and quantitative features of their evolution. This is partly because the set of zero-discord states has zero volume [11], while the volume of separable states is finite. The characteristic property for entanglement evolutions, the possibility for it to undergo sudden death [12–14] (the complete disappearance of entanglement at a certain finite time, while the continuous decoherence of the entangled subsystems is not complete), which is sometimes followed by sudden birth (the reemergence of entanglement after a state is separable for a finite amount of time), is a direct result of the geometry of separable states. Since they have finite volume, there exist separable states which are completely surrounded by other separable states and may not be approached by means of a continuous evolution otherwise than from another separable state. Any evolution which approaches such a state has to display sudden death of entanglement. Since the volume of zero-discord states is not finite, any zero-discord state can be reached directly from a discordant state, and sudden-death-type behavior in the discord evolution is much less likely.

On the other hand, entanglement is symmetric with respect to both entangled subsystems, while the discord does not have to be symmetric with respect to the systems under study [10] (it is fairly common that the measurement of one of the correlated subsystems yields information about its state with less damage to the state itself than the other, some of the geometric measures are artificially symmetrized [15], or that a state is discordant only with respect to one of the subsystems). Another characteristic property of discord evolutions is the occurrence of points of indifferentiability (for which the time dependence of the discord function is continuous but not smooth). This quality of the discord should not be dismissed as an artifact of the mathematical properties of the geometric measures used to quantify the discord, since it has been observed in quantum discord curves calculated using the original discord definition [16,17] and in case of states which do have parity symmetry [18,19].

We focus here on quantum correlations, especially quantum discord, that appear between the system (a qubit or a pair of qubits) and its environment in the course of decoherence of the system. Specifically we consider here the system initialized in a pure state (obviously completely uncorrelated with the environment) that interacts with the environment via pure-dephasing-type coupling that singles out a basis of pointer states, and we consider states of the system that are superpositions of them. Generation of qubit-environment (Q-E) entanglement in this case was a subject of previous works [20–25], and here, building on the results of Ref. [24], we investigate the generation of system-environment discord (with respect both to the environment and to the qubit) during such evolution. In the following, we show that strong quantum correlations described by entanglement and weaker quantum correlations described by the quantum discord with respect to the environment are operationally the same. This means that the class of separable Q-E states that can be reached during such joint evolution is the same as the class of reachable zero-discord states *from the point of view of E*. Hence, all separable Q-E states obtained during the evolution are automatically one-sidedly zero-discord states, so they posses the specifically nondiscordant quality of being a zero-volume set of states. This suppresses the possibility for such Q-E evolutions to display characteristic behaviors for entanglement, such as its sudden death.

On the other hand, when we look at the property of discordance with respect to the system, it turns out to have qualitatively different properties than the discord discussed above. In fact, most of the separable Q-E evolutions are discordant in this sense, and only in very specific situations are zero-discord points possible during the evolution. This means that in terms of weak quantum correlations, two types of evolutions are possible.

After achieving such an understanding of Q-E discord generation, in the last part of the paper we use these results to shed light on issues related to role of system-environment quantum discord in dynamics of open systems. We generalize our results on Q-E evolution to the case of a class of entangled two-qubit states subjected to pure dephasing due to E. Then we construct an example of a system in which only one qubit interacts with E, so that the resulting decoherence is local, and we use the evolution due to the interaction with E to find an example of a state $\hat{\rho}_{\mathrm{SE}}^{\min}$ with zero discord between the qubits and E (zero with respect to the two-qubit subsystem), for which entanglement between the qubits is minimal, but subsequent evolution leads to an *increase* of interqubit entanglement. It is known that the lack of system-environment discord with respect to the system implies that the system's evolution starting from such a state may be described using completely positive (CP) maps [26], so the qubits' evolution starting from a $\hat{\rho}_{\mathrm{SE}}^{\min}$ state of the whole system is CP, but, despite the fact that only one of the qubits is interacting with its local environment, it is does not belong to the LOCC class.

The paper is organized as follows. In Sec. II we introduce the notion of the quantum discord further, including the original definition of the discord, and focusing on the differences and similarities between separable states and zero-discord states. In Sec. II B we state the criteria for zero-discord states following Ref. [27], which we will later use to obtain the main results of this paper. The class of systems under study is described in Sec. III, and the separability criterion specific for this system is the topic of Sec. III A. The equivalence of the class of separable states and zero-discord states with respect to the environment for the class of systems under study is shown in Sec. IV. The properties of the quantum discord with respect to the qubit are discussed in Sec. V, while an extension of the results to entangled two-qubit states is the topic of Sec. VI. In Sec. VII we discuss the implications of our results for the understanding of open quantum systems dynamics.

II. QUANTUM DISCORD

The quantification of quantum discord is in general complicated [28], even in comparison with the stronger measure of quantum correlations, entanglement. In the case of entangled mixed states, some means of quantification of the amount of mixed state entanglement have been available for two decades. This includes entanglement witnesses, a multitude of two-qubit measures [6,29–31], which allow for the calculation of two-qubit or qubit-qutrit entanglement directly from the density matrix.

Contrarily, the first geometric measures of the quantum discord (measures based on the calculation of the smallest distance between a given mixed quantum state and the set of zero-discord states; the distance measures used vary) [15,32–36] and methods of estimating their upper and lower bounds are about five years old [15,37,38]. Note that only the methods for the calculation of the limits on the quantum discord are direct ones, allowing for calculation from the density matrix of the studied system; the calculation of precise values of discord still requires minimization over the set of all zero-discord states.

A. Separable states versus zero-discord states

The class of separable states can be generally represented mathematically as the set of states, which can be written in the form

$$\rho_{\mathrm{AB}}^{\mathrm{sep}} = \sum_{\alpha} p_\alpha \rho_A^\alpha \otimes \rho_B^\alpha. \tag{1}$$

Here the density matrices on the left side of the tensor product correspond to subsystem A, and the ones on the right side correspond to subsystem B. The only constraint is on the parameters of the decomposition p_α, which have to be probabilities, $0 < p_\alpha < 1$ and $\sum_\alpha p_\alpha = 1$. Hence there are no constraints on the states ρ_A^α and ρ_B^α, which do not have to be pure (although there does exist an equivalent definition using projectors) or form an orthonormal basis for subsystem A or B. The lack of the orthonormality requirement is in fact the reason why checking for entanglement between two subsystems is in general complicated.

The class of zero-discord states can be represented in an analogous way [10]. The only difference is that there is an additional constraint on states of one or both subsystems. If the system state has zero discord with respect to subsystem A (B), then there must exist a decomposition of the joint state of systems A and B such that the density matrices $\rho_{A(B)}^\alpha$ in Eq. (1) can be written as projectors,

$$\rho_{A(B)}^\alpha = |a_\alpha\rangle\langle a_\alpha|, \tag{2}$$

where $\{|a_\alpha\rangle\}$ forms an ortonormal set in the subspace of subsystem A (B). If both zero-discord criteria for subsystems A and B are fulfilled, then the state is completely zero-discordant (there is no discord with respect to either subsystem). Note that the set of zero-discord states is obviously a subset of separable states regardless of whether it is discordant with respect to one or both subsystems.

B. The criteria for zero-discord states

Contrarily to the case of entanglement, for which even the determination, if a mixed-state density matrix is entangled or not, is complicated for bipartite entanglement of larger systems (for which at least one is not a qubit or qutrit), the determination, if the quantum discord is present in a system, is fairly straightforward even in the case of two arbitrarily large systems [15,27].

In the following we used the criterion of Ref. [27], which is more suitable for the class of systems under study (both criteria allow us to check if a state is discordant with respect to one of the potentially correlated systems at a time). The criterion introduced in the paper states that a bipartite state (where the parties are of arbitrary dimension N and M) has zero quantum discord with respect to the system M, if and only if all blocks of its density matrix, after the bipartite $(NM) \times (NM)$ density matrix is partitioned into N^2 matrices of dimension $M \times M$ (the particulars of the partition are described below), are normal matrices and commute with each other.

Here the partition is performed starting from a bipartite density matrix

$$\hat{\sigma} = \sum_{kq} \sum_{nm} P_{kq}^{nm} |k\rangle\langle q| \otimes |n\rangle\langle m|, \tag{3}$$

where the indices (and states labeled by them) k,q correspond to one of the subsystems (say, the one of dimension N), while the indices (and states) n,m correspond to the other subsystem (of dimension M). Note that the parameters P_{kq}^{nm} must fulfill a number of conditions for the matrix σ to be a density matrix, but we will not concern ourselves with those here, since in the following a state obtained via a unitary evolution from an initial product state of two density matrices will be considered, which can obviously always be described by a density matrix). The partition of this density matrix into N^2 blocks of dimension $M \times M$ is done as

$$\hat{\sigma}_{kq} = \langle k|\hat{\sigma}|q\rangle \tag{4}$$

for all k,q.

The criterion of normality means that for all k,q

$$[\hat{\sigma}_{kq},\hat{\sigma}_{kq}^\dagger] = 0, \tag{5}$$

while the commutation criterion means that for all k,q and k',q'

$$[\hat{\sigma}_{kq},\hat{\sigma}_{k'q'}] = 0. \tag{6}$$

Both criteria are fulfilled if and only if the state has zero discord with respect to subsystem M (meaning that the state of subsystem of size M can be fully determined by local measurements performed on this subsystem and classical communication alone without disturbing it).

III. THE CLASS OF SYSTEMS UNDER STUDY

We study a class of systems consisting of a qubit and an environment which, when only the qubit is a system of interest, always lead to pure dephasing evolutions of the qubit (the occupations of the qubit remain unchanged). In general, the Hamiltonian of such a system can be written as

$$\hat{H} = \hat{H}_{\mathrm{Q}} + \hat{H}_{\mathrm{E}} + |0\rangle\langle 0| \otimes \hat{V}_0 + |1\rangle\langle 1| \otimes \hat{V}_1. \tag{7}$$

The first term of the Hamiltonian describes the qubit and is given by $\hat{H}_{\mathrm{Q}} = \sum_{i=0,1} \varepsilon_i |i\rangle\langle i|$, the second describes the environment and is arbitrary, while the remaining terms describe the Q-E interaction with the qubit states written on the left side of each term and the environmental operators $\hat{V}_0$ and $\hat{V}_1$ are also arbitrary. Such a pure dephasing model of decoherence is not only theoretically easier to treat than a more general case of arbitrary Q-E coupling, it is also applicable to a very wide class of experimentally relevant qubit systems. For a vast majority of qubits, including almost all the solid-state based ones [39], but also trapped ions [40], pure dephasing of superposition of $|0\rangle$ and $|1\rangle$ pointer states happens on time scales much shorter than relaxation between these states caused by energy exchange with the environment, and basically all the coherence loss can be modeled with the above Hamiltonian. Some specific examples of systems described by Hamiltonian (7) include excitonic qubits coupled to phonons [41–44] (then $\hat{H}_{\mathrm{E}}$ is a Hamiltonian of free bosonic modes, $\hat{V}_0 = 0$ as $|0\rangle$ corresponds to no exciton, and $\hat{V}_1$ is linear in creation and annihilation operators of phonons), qubits based on single spins in quantum dots interacting with spin baths at finite magnetic fields [45,46] (then $\hat{V}_0 = -\hat{V}_1$ so that the coupling can be written as $\propto \hat{V}\hat{\sigma}_z$ where $\hat{V}$ is the operator of the nuclear Overhauser field and $\hat{\sigma}_z$ pertains to spin-1/2 qubit), and spin qubits based on a nitrogen-vacancy center in diamond coupled to a bath of carbon nuclei [47] or electron spins [48,49], for which both $\hat{V}_0 = -\hat{V}_1$ and $\hat{V}_0 = 0$, $\hat{V}_1 \neq 0$ can be realized [50].

The full Q-E evolution operator $\hat{U}(t) = \exp(-i\hat{H}t)$ resulting from the Hamiltonian (7) can be written as

$$\hat{U}(t) = |0\rangle\langle 0| \otimes \hat{w}_0(t) + |1\rangle\langle 1| \otimes \hat{w}_1(t), \quad (8)$$

where we have defined the operators

$$\hat{w}_i(t) = \exp(-i\hat{H}_i t), \quad (9)$$

with $i = 0,1$, and $\hat{H}_i = \hat{H}_E + \hat{V}_i$.

We study the joint state of a qubit and an environment which are initially in a product state $\hat{\sigma}(0) = \hat{\rho}_Q(0) \otimes \hat{R}(0)$ and evolve according to the operator (8). The qubit is initially in a pure state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, with $|\alpha|^2 + |\beta|^2 = 1$ and $\alpha,\beta \neq 0$ (a superposition is needed for dephasing to occur as well as entanglement and discord generation), so the density matrix $\hat{\rho}_Q(0) = |\psi\rangle\langle\psi|$. We impose no restrictions on the initial density matrix of the environment and write it in terms of its eigenstates, $\hat{R}(0) = \sum_n c_n |n\rangle\langle n|$. The time-evolved Q-E density matrix takes the form

$$\hat{\sigma}(t) = \begin{pmatrix} |\alpha|^2 \sum_n c_n |n_0(t)\rangle\langle n_0(t)| & \alpha\beta^* \sum_n c_n |n_0(t)\rangle\langle n_!(t)| \\ \alpha^*\beta \sum_n c_n |n_1(t)\rangle\langle n_0(t)| & |\beta|^2 \sum_n c_n |n_1(t)\rangle\langle n_1(t)| \end{pmatrix}, \quad (10)$$

where the matrix is written in the basis of the eigenstates of the free qubit Hamiltonian, and $|n_i(t)\rangle = \hat{w}_i(t)|n\rangle$ with $\hat{w}_i(t)$ given by Eq. (12).

Note that the density matrix written as in Eq. (10) is already decomposed into four blocks (with respect to the states of the qubit). Each element of the matrix (10), written as it is in block form, is the type of block that allows us to check for the presence of the discord with respect to the environment following the criteria of the previous section.

A. The zero-entanglement criterion for pure dephasing

The problem of separability for the class of systems described above has been solved in Ref. [24]. A joint state of the qubit and its environment (10) which is generated by the evolution operator given by Eq. (8) [which itself comes from the Hamiltonian (7)] is separable at time t if and only if

$$[\hat{R}(0), \hat{w}(t)] = 0, \quad (11)$$

where

$$\hat{w}(t) = \exp(i\hat{H}_0 t)\exp(-i\hat{H}_1 t) = \hat{w}_0(t)\hat{w}_1(t). \quad (12)$$

This criterion can be equivalently stated as

$$\hat{w}_0^\dagger(t)\hat{R}(0)\hat{w}_0(t) = \hat{w}_1^\dagger(t)\hat{R}(0)\hat{w}_1(t). \quad (13)$$

In this form, the criterion is particularly easy to compare with the results of applying the zero-discord criteria to the system under study.

IV. EQUIVALENCE OF SEPARABILITY AND ZERO DISCORD WITH RESPECT TO THE ENVIRONMENT

Using the criterion introduced in Sec. II B to check if the system-environment density matrix given by Eq. (10) is discordant with respect to the environment at time t is uncomplicated. We have the density matrix of the qubit and environment written in such a way that it is already divided into the aforementioned blocks, meaning that each of the four matrices in the subspace of the environment $\hat{\sigma}_{ij}(t) = \langle i|\hat{\sigma}(t)|j\rangle$ is a separate block ($|i\rangle,|j\rangle = |0\rangle,|1\rangle$ are qubit states).

It is straightforward to show that $\hat{\sigma}_{00}(t)$ and $\hat{\sigma}_{11}(t)$ are always normal, since $\hat{\sigma}_{00}(t) = \hat{\sigma}_{00}^\dagger(t)$ and $\hat{\sigma}_{11}(t) = \hat{\sigma}_{11}^\dagger(t)$, so $[\hat{\sigma}_{ii}(t), \hat{\sigma}_{ii}^\dagger(t)] = 0$. For the blocks corresponding to the diagonal elements of the qubit density matrix, $\hat{\sigma}_{01}(t) = \hat{\sigma}_{10}^\dagger(t)$, the part of the normality criterion for zero-discord states, is not so easy to check, and in fact these matrices are not always normal. We will return to this below, since this normality criterion is equivalent to one of the commutation criteria.

Before we continue, let us denote

$$\hat{R}_{ij}(t) = \hat{w}_i(t)\hat{R}(0)\hat{w}_j^\dagger(t). \quad (14)$$

The criterion that all $\hat{\sigma}_{ij}(t)$ must commute obviously reduces to the criterion that all $\hat{R}_{ij}(t)$ must commute. For the density matrix (10), this leads to the following commutation conditions:

$$[\hat{R}_{00}(t), \hat{R}_{11}(t)] = 0, \quad (15a)$$
$$[\hat{R}_{00}(t), \hat{R}_{01}(t)] = \hat{R}_{01}(t)[\hat{R}_{11}(t) - \hat{R}_{00}(t)] = 0, \quad (15b)$$
$$[\hat{R}_{00}(t), \hat{R}_{10}(t)] = [\hat{R}_{00}(t) - \hat{R}_{11}(t)]\hat{R}_{10}(t) = 0, \quad (15c)$$
$$[\hat{R}_{11}(t), \hat{R}_{01}(t)] = [\hat{R}_{11}(t) - \hat{R}_{00}(t)]\hat{R}_{01}(t) = 0, \quad (15d)$$
$$[\hat{R}_{11}(t), \hat{R}_{10}(t)] = \hat{R}_{10}(t)[\hat{R}_{00}(t) - \hat{R}_{11}(t)] = 0, \quad (15e)$$
$$[\hat{R}_{01}(t), \hat{R}_{10}(t)] = \hat{R}_{00}(t)^2 - \hat{R}_{11}(t)^2 = 0, \quad (15f)$$

since

$$\begin{aligned}\hat{R}_{00}(t)\hat{R}_{01}(t) &= \hat{w}_0(t)\hat{R}(0)\hat{w}_0^\dagger(t)\hat{w}_0(t)\hat{R}(0)\hat{w}_1^\dagger(t) \\ &= \hat{w}_0(t)\hat{R}(0)\hat{w}_1^\dagger(t)\hat{w}_1(t)\hat{R}(0)\hat{w}_1^\dagger(t) \\ &= \hat{R}_{01}(t)\hat{R}_{11}(t),\end{aligned} \quad (16)$$

and so on. Note that the criteria of normality for matrices $\hat{\sigma}_{01}$ and $\hat{\sigma}_{10}$ are always equivalent to commutation criterion (15f), since $\hat{R}_{01}^\dagger(t) = \hat{R}_{10}(t)$. Furthermore, all of the conditions (15) are always satisfied when the state (10) is separable, since then $\hat{R}_{00}(t) - \hat{R}_{11}(t) = 0$ as shown in Eq. (13).

This means, quite surprisingly, that the class of separable (zero-entanglement) Q-E states which can be obtained during a pure-dephasing evolution is equivalent to the class of states with zero discord with respect to the environment for this type of evolution, since entangled states are always also discordant [10]. Consequently, for this type of evolution there is little or no difference between entanglement and the environmental quantum discord. The discord may still display qualities resulting from points of indifferentiability, but entanglement evolution is unlikely to display its most characteristic feature, namely, sudden death of entanglement, because for the class of systems under study not only the set of zero-discord states has zero volume, but also the set of separable states has zero volume (even "one-sided" zero-discord states possess the zero-volume quality).

V. SEPARABILITY AND ZERO DISCORD WITH RESPECT TO THE QUBIT

The relationship between separability and the lack of discord with respect to the qubit subspace is much more ambiguous. Since local unitary operations cannot change the amount of discord in a system [51] and specifically no local operations on the environment can change, whether a state is discordant with respect to the qubit or not, as is evident from the definition of "one-sided" zero-discord states in Sec. II A, nor do they change the purity of the reduced density matrix of the qubit, let us work with the Q-E density matrix transformed by an unitary operation on the environment (as we did in Ref. [24]):

$$\tilde{\sigma}(t) = \hat{w}_0^\dagger(t)\hat{\sigma}(t)\hat{w}_0(t). \tag{17}$$

Since separability at time t indicates that there exists a (time-dependent) basis $\{|n'(t)\rangle\}$ in which both the initial density matrix of the environment $\hat{R}(0)$ and the operator $\hat{w}_0^\dagger(t)\hat{w}_1(t)$ are diagonal, they can be written as $\hat{R}(0) = \sum_n c_n'(t)|n'(t)\rangle\langle n'(t)|$ and $\hat{w}_0^\dagger(t)\hat{w}_1(t) = \sum_n \exp(-i\phi_n(t))|n'(t)\rangle\langle n'(t)|$. Consequently the density matrix (17) can be written as

$$\tilde{\sigma}(t) = \sum_n c_n'(t)\begin{pmatrix} |\alpha|^2 & \alpha\beta^* e^{i\phi_n(t)} \\ \alpha^*\beta e^{-i\phi_n(t)} & |\beta|^2 \end{pmatrix} \otimes |n'(t)\rangle\langle n'(t)|. \tag{18}$$

If the separable system density matrix corresponding to time t, Eq. (18), is partitioned into N^2 (where N is the dimension of the environment) 2×2 matrices in terms of the eigenbasis $\{|n'(t)\rangle\}$, then the diagonal matrices [the ones corresponding to $|k\rangle = |q\rangle = |n'(t)\rangle$ in Eq. (4)] are of the form

$$\langle n'(t)|\tilde{\sigma}(t)|n'(t)\rangle = c_n'(t)\begin{pmatrix} |\alpha|^2 & \alpha\beta^* e^{-i\phi_n(t)} \\ \alpha^*\beta e^{i\phi_n(t)} & |\beta|^2 \end{pmatrix}, \tag{19}$$

while the off-diagonal matrices are equal to zero. Hence, all of the matrices of this partition fulfill the normality requirement for the discord, while the commutation requirements are reduced to

$$\begin{aligned}&[\langle n'(t)|\tilde{\sigma}(t)|n'(t)\rangle, \langle m'(t)|\tilde{\sigma}(t)|m'(t)\rangle] \\ &= c_n'(t)c_m'(t)\begin{pmatrix} 2i|\alpha|^2|\beta|^2\sin[\phi_n(t)-\phi_m(t)] & -\alpha\beta^*(|\alpha|^2-|\beta|^2)(e^{i\phi_n(t)}-e^{i\phi_m(t)}) \\ \alpha\beta^*(|\alpha|^2-|\beta|^2)(e^{-i\phi_n(t)}-e^{-i\phi_m(t)}) & 2i|\alpha|^2|\beta|^2\sin[\phi_n(t)-\phi_m(t)] \end{pmatrix} = 0\end{aligned} \tag{20}$$

for all n and m.

First, let us identify the trivial solutions that lead to no pure dephasing of the qubit (when no correlations of any type between a qubit and its environment are formed and the Q-E state remains a product). These include an initial state of the qubit which is not a superposition, i.e., $\alpha = 0$ or $\beta = 0$, and the situation when for all n and m for which $c_n(t)' \neq 0$ and $c_m'(t) \neq 0$, $\phi_n(t) = \phi_m(t) \bmod 2\pi$. Furthermore, the condition stemming from the off-diagonal elements of the matrices (20) that needs to be taken into account when $|\alpha| \neq |\beta|$ implies that for all n and m for which $c_n'(t) \neq 0$ and $c_m'(t) \neq 0$ we must have $\exp(i\phi_n(t)) = \exp(i\phi_m(t))$; if this condition is fulfilled, it is easy to see that the qubit does not undergo pure dephasing (and only a phase shift between the elements of its superposition). Hence pure dephasing is always accompanied by discord generation with respect to the qubit, as long as the initial state of the qubit is not an equal superposition state.

The only case when the qubit can experience pure dephasing due to an interaction with the environment which is not accompanied by discord with respect to the qubit state is if it is initially in an equal superposition state, $|\alpha| = |\beta| = 1/\sqrt{2}$. Then the set of commutation conditions for zero discord are reduced to $\sin[\phi_n(t) - \phi_m(t)] = 0$ for all n and m. To see that such a situation is possible let us study the simplest example, with the dimension of the environment $N = 2$. Imagine that at a certain time t the exponential factors are $\exp[i\phi_0(t)] = 1$ and $\exp[i\phi_1(t)] = -1$, respectively. The level of coherence of the qubit (the amplitude of the off-diagonal element of its reduced density matrix) is governed by the function $|c_0'(t) - c_1'(t)|$ and the qubit is fully coherent for $|c_0'(t) - c_1'(t)| = 1$ and in a completely mixed state for $|c_0'(t) - c_1'(t)| = 0$. Obviously, regardless of the values of $c_0'(t)$ and $c_1'(t)$, there is no discord of any kind between the qubit and the environment, but only for $c_0'(t) = 0$ or $c_1'(t) = 0$ is the qubit fully coherent, while the coherence of the qubit depends only on the mixedness of the initial density matrix of the environment. In the case of a single-qubit environment it is not possible for a pure dephasing evolution to have zero-qubit discord for all times, because this would require the initial environmental state to be pure, and for such a state the dephasing of the qubit is equivalent to creation of Q-E entanglement [52].

In general, in order to have zero discord with respect to the qubit initialized in an equal superposition state which undergoes nonentangling evolution the following conditions have to be met at time t. For all n and m that correspond to nonzero coefficients $c_n'(t)$ and $c_m'(t)$, the phase differences fulfill $\phi_n(t) - \phi_m(t) = p\pi$, where p is an integer. For an environment with $N > 2$ this implies a rigid condition on all phases corresponding to nonzero coefficients $c_n'(t)$, which must all have the form $\phi_n(t) = \phi_0(t) + q\pi$, again with integer q.

The situation is peculiar, since decoherence is almost always accompanied by a buildup of the quantum discord with respect to the qubit. The one (very notable) exception allows for discord-less dephased states, when the qubit state is of high symmetry and this symmetry is mirrored by the state of the environment. In this case any level of decoherence is possible, so it is not only a minor aberration, but a different type of decoherence process in terms of correlations with an environment.

VI. ENHANCEMENT OF TWO-QUBIT ENTANGLEMENT UNDER LOCAL DECOHERENCE

It has been recently shown [26] that lack of the quantum discord with respect to one of the subsystems in an initial state implies complete positivity of the reduced dynamics of this subsystem. Hence, in the case of evolution of the type discussed here, in the case of zero Q-E entanglement being generated, and thus zero discord with respect to the environment being generated, the evolution of the environment can be described by CP maps not only from the studied initial product state but also setting the initial time to any time t. As we have shown in the previous section this is not usually the case for the evolution of the qubit, which can certainly be described by CP maps from the initial product state, but nothing can be said for most evolved states. Note that the full understanding of conditions for initial system-environment correlations that make the subsequent evolution of the reduced state of the system completely positive is a subject of ongoing research [53–56].

The formalism presented in Ref. [24] in general cannot be used to study two-qubit states undergoing pure dephasing due to an interaction with an environment, but it turns out that it is viable if the initial qubit state is a superposition of only two states that are product states in the pointer states bases of the qubits (i.e., the bases singled out by the pure dephasing couplings to the respective environments). Such a state is "operationally" a two-level system, since during pure dephasing it will never leave the subspace of these two states, and the two-qubit entanglement is then simply proportional to the modulus of the single nonzero coherence present in the reduced density matrix of the qubits [57–60]. This allows for the study of the generation of entanglement between any initial two-qubit Bell state and its environment, while the state undergoes pure dephasing and its entanglement is diminished.

Furthermore, if the qubits are initialized in a pure state, each qubit interacts with a separate environment, the initial state of these environments is a product of density operators of the two environments, and there is no interaction between the qubits and between the environments, then the evolution of each qubit is local, so the qubits evolve under LOCC. In the case of a product initial state of the two-qubit state and the environments, it is known that such evolution cannot lead to enhancement of entanglement between the initial qubit state and any qubit state at time t; this llows from definition of entanglement as a quantity that cannot be increased by LOCC [6,7].

However, it has to be stressed that while the above-listed conditions for LOCC dynamics are too strong (they are definitely sufficient, but not all of them are necessary), breaking of any of them could make the evolution become nonlocal. Full understanding of the necessary conditions is hampered by the fact that LOCC transformations are notoriously difficult to characterize (see Refs. [6,61] and references therein). It is easier to consider a larger set (containing LOCC within it) of so-called separable maps that are CP and for which all the Kraus operators defining them can be written as products of operators acting on relevant local subsystems. This, however, comes at a price: there exist separable maps that can increase entanglement of states [62] that are neither "generically" separable nor pure [63], e.g., certain states that were obtained from pure entangled states by subjecting them to decoherence.

We have reminded the reader about those known results in order to stress the fact that the mathematical conditions for an evolution not to increase entanglement are highly nontrivial, and consequently simple intuitions about what kind of evolution is "local" (and thus cannot increase entanglement) often prove wrong. It is known that local operations can enhance entanglement with respect to an initial state having correlations between the entangled system and the environment (for an example see Ref. [64]), if these correlations make the subsequent dynamics not completely positive, and thus not belonging to LOCC, as LOCC is a subset of separable CP operations. As shown recently, nonzero discord between the qubits and their environment can be [26] (but *does not have to be*; see Ref. [55]) a correlation that leads to subsequent non-CP dynamics. Let us see then if the two-qubit generalization of the previously obtained results about system-environment discord generated during decoherence can shed some light on the behavior of two-qubit entanglement dynamics.

To this end let us study the initial two-qubit Bell state $|\psi\rangle = 1/\sqrt{2}(|00\rangle + |11\rangle)$. The choice of the Bell state is arbitrary. We assume (for simplicity) that only one of the qubits interacts with an environment and that this environment is a qubit itself in some initial state $R(0) = c_0|0\rangle\langle 0| + c_1|1\rangle\langle 1|$. The product initial state of the whole state system is therefore $\hat{\sigma}(0) = |\psi\rangle\langle\psi| \otimes R(0)$. The Hamiltonian of this system is

$$\hat{H} = \varepsilon_A|1\rangle_{AA}\langle 1| + \varepsilon_B|1\rangle_{BB}\langle 1| + |1\rangle_{AA}\langle 1| \otimes \hat{V}_A + \hat{H}_E, \tag{21}$$

where the indices A and B differentiate between the qubits, $\hat{V}_A$ is an operator acting in the subspace of the environment, and $\hat{H}_E$ is the free Hamiltonian of the environment. The interaction of qubit A with the environment has been asymmetrized for convenience, since the aim of this section is to show an exemplary evolution of a certain type and not to quantify all possible evolutions of this type.

Although this Hamiltonian is of larger dimensionally in terms of the qubits than the Hamiltonian of Eq. (7), the resulting evolution is equivalent to the evolution discussed in Sec. III, if the assumptions introduced in the previous paragraph are taken into account. Although the evolution operator is different and is equal to

$$\begin{aligned}\hat{U}(t) = &|00\rangle\langle 00| \otimes \mathbb{I} + |01\rangle\langle 01| \otimes \mathbb{I}\\ &+ |10\rangle\langle 10| \otimes \hat{w}(t) + |11\rangle\langle 11| \otimes \hat{w}(t),\end{aligned} \tag{22}$$

where $\hat{w}(t) = \exp[-i(\hat{H}_E + \hat{V})t]$, the density matrix of the whole system evolves according to

$$\hat{\sigma}(t) = \frac{1}{2}\begin{pmatrix} \hat{R}(0) & 0 & 0 & \hat{R}(0)\hat{w}(t) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hat{w}^\dagger(t)\hat{R}(0) & 0 & 0 & \hat{w}^\dagger(t)\hat{R}(0)\hat{w}(t) \end{pmatrix}, \tag{23}$$

which is the same as in case of a single qubit interacting with the type of environment under study.

Let us now additionally assume that the evolution is nonentangling (as always in this paper), so it is possible to write $\hat{R}(0)$ and $\hat{w}(t)$ in the common eigenbasis $\{|n'(t)\rangle\}$ at every time t. Hence, the state $\hat{\sigma}(t)$ can be written in the form

$$\hat{\sigma}(t) = \sum_{n=0}^{1} c'_n(t)|\psi_n(t)\rangle\langle\psi_n(t)| \otimes |n'(t)\rangle\langle n'(t)|, \tag{24}$$

where $|\psi_n(t)\rangle = 1/\sqrt{2}(|00\rangle + e^{i\phi_n(t)}|11\rangle)$ and the factors $e^{i\phi_n(t)}$ are eigenvalues of $\hat{w}(t)$ corresponding to the eigenstates $|n'(t)\rangle$ appropriately. It is straightforward to quantify interqubit

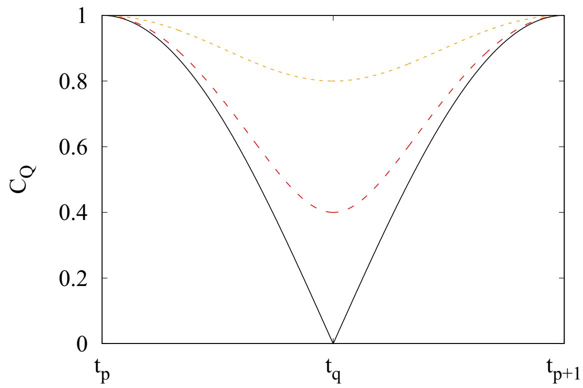

FIG. 1. Full cycle (evolution from t_p to t_{p+1}) of interqubit entanglement when only one of the qubits interacts with an $N=2$ environment for $c_0 = 0.5$ (black solid line) $c_0 = 0.7$ (red dashed line), and $c_0 = 0.9$ (orange dotted line).

entanglement in this state (after tracing out the degrees of freedom of the environment) and the entanglement measure concurrence [29] of such a state is equal to $C_Q = |c_0'(t)e^{i\phi_0(t)} + c_1'(t)e^{i\phi_1(t)}|$.

In the simplest case, the basis $\{|n'(t)\rangle\}$ is time independent and is the same as the eigenbasis of the initial state of the environment, $\{|0\rangle,|1\rangle\}$, so $c_0'(t) = c_0$ and $c_1'(t) = c_1$, while the time dependence of the phase factors reduces to $\phi_n(t) = \varphi_n t$. Here zero discord with respect to the qubit system is obtained only in two situations: first, at times t_p when $|\psi_0(t_p)\rangle = |\psi_1(t_p)\rangle$, so $e^{i\varphi_0 t_p} = e^{i\varphi_1 t_p}$ and the qubit is in a pure, maximally entangled state, and second, at times t_q when $\langle\psi_0(t_q)|\psi_1(t_q)\rangle = 0$, so $e^{i\varphi_0 t_q} = -e^{i\varphi_1 t_q}$ and the qubit decoherence is maximal while interqubit entanglement is minimal. Times t_p and t_q appear interchangeably, since $t_p = 2\pi p/|\varphi_1 - \varphi_0|$ and $t_q = 2\pi(q+1)/|\varphi_1 - \varphi_0|$, with $p,q = 0,1,2\ldots,$ and the evolution of interqubit entanglement is periodically repeated every $2\pi/|\varphi_1 - \varphi_0|$. The evolution of such entanglement from a certain time t_p to time t_{p+1} (capturing a full cycle of entanglement evolution) is shown in Fig. 1 for three different initial states of the environment (the time t_q which appears midcycle is marked on the figure).

If we now choose a certain time $\tau = t_q$ as a new initial time, the evolution of the new initial state $\hat{\sigma}(\tau)$, which has minimal entanglement (zero entanglement for $c_0 = c_1 = 1/2$) to any later state $t+\tau$ can be described using CP maps, since the state is zero-discordant with respect to the two qubits [26]. Note that all later states have greater or equal interqubit entanglement than the new initial state, so the evolution discussed in this section, which *could easily be mistakenly believed to be local, as it occurs due to interaction with an environment of only one of the qubits*, is also entangling, and thus it does not belong to LOCC. Apparently, the initial nonlocality (entanglement) of the qubits' state makes the total system state at time t_q incompatible with subsequently local dynamics, although, as we have shown, this incompatibility does not follow from nonzero discord. Finally, let us note that we have so far failed at finding a separable representation of the CP evolution of the qubits from time t_q onwards, but have not proven that such a representation is impossible. It remains then an open question if the example evolution described above is CP and separable, but not LOCC, or if it is simply CP but nonseparable. Other examples of bipartite system and environment evolutions which lead to similar effects of entanglement increase (but not during pure dephasing) can be found in Refs. [65,66].

It is interesting to further note that at all times the entanglement between one of the qubits and a subsystem containing the other qubit and the environment is maximal. This is easily checked by calculating the entanglement measure negativity [67]. This fact is in full agreement with a theorem proven in Ref. [65], which for the scenario studied here reduces to the following (all conditions for the applicability of the theorem are fulfilled): If there is no discord between the two-qubit subsystem and the environment with respect to the environment, then entanglement between a single qubit and the rest of the system cannot increase with time. Since the evolution of the whole system periodically returns to its initial state, entanglement of such partitions cannot change and remains constant.

VII. CONCLUSION

We have studied pure dephasing evolution of a qubit, intialized in a pure state, interacting with an environment and found that, if no Q-E entanglement is generated at a given time, then automatically no Q-E quantum discord with respect to the environment is generated. Hence, the set of separable states which can be obtained due to an evolution described by the class of Hamiltonians studied is zero-volume, and behaviors such as sudden death of Q-E entanglement are unlikely. Furthermore, the evolution of the environment alone between two arbitrary times may be described using completely positive maps, as follows from connection between zero discord with respect to one subsystem and complete positivity of subsequent evolution of the reduced state of this system [26].

We have also looked at the Q-E quantum discord with respect to the qubit and it turns out that the situation is very different here. For times at which the qubit and the environment are separable, this type of discord is usually still present in the system. It is only possible for such discord to be zero if the qubit is initially in an equal superposition state. Then the discord can vanish at certain times when the relative phases of the environmental states evolving due to the interaction with the qubit specifically align, something that can be expected to happen only for rather small environments. Furthermore, an evolution for which this type of quantum correlations would *never* appear is impossible if we ignore the trivial cases of evolutions not leading to any decoherence.

Last, we were able to compare an exemplary evolution of two-qubit entanglement under the influence of a *local* interaction with an environment, with Q-E discord generation. We have shown that such an evolution which displays zero-discordant points in time with respect to the two qubit subsystem is possible (but not common). This means that the evolution starting from one of the zero-discord times can be described using completely positive maps. Since at such points the qubits are either fully entangled (as they return at these times to their initial state) or have minimal entanglement possible, the evolution starting from times corresponding to the former case are trivial, but those corresponding to the latter situation lead to enhancement of interqubit entanglement due to a local interaction while the evolution can be described by completely positive maps. This interesting observation illustrates how hard it is to identify a LOCC evolution when one abandons the common assumption of complete lack of correlations between the initial states of the entangled system and the environment.

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