Quantum state concentration and classification of multipartite entanglement

S. M. Zangi,[1] Jun-Li Li,[1,2] and Cong-Feng Qiao[1,2,3,*]

[1]*Department of Physics, University of Chinese Academy of Sciences, YuQuan Road 19A, Beijing 100049, China*
[2]*Key Laboratory of Vacuum Physics, University of Chinese Academy of Sciences, YuQuan Road 19A, Beijing 100049, China*
[3]*Department of Physics & Astronomy, York University, Toronto, Ontario, Canada M3J 1P3*

(Received 22 July 2017; published 3 January 2018)

Entanglement is a unique feature of quantum theory and has tremendous potential for application. Nevertheless, the complexity of quantum entanglement grows exponentially with an increase in the number of entangled particles. Here we introduce a quantum state concentration scheme which decomposes the multipartite entangled state into a set of bipartite and tripartite entangled states. It is shown that the complexity of the entanglement induced by the large number of particles is transformed into the high dimensions of bipartite and tripartite entangled states for pure quantum systems. The results not only simplify the tedious work of verifying the (in)equivalence of multipartite entangled states, but also are instructive in the quantum many-body problem involving multipartite entanglement.

DOI: 10.1103/PhysRevA.97.012301

I. INTRODUCTION

Entanglement is regarded as an essential physical resource of quantum information sciences, which are responsible for the so-called second quantum revolution [1]. Besides the development of quantum algorithms [2,3] and quantum computation [4], every study related to many-body quantum system [5] would benefit from a deeper understanding of multipartite entanglement. Entanglement may be classified based on the different tasks it performs in quantum information processing, which forms the basis of the qualitative and quantitative characterizations of multipartite entanglement [6]. Though an enormous amount of work in the literature has been dedicated to this subject [7,8], a very limited amount of information about multipartite entanglement has been obtained. This is because the complexity of characterizing entanglement using classical parameters, i.e., the coefficients of the quantum state in decomposition bases, increases dramatically with the number of particles and dimensions.

Two superficially different entangled states may be used to implement the same quantum information task identically if they are equivalent under local unitary (LU) operations and different performances if they are equivalent under invertible local operations [stochastic local operations and classical communication (SLOCC)]. The LU equivalence of arbitrary multipartite entangled states could be understood via the high-order singular-value decomposition (HOSVD) [9,10] and an alternative method also exists for multiqubit states [11,12]. However, only the states with specific symmetries were explored by effective methods under SLOCC [13,14]. While the coefficient matrix method is a practical but rather coarse-grained classification method for multipartite entanglement [15], invariant polynomials encountered in distinguishing the inequivalent classes under SLOCC usually involve cumbersome rational expressions [16]. A recent study shows that four-partite entanglement may be well understood through its subsystem's entanglement [17]. Then one may naturally ask whether general multipartite entanglement could also be understood by the entangled subsystems, rather than by the classical parameters (coefficients of the quantum state) alone.

In this paper we suggest a splitting scheme for the study of multipartite entanglement, which nontrivially generalizes the method of [17] to arbitrary multipartite states. By introducing virtual particles and performing a sequence of high-order singular-value decompositions, a multipartite entangled pure state is transformed into a set of states with only bipartite and tripartite entangled states. This set of states, which we call core entangled states, forms a hierarchical structure. The concentration of multipartite entanglement to the core entangled states exhibits a structure similar to that of the tree tensor network state [18]. By applying entanglement classification we find that two multipartite states are equivalent under LU operation or SLOCC if and only if their core entangled states in each hierarchy are equivalent under LU operation and SLOCC, respectively.

II. QUANTUM STATE CONCENTRATION

An arbitrary $(I_1 \times I_2 \times \cdots \times I_N)$-dimensional multipartite quantum state has the form

$$|\Psi\rangle = \sum_{i_1,i_2,\ldots,i_N=1}^{I_1,I_2,\ldots,I_N} \psi_{i_1 i_2 \cdots i_N} |i_1\rangle |i_2\rangle \cdots |i_N\rangle, \tag{1}$$

where the complex numbers $\psi_{i_1 i_2 \cdots i_N} \in \mathbb{C}$ are coefficients of the state in the orthonormal basis $\{|i_1\rangle, |i_2\rangle, \ldots, |i_N\rangle\}$. In this form, the quantum state may be regarded as a high-order tensor Ψ whose tensor elements are $\psi_{i_1 i_2 \cdots i_N}$ and the inner

*Author to whom correspondence should be addressed: qiaocf@ucas.ac.cn

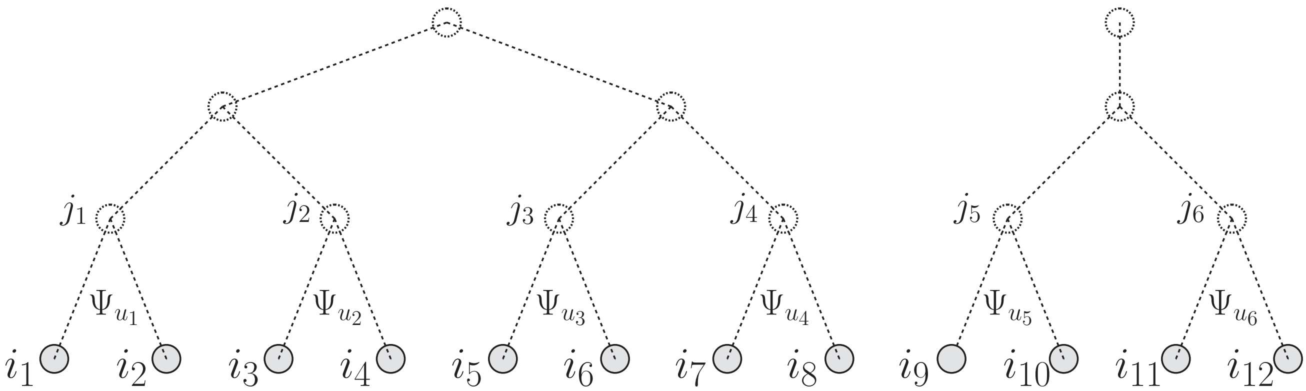

FIG. 1. A 12-partite entangled state is first transformed into six tripartite states and one six-partite state: $(\Psi_{u_1},\Psi_{u_2},\ldots,\Psi_{u_6},\Omega_{j_1j_2\cdots j_6})$. Further rescaling may turn the 12-partite state into ten tripartite states and one bipartite state.

product of two states of the same quantum system is defined as $\langle\Psi'|\Psi\rangle = \langle\psi'_{i_1\cdots i_N}|\psi_{i_1\cdots i_N}\rangle \equiv \sum_{i_1,i_2,\ldots,i_N=1}^{I_1,I_2,\ldots,I_N}\psi'^{*}_{i_1i_2\cdots i_N}\psi_{i_1i_2\cdots i_N}$. We group every two particles into a composite one, i.e., $(i_1i_2)(i_3i_4)\cdots(i_{N-1}i_N)$, and make the map $(i_{2k-1}i_{2k})\mapsto j_k$ such that $j_k=(i_{2k-1}-1)I_{2k}+i_{2k}$ (we may set $j_{(N+1)/2}=i_N$ for N odd). This rescaling of the quantum state can be expressed as

$$
\begin{aligned}
|\Psi\rangle &= \sum_{i_1,i_2,\ldots,i_N=1}^{I_1,I_2,\ldots,I_N}\psi_{(i_1i_2)(i_3i_4)\cdots(i_{N-1}i_N)}|i_1i_2\rangle|i_3i_4\rangle\cdots|i_{N-1}i_N\rangle \\
&= \sum_{j_1,j_2,\ldots,j_M=1}^{J_1,J_2,\ldots,J_M}\psi_{j_1j_2\cdots j_M}|j_1\rangle|j_2\rangle\cdots|j_M\rangle. \qquad (2)
\end{aligned}
$$

Now Ψ may be regarded as an M-partite quantum state rescaled from the N-partite state.

For an M-order tensor Ψ with dimensions of $J_1\times J_2\times\cdots\times J_M$, its kth mode matrix unfolding is represented by $\Psi_{(k)}$, which is a $[J_k\times(J_{k+1}\cdots J_MJ_1J_2\cdots J_{k-1})]$-dimensional matrix with matrix elements $\psi_{j_k(j_{k+1}\cdots j_Mj_1j_2\cdots j_{k-1})}$ [19]. The HOSVD of the M-partite state Ψ is

$$
\Psi = U^{(1)}\otimes U^{(2)}\otimes\cdots\otimes U^{(M)}\Omega, \qquad (3)
$$

where unitary matrices $U^{(k)}=(\vec{u}_1^{\,(k)},\ldots,\vec{u}_{J_k}^{\,(k)})$ are composed of the left singular vectors of $\Psi_{(k)}$ and Ω is called the core tensor of Ψ [9,19]. The core tensor Ω has the tensor elements $\omega_{j_1j_2\cdots j_M}$ and is all orthogonal, i.e., $\langle\omega_{j_1\cdots j_k=\alpha\cdots j_M}|\omega_{j_1\cdots j_k=\beta\cdots j_M}\rangle=\delta_{\alpha\beta}$, $k\in\{1,\ldots,M\}$. Equation (3) can also be written in the form of tensor elements

$$
\Psi = \sum_{j_1,j_2,\ldots,j_M=1}^{r_1,r_2,\ldots,r_M}\omega_{j_1j_2\cdots j_M}\,\vec{u}_{j_1}^{\,(1)}\circ\vec{u}_{j_2}^{\,(2)}\circ\cdots\circ\vec{u}_{j_M}^{\,(M)}. \qquad (4)
$$

Here r_k is the local rank of the kth mode matrix unfolding of Ω and $\vec{u}_{j_k}^{\,(k)}$ are $(I_{2k-1}\times I_{2k})$-dimensional orthonormal vectors for $j_k\in\{1,\ldots,r_k\}$, with $\circ$ being the direct product. The singular vectors in the unitary matrix $U^{(k)}$ can be grouped into two parts according to the rank r_k,

$$
\begin{aligned}
&U^{(k)}=\left(U_1^{(k)},U_0^{(k)}\right), \quad \text{where} \quad U_1^{(k)}\equiv\left(\vec{u}_1^{(k)},\ldots,\vec{u}_{r_k}^{(k)}\right),\\
&U_0^{(k)}\equiv\left(\vec{u}_{r_k+1}^{(k)},\ldots,\vec{u}_{J_k}^{(k)}\right). \qquad (5)
\end{aligned}
$$

We define the wrapping of an $(I_1\times I_2)$-dimensional vector $\vec{u}$ into an $I_1\times I_2$ matrix as [17]

$$
\mathcal{W}(\vec{u})\equiv\begin{pmatrix} u_1 & u_{I_1+1} & \cdots & u_{(I_2-1)I_1+1}\\ u_2 & u_{I_1+2} & \cdots & u_{(I_2-1)I_1+2}\\ \vdots & \vdots & \ddots & \vdots\\ u_{I_1} & u_{2I_1} & \cdots & u_{I_2I_1}\end{pmatrix} \qquad (6)
$$

and the vectorization of a matrix is defined as $\mathcal{V}[\mathcal{W}(\vec{u})]\equiv\vec{u}$. An $(r\times I_1\times I_2)$-dimensional tripartite pure state can be expressed in tuples of matrices, that is, $\{\Gamma_1,\ldots,\Gamma_r\}$, where $\Gamma_i\in\mathbb{C}^{I_1\times I_2}$ [17,20]. Hence, by wrapping the $(I_{2k-1}\times I_{2k})$-dimensional vector $\vec{u}_{j_k}^{(k)}$ into an $I_{2k-1}\times I_{2k}$ matrix, we get an $r_k\times I_{2k-1}\times I_{2k}$ tripartite state Ψ_{u_k} and its complementary state $\overline{\Psi}_{u_k}$ from the unitary matrix $U^{(k)}=(\vec{u}_1^{(k)},\ldots,\vec{u}_{r_k}^{(k)},\vec{u}_{r_k+1}^{(k)},\ldots,\vec{u}_{J_k}^{(k)})$, i.e.,

$$
\begin{aligned}
\Psi_{u_k}&\equiv\left(\mathcal{W}(\vec{u}_1^{(k)}),\ldots,\mathcal{W}(\vec{u}_{r_k}^{(k)})\right),\\
\overline{\Psi}_{u_k}&\equiv\left(\mathcal{W}(\vec{u}_{r_k+1}^{(k)}),\ldots,\mathcal{W}(\vec{u}_{J_k}^{(k)})\right), \qquad (7)
\end{aligned}
$$

where $\mathcal{W}(\vec{u}_i^{(k)})\in\mathbb{C}^{I_{2k-1}\times I_{2k}}$ are $(I_{2k-1}\times I_{2k})$-dimensional complex matrices [17].

The N-partite state Ψ now is rescaled and decomposed into M tripartite states and one M-partite state

$$
\Psi = \left(\Psi_{u_1},\Psi_{u_2},\ldots,\Psi_{u_M},\Omega_r\right). \qquad (8)
$$

Here Ψ_{u_k} are $(r_k\times I_{2k-1}\times I_{2k})$-dimensional tripartite states as defined in Eq. (7) and Ω_r is an $(r_1\times r_2\times\cdots\times r_M)$-dimensional M-partite state whose coefficients are $\omega_{j_1j_2\cdots j_M}$. Further rescaling of Ω_r as in Eq. (2) and then wrapping the singular vectors would lead to another set of tripartite states where one may get a hierarchical structure of tripartite states for the multipartite entangled state in the end (see Fig. 1). We call this decomposition of a multipartite entangled state into bipartite and tripartite entangled states the quantum state concentration, which exhibits a form quite similar to the tree tensor network state [18]. To exemplify the application of the scheme in the quantum many-body problem, we apply the quantum state concentration technique to the multipartite entanglement classification.

For the equivalence (under LU operation or SLOCC) of two N-partite states Ψ' and Ψ, we have the following theorem, which is a multipartite generalization of Ref. [17].

Theorem 1. Two N-partite entangled states Ψ' and Ψ are equivalent if and only if the quantum states in the decompositions $\Psi' = (\Psi_{u'_1},\ldots,\Psi_{u'_M},\Omega'_r)$ and $\Psi = (\Psi_{u_1},\ldots,\Psi_{u_M},\Omega_r)$ are equivalent in the following way:

$$\Psi_{u'_k} = P^{(k)} \otimes A_{2k-1} \otimes A_{2k} \Psi_{u_k} \ \forall\, k \in \{1,\ldots,M\}, \tag{9}$$

$$\Omega_r = P^{(1)} \otimes \cdots \otimes P^{(M)} \Omega'_r, \tag{10}$$

where A_i and $P^{(k)}$ are all invertible (or unitary) matrices for SLOCC (or LU) equivalence.

Proof. First, if $\Psi' = A_1 \otimes \cdots \otimes A_N\, \Psi$, then the HOSVD of the rescaled M-partite Ψ' and Ψ is equivalent to

$$U'^{(1)} \otimes \cdots \otimes U'^{(M)}\Omega' = (A_1 \otimes A_2)U^{(1)} \otimes \cdots \otimes (A_{2M-1} \otimes A_{2M})U^{(M)}\Omega. \tag{11}$$

Here Ω' and Ω have the nonzero parts of Ω'_r and Ω_r, respectively. Substituting the QR factorization $(A_{2k-1} \otimes A_{2k})U^{(k)} = Q_{u_k}R_{u_k}$ into Eq. (11) and applying HOSVD to $R_{u_1} \otimes \cdots \otimes R_{u_M}\Omega$, we get

$$R_{u_1} \otimes \cdots \otimes R_{u_M}\Omega = X_{u_1} \otimes \cdots \otimes X_{u_M}\Omega', \tag{12}$$

$$U'^{(k)} = Q_{u_k}X_{u_k} \ \forall\, k \in \{1,\ldots,M\}, \tag{13}$$

where X_{u_k} are unitary matrices. Equations (12) and (13) also give

$$\Omega = \left(R_{u_1}^{-1}X_{u_1}\right) \otimes \cdots \otimes \left(R_{u_M}^{-1}X_{u_M}\right)\Omega', \tag{14}$$

$$U'^{(k)} = (A_{2k-1} \otimes A_{2k})U^{(k)}\left(R_{u_k}^{-1}X_{u_k}\right) \forall\, k \in \{1,\cdots,M\}. \tag{15}$$

If we let $\widetilde{P}^{(k)} \equiv R_{u_k}^{-1}X_{u_k}$, because Ω' and Ω have the same local ranks of r_k, Eq. (14) leads to

$$\widetilde{P}^{(k)} = \begin{pmatrix} P^{(k)} & Y^{(k)} \\ 0 & \overline{P}^{(k)} \end{pmatrix}. \tag{16}$$

Here $P^{(k)} \in \mathbb{C}^{r_k \times r_k}$ and $\overline{P}^{(k)} \in \mathbb{C}^{(I_{2k-1}\times I_{2k}-r_k)\times(I_{2k-1}\times I_{2k}-r_k)}$ are invertible matrices and $\widetilde{P}^{(k)}$ are unitary if all matrices A_j are unitary, which is easy to see from Eq. (15). As the tensor elements $\omega'_{j_1 j_2 \cdots j_M}$ and $\omega_{j_1 j_2 \cdots j_M}$ of the core tensors Ω and Ω' are nonzero only for $1 \leqslant j_k \leqslant r_k \ \forall\, k \in \{1,\ldots,M\}$, Eqs. (14) and (16) lead to Eq. (10). In addition, substituting Eq. (16) in Eq. (15), we have

$$\left(\vec{u}_1^{\,\prime(k)}, \vec{u}_2^{\,\prime(k)}, \ldots, \vec{u}_{r_k}^{\,\prime(k)}\right) = A_{2k-1} \otimes A_k\left(\vec{u}_1^{(k)}, \vec{u}_2^{(k)}, \ldots, \vec{u}_{r_k}^{(k)}\right)P^{(k)}, \tag{17}$$

where $\vec{u}_i^{\,\prime(k)}$ and $\vec{u}_i^{(k)}$ are from $U'^{(k)} = (U_1'^{(k)}, U_0'^{(k)})$ and $U^{(k)} = (U_1^{(k)}, U_0^{(k)})$ based on the definition in Eq. (5). The wrapping operations make $(\mathcal{W}(\vec{u}_1^{\,\prime(k)}), \ldots, \mathcal{W}(\vec{u}_{r_k}^{\,\prime(k)})) = \Psi_{u'_k}$ and $(\mathcal{W}(\vec{u}_1^{(k)}), \ldots, \mathcal{W}(\vec{u}_{r_k}^{(k)})) = \Psi_{u_k}$, therefore Eq. (17) is equivalent to Eq. (9).

Second, Eq. (9) may be expressed in the form

$$U_1'^{(k)} = (A_{2k-1} \otimes A_{2k})U_1^{(k)}P^{(k)} \ \forall\, k \in \{1,\ldots,M\}, \tag{18}$$

where $U_1'^{(k)} = (\vec{u}_1^{\,\prime(k)}, \ldots, \vec{u}_{r_k}^{\,\prime(k)})$ and $U_1^{(k)} = (\vec{u}_1^{(k)}, \ldots, \vec{u}_{r_k}^{(k)})$ are from $U'^{(k)} = (U_1'^{(k)}, U_0'^{(k)})$ and $U^{(k)} = (U_1^{(k)}, U_0^{(k)})$. It is legitimate to construct the matrix $\widetilde{P}^{(k)} = \left(\begin{smallmatrix} P^{(k)} & Y^{(k)} \\ 0 & \overline{P}^{(k)} \end{smallmatrix}\right)$ such that

$$U'^{(k)} = (A_{2k-1} \otimes A_{2k})U^{(k)}\begin{pmatrix} P^{(k)} & Y^{(k)} \\ 0 & \overline{P}^{(k)} \end{pmatrix}, \tag{19}$$

where $\overline{P}^{(k)}$ is invertible (unitary when A_j are unitary). The decomposition $\Psi' = (\Psi_{u'_1},\ldots,\Psi_{u'_M},\Omega'_r)$ can be expressed as

$$\begin{aligned} \Psi' &= U'^{(1)} \otimes \cdots \otimes U'^{(M)}\Omega' \\ &= (A_1 \otimes A_2)U^{(1)}\widetilde{P}^{(1)} \otimes \cdots \otimes (A_{2M-1} \otimes A_{2M})U^{(M)}\widetilde{P}^{(M)}\Omega' \\ &= (A_1 \otimes A_2)U^{(1)} \otimes \cdots \otimes (A_{2M-1} \otimes A_{2M})U^{(M)}\Omega \\ &= A_1 \otimes A_2 \otimes \cdots \otimes A_{2M}\Psi. \end{aligned} \tag{20}$$

Here Eq. (19) is used in the second equality and Eq. (10) is used in the third equality. Therefore, Ψ' and Ψ are equivalent under SLOCC or LU operation when A_i are invertible or unitary. Q.E.D.

Theorem 1 decomposes the N-partite entangled state into M tripartite states and one M-partite state, where $M = \lceil \frac{N}{2} \rceil$ is the smallest integer greater than or equal to $N/2$. The M-partite state could be further rescaled and turn into another set of $\lceil \frac{M}{2} \rceil$ tripartite states and one $\lceil \frac{M}{2} \rceil$-partite entangled state. Along this line, one may finally get a hierarchy of tripartite entangled states and one bipartite entangled state (see Fig. 1). This scheme therefore reduces the entanglement classifications of a multipartite state to that of only bipartite and tripartite states and makes the tripartite entanglement a key ingredient of quantum entanglement.

The fact that the set of tripartite and bipartite entangled states represents faithfully multipartite entanglement can be understood as follows. The number of parameters needed to characterize the entanglement classes under SLOCC for an $I_1 \times I_2 \times \cdots \times I_N$ quantum state is [6]

$$\mathcal{N}_{I_1\times\cdots\times I_N} = 2(I_1 I_2 \cdots I_N - 1) - 2\sum_{i=1}^{N}\left(I_i^2 - 1\right). \tag{21}$$

In the decomposition $\Psi = (\Psi_{u_1},\ldots,\Psi_{u_M},\Omega_r)$ of Theorem 1, the number of parameters becomes $\mathcal{N}_3 + \mathcal{N}_M$, where

$$\mathcal{N}_3 = \sum_{k=1}^{M}\left[2(r_k I_{2k-1} I_{2k} - 1) - 2\left(I_{2k-1}^2 + I_{2k}^2 - 2\right)\right], \tag{22}$$

$$\mathcal{N}_M = 2(r_1 r_2 \cdots r_M - 1) - 2\sum_{k=1}^{M}\left(r_k^2 - 1\right). \tag{23}$$

Here $2(I_{2k-1}^2 + I_{2k}^2 - 2)$ are induced by A_{2k-1} and A_{2k} in the M tripartite entangled states and $2\sum_{k=1}^{M}(r_k^2 - 1)$ are induced by $P^{(k)}$ in the M-partite entangled states, according to Eqs. (9) and (10) in Theorem 1. The number $\mathcal{N}_3 + \mathcal{N}_M$ equals $\mathcal{N}_{I_1\times\cdots\times I_N}$ in the worst case of $r_k = I_{2k-1}I_{2k}$ in the rescaling process. Along this line, we will finally get a set of states with bipartite and tripartite entangled states only and the complexity of characterizing the entanglement of multipartite states is transformed into the large numbers and high dimensions of the tripartite and bipartite entangled states in the set.

To illustrate how the parameters in the multipartite state transform under the decomposition of Theorem 1, we present explicit examples of a four-qubit and a six-qubit state. Considering the four-qubit state $|\Psi\rangle = a_1|0001\rangle + a_2|0010\rangle + a_3|0100\rangle + a_4|1000\rangle$, where we assume $a_i \in \mathbb{R}$ for the sake of illustration, the state contains three independent real parameters (four parameters with one normalization constraint). The four particles may be grouped as

$$|\Psi\rangle = a_1|(00)(01)\rangle + a_2|(00)(10)\rangle + a_3|(01)(00)\rangle + a_4|(10)(00)\rangle = a_1|01\rangle + a_2|02\rangle + a_3|10\rangle + a_4|20\rangle. \tag{24}$$

The last line in Eq. (24) is a bipartite state of 4×4 and can be represented by a matrix whose singular-value decomposition is

$$\Psi = \begin{pmatrix} 0 & a_1 & a_2 & 0 \\ a_3 & 0 & 0 & 0 \\ a_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= U\Lambda V^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{a_3}{\sqrt{a_3^2+a_4^2}} & 0 & \frac{-a_4}{\sqrt{a_3^2+a_4^2}} \\ 0 & \frac{a_4}{\sqrt{a_3^2+a_4^2}} & 0 & \frac{a_3}{\sqrt{a_3^2+a_4^2}} \\ 0 & 0 & 1 & 0 \end{pmatrix} \times \begin{pmatrix} \sqrt{a_1^2+a_2^2} & 0 & 0 & 0 \\ 0 & \sqrt{a_3^2+a_4^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{a_1}{\sqrt{a_1^2+a_2^2}} & 0 & 0 & \frac{-a_2}{\sqrt{a_1^2+a_2^2}} \\ \frac{a_2}{\sqrt{a_1^2+a_2^2}} & 0 & 0 & \frac{a_1}{\sqrt{a_1^2+a_2^2}} \\ 0 & 0 & 1 & 0 \end{pmatrix}^\dagger. \tag{25}$$

Based on Eq. (7), we obtained one bipartite state $\psi_\Lambda = \mathrm{diag}\{\sqrt{a_1^2+a_2^2}, \sqrt{a_3^2+a_4^2}\}$ and two tripartite states

$$\psi_u = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \frac{1}{\sqrt{a_3^2+a_4^2}} \begin{pmatrix} 0 & a_4 \\ a_3 & 0 \end{pmatrix} \right\}, \qquad \psi_v = \left\{ \frac{1}{\sqrt{a_1^2+a_2^2}} \begin{pmatrix} 0 & a_2 \\ a_1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}, \tag{26}$$

where there is one free parameter in each of them (note that $\frac{a_3^2}{a_3^2+a_4^2} + \frac{a_4^2}{a_3^2+a_4^2} = 1$). In this example, the parameters in the multipartite entangled state $|\psi\rangle$ are evenly distributed among the decomposed tripartite and bipartite entangled states. As the number of core entangled states grows, there will be fewer parameters in each individual decomposed state, which results in a simplification to the practical entanglement classification.

Considering the six-qubit quantum state $|\Phi\rangle = b_1|000000\rangle + b_2|010101\rangle + b_3|101010\rangle + b_4|111111\rangle$ with $b_i \in \mathbb{R}$, we may group the six particles as follows:

$$\begin{aligned} |\Phi\rangle &= b_1|(00)(00)(00)\rangle + b_2|(01)(01)(01)\rangle \\ &\quad + b_3|(10)(10)(10)\rangle + b_4|(11)(11)(11)\rangle \\ &= b_1|000\rangle + b_2|111\rangle + b_3|222\rangle + b_4|333\rangle, \end{aligned} \tag{27}$$

where the last line represents a tripartite state of 4×4×4. An HOSVD to this tripartite state leads to

$$\phi_{u_k} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \quad k \in \{1,2,3\} \tag{28}$$

$$|\phi_\Omega\rangle = b_1|000\rangle + b_2|111\rangle + b_3|222\rangle + b_4|333\rangle. \tag{29}$$

That is, we get three 4×2×2 entangled states ϕ_{u_1}, ϕ_{u_2}, and ϕ_{u_3} and one 4×4×4 state $|\phi_\Omega\rangle$. Further decomposition of Eq. (29) may be performed according to the grouping of $|\phi_\Omega\rangle = b_1|(00)0\rangle + b_2|(11)1\rangle + b_3|(22)2\rangle + b_4|(33)3\rangle$. However, we may stop at Eq. (29), as we have already decomposed the multipartite state into only tripartite states. In this example, all the parameters in $|\Phi\rangle$ are transformed and concentrated into the high-dimensional 4×4×4 tripartite state and there is no parameter in the other three tripartite entangled states ϕ_{u_k}, $k \in \{1,2,3\}$.

These two explicit examples provide an understanding of how our method works. The parameters of the multipartite entangled state are redistributed and/or concentrated into the core entangled states, which are at most tripartite entangled. In the following we present two practical corollaries for verifying the equivalence of tripartite entanglement under SLOCC and LU operation. The realignment of a matrix $A \in \mathbb{C}^{(I_1\times I_2)\times(I_1\times I_2)}$ according to the factorization of $I_1\times I_2$ is defined as [21]

$$\begin{aligned} \mathcal{R}(A) \equiv \big(&\mathcal{V}(A_{11}),\ldots,\mathcal{V}(A_{I_1 1}),\mathcal{V}(A_{12}),\ldots, \\ &\mathcal{V}(A_{I_1 2}),\ldots,\mathcal{V}(A_{I_1 I_1})\big)^{\mathrm{T}}, \end{aligned}$$

where $\mathcal{R}(A) \in \mathbb{C}^{(I_1\times I_1)\times(I_2\times I_2)}$ and $A_{ij} \in \mathbb{C}^{I_2\times I_2}$ are the submatrices of A,

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1I_1} \\ A_{21} & A_{22} & \cdots & A_{2I_1} \\ \vdots & \vdots & \ddots & \vdots \\ A_{I_1 1} & A_{I_1 2} & \cdots & A_{I_1 I_1} \end{pmatrix}. \tag{30}$$

For two $r\times I_1\times I_2$ genuine tripartite entangled states $\Psi_{u'} = (\mathcal{W}(\vec{u}'_1),\ldots,\mathcal{W}(\vec{u}'_r))$ and $\Psi_u = (\mathcal{W}(\vec{u}_1),\ldots,\mathcal{W}(\vec{u}_r))$, we may construct their complementary states, i.e., $\overline{\Psi}_{u'} = (\mathcal{W}(\vec{u}'_{r+1}),\ldots,\mathcal{W}(\vec{u}'_{I_1\times I_2}))$ and $\overline{\Psi}_u = (\mathcal{W}(\vec{u}_{r+1}),\ldots,\mathcal{W}(\vec{u}_{I_1\times I_2}))$, where $\vec{u}'_i$ and $\vec{u}_i$ are $(I_1\times I_2)$-dimensional vectors and $U' = (\vec{u}'_1,\ldots,\vec{u}'_{I_1\times I_2})$ and $U = (\vec{u}_1,\ldots,\vec{u}_{I_1\cdot I_2})$ are invertible matrices [17]. We have the following corollaries.

Corollary 1. Two $(r\times I_1\times I_2)$-dimensional entangled quantum states $\Psi_{u'}$ and Ψ_u are equivalent under local operators, i.e., $|\Psi_{u'}\rangle = P\otimes A_1\otimes A_2\,|\Psi_u\rangle$, if and only if there exist $\widetilde{P} = \begin{pmatrix} P & Y \\ 0 & \overline{P} \end{pmatrix} \in \mathbb{C}^{(I_1\times I_2)\times(I_1\times I_2)}$ such that

$$\mathrm{rank}[\mathcal{R}(U\widetilde{P}U'^{-1})] = 1. \tag{31}$$

Here $\mathcal{R}$ is the matrix realignment according to the factorization of $I_1 \times I_2$; $\widetilde{P}$ and $U\widetilde{P}U'^{-1}$ are invertible (unitary) for SLOCC (LU) equivalences.

Proof. It has been shown that $\Psi_{u'}$ and Ψ_u are equivalent under P, A_1, and A_2 if and only if [17]

$$(U_1', U_0') = (A_1 \otimes A_2)(U_1, U_0)\begin{pmatrix} P & Y \\ 0 & P \end{pmatrix}. \tag{32}$$

Therefore, $(A_1^{-1} \otimes A_2^{-1}) = U\widetilde{P}U'^{-1}$. According to Lemma 3 of Ref. [22], $U\widetilde{P}U'^{-1}$ is a direct product of two unitary or invertible matrices if and only if $U\widetilde{P}U'^{-1}$ is unitary or invertible and $\mathcal{R}(U\widetilde{P}U'^{-1})$ has rank 1. ■

Corollary 2. Two $(r \times I_1 \times I_2)$-dimensional entangled quantum states $\Psi_{u'}$ and Ψ_u are equivalent under local operators, i.e., $|\Psi_{u'}\rangle = P \otimes A_1 \otimes A_2 |\Psi_u\rangle$, if and only if there exist $\widetilde{P} = \begin{pmatrix} P & Y \\ 0 & P \end{pmatrix} \in \mathbb{C}^{(I_1 \times I_2) \times (I_1 \times I_2)}$ such that

$$\mathcal{F}[\mathcal{W}(U\widetilde{P}U'^{-1}\vec{a})] = \mathcal{F}[\mathcal{W}(\vec{a})] \ \forall \vec{a}. \tag{33}$$

Here $\vec{a}$ is an arbitrary $(I_1 \times I_2)$-dimensional vector; for SLOCC equivalence, $\mathcal{F}$ denotes the rank; for LU equivalence, $\widetilde{P}$ should be unitary and $\mathcal{F}$ denotes a concave, symmetric, and strictly increasing function on singular values of matrices with $\mathcal{F}(0) = 0$.

Proof. The operator $\Phi = U\widetilde{P}U'^{-1}$ induces a linear map $\varphi : \mathbb{C}^{I_1 \times I_2} \mapsto \mathbb{C}^{I_1 \times I_2}$ for the wrapping $\mathcal{W} : \mathcal{W}(\Phi\vec{a}) = \varphi[\mathcal{W}(\vec{a})]$ [17]. The proof the Corollary 2 can be carried out straightforwardly by the application of linear preserver problem with local ranks [23] and matrix norms [24]. ■

With the state concentration technique, the verification of SLOCC and LU equivalences of multipartite entanglement turns to the bipartite and tripartite entanglement classifications. Corollaries 1 and 2 further simplify the verification of equivalent relations for tripartite entanglement. Note that the proposed method employs only linear equations in the verification procedure [see Eq. (31)] and detailed information of the connecting matrices, i.e., $A_1, \ldots, A_N$, is not a prerequisite for both SLOCC and LU equivalences of two tripartite entangled states [17].

III. CONCLUSION

The characterization of multipartite entanglement is a longstanding tough issue in quantum information, due to the dramatic increase in the number of parameters characterizing it. In this work a quantum state concentration technique is introduced, which turns the multipartite entangled state into a set of bipartite and tripartite entangled states, and the complexity of the entanglement characterization for multiple particles is transformed into that of large numbers and high dimensions of tripartite and bipartite entangled states in the set. By exploring the method, the classification of multipartite entanglement under SLOCC or LU operations is accomplished by classifying only the core entangled states, i.e., tripartite and bipartite entangled states. The results indicate that multipartite entanglement is no more complex than the tripartite entangled states of high enough dimensions. Considering the implicit relation to the tree tensor network state, the scheme presented here may also be instructive in other studies concerning quantum multipartite states, e.g., condensed matter physics [5] and quantum chemistry [25].

ACKNOWLEDGMENTS

This work was supported in part by the Ministry of Science and Technology of the People's Republic of China (Grant No. 2015CB856703), the Strategic Priority Research Program of the Chinese Academy of Sciences (Grant No. XDB23030100), and the National Natural Science Foundation of China (Grants No. 11375200 and No. 11635009). S.M.Z. was also supported in part by a CAS-TWAS Fellowship.

[1] I. Georgescu, Foundations of quantum mechanics, Nat. Phys. **10**, 253 (2014).

[2] R. Jozsa and N. Linden, On the role of entanglement in quantum-computational speed-up, Proc. R. Soc. A **459**, 2011 (2003).

[3] D. Bruß and C. Macchiavello, Multipartite entanglement in quantum algorithms, Phys. Rev. A **83**, 052313 (2011).

[4] H. J. Briegel, D. E. Browne, W. Dür, R. Raussendorf, and M. Van den Nest, Measurement-based quantum computation, Nat. Phys. **5**, 19 (2009).

[5] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Entanglement in many-body systems, Rev. Mod. Phys. **80**, 517 (2008).

[6] W. Dür, G. Vidal, and J. I. Cirac, Three qubits can be entangled in two inequivalent ways, Phys. Rev. A **62**, 062314 (2000).

[7] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, Rev. Mod. Phys. **81**, 865 (2009).

[8] O. Gühne and G. Tóth, Entanglement detection, Phys. Rep. **474**, 1 (2009).

[9] B. Liu, J.-L. Li, X. Li, and C.-F. Qiao, Local Unitary Classification of Arbitrary Dimensional Multipartite Pure States, Phys. Rev. Lett. **108**, 050501 (2012).

[10] J.-L. Li and C.-F. Qiao, Classification of arbitrary multipartite entangled states under local unitary equivalence, J. Phys. A: Math. Theor. **46**, 075301 (2013).

[11] B. Kraus, Local Unitary Equivalence of Multipartite Pure States, Phys. Rev. Lett. **104**, 020504 (2010).

[12] B. Kraus, Local unitary equivalence and entanglement of multipartite pure states, Phys. Rev. A **82**, 032121 (2010).

[13] F. Verstraete, J. I. Cirac, J. I. Latorre, E. Rico, and M. M. Wolf, Renormalization-Group Transformations on Quantum States, Phys. Rev. Lett. **94**, 140601 (2005).

[14] T. Bastin, S. Krins, P. Mathonet, M. Godefroid, L. Lamata, and E. Solano, Operational Families of Entanglement Classes for Symmetric N-Qubit States, Phys. Rev. Lett. **103**, 070503 (2009).

[15] X. Li and D. Li, Classification of General n-Qubit States under Stochastic Local Operations and Classical Communication in

Terms of the Rank of Coefficient Matrix, Phys. Rev. Lett. **108**, 180502 (2012).

[16] G. Gour and N. R. Wallach, Classification of Multipartite Entanglement of All Finite Dimensionality, Phys. Rev. Lett. **111**, 060502 (2013).

[17] S. M. Zangi, J.-L. Li, and C.-F. Qiao, Entanglement classification of four-partite states under the SLOCC, J. Phys. A: Math. Theor. **50**, 325301 (2017).

[18] Y. Shi, L. Duan, and G. Vidal, Classical simulation of quantum many-body systems with a tree tensor network, Phys. Rev. A **74**, 022320 (2006).

[19] L. De Lathauwer, B. De Moor, and J. Vandewalle, A multilinear singular value decomposition, SIAM J. Matrix Anal. Appl. **21**, 1253 (2000).

[20] J.-L. Li, S.-Y. Li, and C.-F. Qiao, Classification of the entangled states $L \times N \times N$, Phys. Rev. A **85**, 012301 (2012).

[21] C. F. Van Loan, The ubiquitous Kronecker product, J. Comput. Appl. Math. **123**, 85 (2000).

[22] L.-L. Sun, J.-L. Li, and C.-F. Qiao, Classification of the entangled states of $2 \times L \times M \times N$, Quantum Inf. Process. **14**, 229 (2015).

[23] C.-K. Li and S. Pierce, Linear preserver problems, Am. Math. Mon. **108**, 591 (2001).

[24] M. Marcus and W. R. Gordon, A generalization of unitary group, Linear Algebra Appl. **3**, 225 (1970).

[25] N. Nakatani and G. K.-L. Chan, Efficient tree tensor network states (TTNS) for quantum chemistry: Generalizations of the density matrix renormalization group algorithm, J. Chem. Phys. **138**, 134113 (2013).