Quantum metrology at level anticrossing

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We address parameter estimation in two-level systems exhibiting level anticrossing and prove that universally optimal strategies for parameter estimation may be designed. In fact, we find a parameter-independent measurement scheme, leading to the ultimate quantum precision, independently of the value of the parameter of interest. Optimal estimation may be achieved also at high temperature, depending on the structure of the two-level Hamiltonian. Finally, we discuss parameter estimation based on dynamical strategies, and a number of specific applications.

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I. INTRODUCTION

The von Neumann–Wigner theorem, also known as the no-crossing rule, describes a characteristic phenomenon occurring in systems with a parameter-dependent Hamiltonian [1,2]. The theorem states that, if a Hamiltonian depends on n real parameters, then the eigenvalues cannot be degenerate, apart from a $(n-2)$-dimensional manifold in the parameter space. For a system Hamiltonian depending on a single parameter, this means that the eigenvalues in general cannot cross as a function of the parameter itself. Level anticrossing, also referred to as level repulsion, plays a relevant role in several branches of quantum physics and chemistry [3–7] and frequently arises in the study of condensed-matter systems. In systems with parameter-dependent Hamiltonians and level anticrossing, small perturbations to the parameter may induce relevant changes in the system ground state [8,9], which are possibly reflected in large variations of some accessible observable. A level anticrossing, which is also connected to the creation of resonances [10,11] and to the onset of chaos [12–15], may thus represent a resource for the characterization of Hamiltonians and, in turn, for the precise quantum-enhanced estimation of parameters [16,17], i.e., for *quantum metrology*.

In this paper, we address quantum parameter estimation based on the properties of the ground state and of the thermal-equilibrium states of parameter-dependent two-level Hamiltonians [18–24]. In particular, we analytically show that universal optimal estimation may be achieved; that is, the ultimate precision permitted by quantum mechanics may be obtained by a class of parameter-independent measurement schemes. This applies to the case where the parameter-dependent state coincides with the Hamiltonian ground state, as well as in the high-temperature limit. Such a result is of metrological interest since, in general, the optimal observable depends on the value of the unknown parameter and some form of *a priori* knowledge of such a parameter is required, or should be gained by adaptive schemes, in order to design the best estimation scheme. Besides, such universality applies to the case of level anticrossing, where precise parameter estimation is typically possible, and to situations where complex systems may be regarded as effective two-level systems.

The paper is structured as follows: In Sec. II we introduce the notation and the basic tools to analyze two-level systems with parameter-dependent Hamiltonian. In Sec. III we discuss the ultimate quantum bounds to the precision of parameter estimation, whereas in Sec. IV we show how those limits may be achieved by parameter-independent measurement schemes, including estimation at finite temperature. We also discuss dynamical strategies and show under which conditions they allow precise parameter estimation. Section V is devoted to some examples and Sec. VI closes the paper with some concluding remarks.

II. THE SYSTEM

Let us consider a two-level system governed by a parameter-dependent Hamiltonian of the form

$$\mathcal{H} = \begin{pmatrix} \omega_1(\lambda) & \gamma(\lambda) \\ \gamma^*(\lambda) & \omega_2(\lambda) \end{pmatrix}. \tag{1}$$

The parameter λ is the quantity of interest, which is initially unknown and whose value has to be estimated by measuring some observable of the system. We assume that $\lambda \in \Lambda$, where Λ is a generic subset of the real field. The eigenvalues of $\mathcal{H}$ and the energy gap are given by

$$h_\pm(\lambda) = \omega_0(\lambda) \pm \sqrt{|\gamma(\lambda)|^2 + \Delta^2(\lambda)}, \tag{2}$$

$$h_+ - h_- = 2\Delta(\lambda)\sqrt{1+|x|^2}, \tag{3}$$

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where we have introduced the adimensional quantity $x \equiv \gamma(\lambda)/\Delta(\lambda)$ and the additional parameters

$$\omega_0(\lambda) = \tfrac{1}{2}[\omega_2(\lambda) + \omega_1(\lambda)], \tag{4}$$

$$\Delta(\lambda) = \tfrac{1}{2}[\omega_2(\lambda) - \omega_1(\lambda)]. \tag{5}$$

In the following, without loss of generality, we assume that $\omega_2(\gamma) > \omega_1(\gamma) > 0$, i.e., $\Delta(\lambda) > 0$, and $\gamma(\lambda) \in \mathbb{R}\ \forall\ \lambda$. As a result, from Eqs. (2) and (3) it follows that $h_- < h_+$, $\forall\ \lambda$, i.e., no level crossing occurs for any value of the parameter of interest λ. To simplify the notation, we drop hereafter the explicit dependence on λ of the parameters introduced so far. The Hamiltonian may be thus rewritten as follows:

$$H = \omega_0\sigma_0 - \Delta\sigma_3 + \gamma\sigma_1, \tag{6}$$

where σ_k (with $k = 0, \ldots, 3$) denote the Pauli matrices. The terms $\gamma\sigma_1$ and $\Delta\sigma_3$ are usually referred to as the *transverse* and the *longitudinal* parts of the Hamiltonian, respectively. The projectors over the eigenvectors $|\psi_\pm\rangle$ of $\mathcal{H}$ may be expressed as

$$P_\pm \equiv |\psi_\pm\rangle\langle\psi_\pm| = \frac{1}{2}\left[\mathbb{I} \pm \frac{\Delta\sigma_3 - \gamma\sigma_1}{\sqrt{\gamma^2 + \Delta^2}}\right] = \frac{1}{2}\left[\mathbb{I} \pm \frac{\sigma_3 - x\sigma_1}{\sqrt{1 + x^2}}\right] \tag{7}$$

and are independent of ω_0.

III. ULTIMATE BOUNDS TO PRECISION

To gain information about the value of the parameter λ, which may not correspond to an observable, one performs repeated measurements on the system and suitably processes the obtained data. The optimal measurement, i.e., the one that allows the most precise parameter estimation, corresponds to the spectral measure of the so-called symmetric logarithmic derivative (SLD) L_λ. This is defined by the Lyapunov-like equation

$$\partial_\lambda\rho_\lambda = \tfrac{1}{2}(L_\lambda\rho_\lambda + \rho_\lambda L_\lambda), \tag{8}$$

where ρ_λ is the (parameter-dependent) state of the system [25,26]. At zero temperature the system is in its ground state and the SLD reduces to

$$L_\lambda = 2(|\partial_\lambda\psi_-\rangle\langle\psi_-| + |\psi_-\rangle\langle\partial_\lambda\psi_-|) = \frac{\partial_\lambda x}{(1 + x^2)^{\frac{3}{2}}}(\sigma_1 - x\sigma_3). \tag{9}$$

By measuring L_λ on repeated preparations of the system, one collects the required data and then builds an estimator for the unknown quantity λ, i.e., a function $\hat{\lambda}(\chi)$ of the data sample $\chi = \{x_1, x_2, \ldots, x_M\}$ that ideally returns the value of the parameter. The precision of the overall estimation strategy can be identified with the variance of the estimator. An efficient estimator (e.g., the maximum-likelihood or the Bayesian estimator) has a variance saturating the quantum Cramér–Rao bound, $\mathrm{Var}(\hat{\lambda}) = 1/MH(\lambda)$ in the limit $M \gg 1$, where M is the number of measurements and $H(\lambda) = \mathrm{Tr}[\rho_\lambda L_\lambda^2] = \langle\psi_-|L_\lambda^2|\psi_-\rangle$ is the so-called quantum Fisher information (QFI) [27–42]. Notice that the optimal measurement and the corresponding precision explicitly depend on the value of λ. Therefore, an approximate *a priori* knowledge of λ is generally required in order to identify the observable that is optimal in the relevant range of values.

In the present case, one can obtain the QFI starting from Eq. (9). The resulting expression is given by

$$H(\lambda) = \frac{(\partial_\lambda x)^2}{(1 + x^2)^2} = 16\left(\frac{\Delta}{h_+ - h_-}\right)^4 (\partial_\lambda x)^2. \tag{10}$$

The last expression well illustrates the connections with the level anticrossing, which are further discussed in Sec. V A. The same result may be obtained by expressing the QFI in terms of the ground-state fidelity [43–47]:

$$H(\lambda) = \lim_{\delta\lambda\to 0} 4\left(\frac{1 - |\langle\psi_-(\lambda + \delta\lambda)|\psi_-(\lambda)\rangle|}{\delta\lambda^2}\right). \tag{11}$$

By using Eq. (11), it may be proved that the QFI of *any* linear superposition, with parameter-independent coefficients, of the Hamiltonian eigenstates, $|\psi_\theta\rangle = \cos\theta|\psi_-\rangle + \sin\theta|\psi_+\rangle$, is equal to that of the ground state (see Sec. IV B for a further discussion).

As is apparent from the above expressions, the QFI, and thus the upper bound to the precision of any estimation scheme, does not depend on ω_0. Notice also that, if either $\Delta = 0$ or $\gamma = 0\,\forall\,\lambda$, then $H(\lambda) = 0$, and thus no estimation strategy is possible. This behavior may be understood by looking at Eq. (7), which shows that, for $\Delta = 0$ or $\gamma = 0$, the density operator corresponding to the eigenstates of the system becomes $P_\pm(\lambda) = \frac{1}{2}[\mathbb{I} \mp \mathrm{sgn}(\gamma)\sigma_1]$ and $P_\pm(\lambda) = \frac{1}{2}[\mathbb{I} \pm \mathrm{sgn}(\Delta)\sigma_3]$, respectively. In both cases, the ground state is independent of λ (except for the crossing points, if any), and no information on the parameter may be gained by performing measurements on the system.

IV. UNIVERSALLY OPTIMAL ESTIMATION BY PROJECTIVE MEASUREMENTS

Since the SLD depends on the unknown value of the parameter, a question arises of whether the ultimate precision may be actually achieved without any *a priori* information. As we will see, universal estimation based on a single experimental set up, which allows a given, parameter-independent measurement, may indeed be obtained.

A generic (projective) measurement on a two-level system is described by the operatorial measure $\{\Pi, \mathbb{I} - \Pi\}$, where

$$\Pi = \tfrac{1}{2}(\mathbb{I} + \mathbf{r}\cdot\sigma), \tag{12}$$

and $|\mathbf{r}| = 1$. The probabilities corresponding to the two possible outcomes are given by

$$q_0(\lambda) \equiv q(\lambda) = \mathrm{Tr}[\rho_\lambda\Pi] = \frac{1}{2}\left(1 + \frac{xr_1 - r_3}{\sqrt{1 + x^2}}\right), \tag{13}$$

$$q_1(\lambda) = \mathrm{Tr}[\rho_\lambda(\mathbb{I} - \Pi)] = 1 - q(\lambda). \tag{14}$$

The variance of any estimator based on the measurement of a specific observable is bounded by the classical Cramér–Rao bound, $\mathrm{Var}(\hat{\lambda}) \geqslant 1/MF(\lambda)$ [48]. The efficient estimators are those saturating the bound, where $F(\lambda)$ is the Fisher

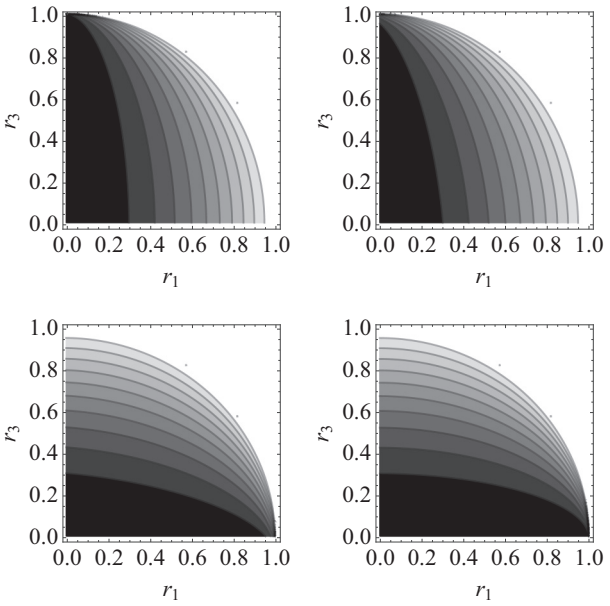

FIG. 1. Plot of the function $g_\lambda(r_1,r_3)$, corresponding to the ratio between Fisher and quantum Fisher information at zero temperature, for fixed values of $x = \gamma/\Delta$. The values of x are 0.01, 0.1, 10, 100, going from the top-left to the bottom-right panel. The function is defined only in the region $r_1^2 + r_3^2 \leqslant 1$. Darker regions correspond to lower values of g, from 0 (black) to 1 (white).

information of the probability distribution $q_k(\lambda)$:

$$F(\lambda) = \sum_k \frac{(\partial_\lambda q_k)^2}{q_k} = \frac{(\partial_\lambda q)^2}{q(1-q)} = H(\lambda) g_\lambda(r_1,r_3), \quad (15)$$

with

$$g_\lambda(r_1,r_3) \equiv g(x,r_1,r_3) = \frac{(r_1 + x r_3)^2}{1 + x^2 - (x r_1 - r_3)^2}. \quad (16)$$

As expected from the quantum Cramér–Rao theorem, we have $F(\lambda) \leqslant H(\lambda)$, i.e., $g_\lambda(r_1,r_3) \leqslant 1 \ \forall \ \lambda,r_1,r_3$ (see Fig. 1). On the other hand, whenever $r_2 = 0$ the above inequality is saturated, $F(\lambda) = H(\lambda)$. Therefore, any observable of the form $\sigma_\theta = \sigma_1 \sin\theta + \sigma_3 \cos\theta$ leads to an optimal estimation. In other words, Eq. (15) and the following arguments show that universal optimal estimation, leading to maximum precision for any value of λ, may be achieved by parameter-independent measurements.

Let us now discuss robustness of the estimation strategy. We have shown above that the optimal (projective) measurement corresponds to the choice $r_2 = 0$, and thus to any pair (r_1,r_3) satisfying $r_1^2 + r_3^2 = 1$, for the expression of the operator measure Π in Eq. (12). On the other hand, some observables may be better than others in practical implementations, depending on the relative values of γ and Δ, i.e., the value of x. Indeed, as is apparent from the upper panels of Fig. 1, if $|\gamma| \ll \Delta$ in the whole range of variation of λ, then $F(\lambda) \simeq H(\lambda)$ also if some imperfections lead to the measurement of a slightly different observable, with respect to the optimal one σ_θ, provided that $r_2 \gtrsim 0$, $r_3 \gtrsim 0$, $r_1 \lesssim 1$. A similar result may be obtained for $|\gamma| \gg \Delta$ in the whole range of variation of λ, where the roles of r_1 and r_3 are exchanged. This may be seen from the lower panels in Fig. 1, and also from the symmetry $g(x,r_1,r_3) = g(1/x,r_3,r_1)$ of the function g.

Notice also that, for any binary measurement $\{\Pi, \mathbb{I} - \Pi\}$ described by POVM like that in Eq. (12), the classical Cramér–Rao bound $\mathrm{Var}(\hat{\lambda}) \geqslant 1/MF(\lambda)$ may be saturated by any inversion estimator [49,50]. Together with the possibility of implementing optimal observables, this makes universal optimal estimation achievable with current technology in quantum metrology experiments involving level anticrossing.

A. Estimation at finite temperature

If the system is not at zero temperature, the equilibrium state is given by

$$\rho_{\lambda,\beta} = p_+ P_+ + p_- P_-, \quad (17)$$

where β is the inverse temperature and the projectors $P_\pm$ over the eigenvectors $|\psi_\pm\rangle$ are given in Eq. (7). The probabilities $p_\pm = e^{-\beta h_\pm}/Z$ are obtained from the eigenvalues $h_\pm$ [Eq. (2)] and from the partition function

$$Z = e^{-\beta h_+} + e^{-\beta h_-} = 2e^{-\beta\omega_0} \cosh(\beta\sqrt{\gamma^2 + \Delta^2}). \quad (18)$$

Using the above expressions, we can make the dependence on the occupation probabilities on the Hamiltonian parameters explicit:

$$p_\pm = \tfrac{1}{2}[1 \pm \tanh(\beta\sqrt{\gamma^2 + \Delta^2})], \quad (19)$$

which leads to

$$\rho_{\lambda,\beta} = \frac{1}{2}\left[\mathbb{I} - \tanh(\beta\Delta\sqrt{1+x^2})\frac{\sigma_3 - x\sigma_1}{\sqrt{1+x^2}}\right]. \quad (20)$$

This density operator corresponds to a mixed state, with purity

$$\mu_{\lambda\beta} = \mathrm{Tr}\left[\rho_{\lambda\beta}^2\right] = \tfrac{1}{2}[1 + \tanh^2(\beta\Delta\sqrt{1+x^2})]. \quad (21)$$

The quantum Fisher information is now given by the sum of two terms, $H_\beta(\lambda) = H_C(\lambda) + H_Q(\lambda)$, which are usually referred to as the *classical* and the *quantum* parts of the QFI [51,52]. The classical part corresponds to the Fisher information of the spectral eigenmeasure:

$$H_C(\lambda) = \frac{(\partial_\lambda p_+)^2}{p_+ p_-} = \frac{\beta^2(\gamma\partial_\lambda\gamma + \Delta\partial_\lambda\Delta)^2}{\gamma^2 + \Delta^2} k_C(\beta,\lambda), \quad (22)$$

where the adimensional function k_C is given by

$$k_C(\beta,\lambda) = \frac{1}{\cosh^2(\beta\sqrt{\gamma^2 + \Delta^2})}. \quad (23)$$

The quantum part H_Q takes into account the contribution coming from the dependence of the eigenvectors on λ:

$$H_Q(\lambda) = 2\sum_{j,k=\pm} |\langle\psi_j|\partial_\lambda\psi_k\rangle|^2 \frac{(p_j - p_k)^2}{p_j + p_k} = H_0(\lambda) k_Q(\beta,\lambda), \quad (24)$$

where the temperature dependence is all in the hyperbolic function

$$k_Q(\beta,\lambda) = \tanh^2(\beta\sqrt{\gamma^2 + \Delta^2}), \quad (25)$$

while $H_0(\lambda)$ is the zero-temperature QFI reported in Eq. (10). In the low-temperature limit, corresponding to $1/\beta \ll |\gamma|, \Delta$,

we have

$$k_Q(\beta,\lambda) \simeq 1, \quad k_C(\beta,\lambda) \simeq 0. \tag{26}$$

At high temperatures, where $1/\beta \gg |\gamma|, \Delta$, one has instead

$$k_Q(\beta,\lambda) \simeq 0, \quad k_C(\beta,\lambda) \simeq 1. \tag{27}$$

Equations (26) and (27) show that the quantum term H_Q is predominant compared to the classical term H_C in the low-temperature regime, whereas the opposite takes place at high temperatures.

To discuss the dependence of the achievable precision on temperature (for a given measurement), we now calculate the Fisher information. Given a generic projective measurement, the probabilities $q_{k=0,1}$ corresponding to the possible outcomes can be derived from

$$q_\beta(\lambda) = \mathrm{Tr}[\rho_{\beta\lambda}\Pi] = 1 - p_- + q(\lambda)(2p_- - 1) \tag{28}$$

$$= \frac{1}{2} + \left[q(\lambda) - \frac{1}{2}\right] \tanh(\beta\sqrt{\gamma^2+\Delta^2}), \tag{29}$$

where $q(\lambda)$ is the zero-temperature probability given in Eq. (13). The statistics of the measured observable thus converges slowly to that characterizing the ground state, as one approaches the zero-temperature limit. In the high-temperature limit one can expand the Fisher information up to second order in the small quantity $\beta(\gamma^2+\Delta^2)^{1/2}$, and thus obtain

$$F_\beta(\lambda) = \frac{(\partial_\lambda q_\beta)^2}{q_\beta(1-q_\beta)} \simeq (r_1\partial_\lambda\gamma - r_3\partial_\lambda\Delta)^2\beta^2. \tag{30}$$

The Fisher information of Eq. (30) should be compared with the QFI, whose low-temperature expansion reads as follows:

$$H_\beta(\lambda) \simeq [(\partial_\lambda\gamma)^2 + (\partial_\lambda\Delta)^2]\beta^2. \tag{31}$$

When only the transverse or only the longitudinal part of the Hamiltonian depends on the parameter λ (i.e., if either $\partial_\lambda\gamma = 0$ or $\partial_\lambda\Delta = 0$), then the condition $H_\beta(\lambda) \simeq F_\beta(\lambda)$ is achieved respectively for $r_3 = 1$ or $r_1 = 1$. Therefore, the optimal observable is well defined and does not depend on the specific value of λ: the universal optimal estimation is thus achievable also at high temperature.

B. Dynamical estimation strategies

One may wonder whether relaxing the restriction to ground or thermal equilibrium states, in particular through the application of unitary transformation, may improve precision for some class of estimation strategies. In fact, general considerations about unitary families of states suggest the opposite [26], i.e., that no improvement may be achieved in this way. To investigate this point further, we consider hereafter two different scenarios. In the first one, the system can be prepared in any desired parameter-dependent state $|\varphi_\lambda\rangle$, and then undergo a parameter-independent unitary transformation U_0. Such transformation, which can always be thought of as the time evolution induced by a suitable Hamiltonian, eventually leads to the state $|\psi_\lambda\rangle = U_0|\varphi_\lambda\rangle$. The resulting expression of the SLD is given by

$$L_{\psi_\lambda} = 2(|\partial_\lambda\psi_\lambda\rangle\langle\psi_\lambda| + |\psi_\lambda\rangle\langle\partial_\lambda\psi_\lambda|) = U_0 L_{\varphi_\lambda} U_0^\dagger. \tag{32}$$

As a result, the QFI of the initial and of the final states coincide, being

$$H_{\psi_\lambda} = \langle\psi_\lambda|L^2_{\psi_\lambda}|\psi_\lambda\rangle = \langle\varphi_\lambda|L^2_{\varphi_\lambda}|\varphi_\lambda\rangle = H_{\varphi_\lambda}. \tag{33}$$

A parameter-independent time evolution therefore does not improve the precision of the parameter estimation.

In the second scenario that we consider, the system is initialized in a parameter-independent state $|\psi_0\rangle$ and turned into a parameter-dependent state by the application of a time-evolution operator, $|\psi_\lambda\rangle = e^{-i\mathcal{H}t}|\psi_0\rangle$, where $\mathcal{H}$ is the Hamiltonian defined in Sec. II. Without loss of generality, we assume that $|\psi_0\rangle = |\uparrow\rangle$, that the quantization axis defined by the parameters Δ and γ lies in the xz plane, and that it forms an angle θ with the z axis. As a result, the parameter-dependent state is

$$|\psi_\lambda\rangle = [\cos(\delta t) - i\cos\theta\sin(\delta t)]|\uparrow\rangle - i\sin(\delta t)\sin\theta|\downarrow\rangle, \tag{34}$$

where $\theta \equiv -\arctan(\gamma/\Delta)$ and $\delta \equiv (h_+ - h_-)/2$ is half the energy gap between the two eigenstates. The derivative of such a state with respect to λ has the following overlaps with the basis states:

$$\langle\uparrow|\partial_\lambda\psi_\lambda\rangle = -t(\partial_\lambda\delta)[\sin(\delta t) + i\cos(\delta t)\cos\theta] + i(\partial_\lambda\theta)\sin(\delta t)\sin\theta, \tag{35}$$

$$\langle\downarrow|\partial_\lambda\psi_\lambda\rangle = -it(\partial_\lambda\delta)\cos(\delta t)\sin\theta - i(\partial_\lambda\theta)\sin(\delta t)\cos\theta. \tag{36}$$

Plugging the above expressions into that of the QFI for pure states, one obtains, after some algebra, the following equation:

$$H(\lambda) = 4t^2(\partial_\lambda\delta)^2\sin^2\theta \tag{37}$$

$$+ 4(\partial_\lambda\theta)^2\sin^2(\delta t)[1 - \cos^2(\delta t)\sin^2\theta] \tag{38}$$

$$+ 8t(\partial_\lambda\delta)(\partial_\lambda\theta)\sin(\delta t)\cos(\delta t)\sin\theta\cos\theta. \tag{39}$$

The sensitivity with which the parameter λ can be estimated thus results from three main terms. The first one increases quadratically with time, results from the dependence on λ of the energy gap δ, and is maximal if the spin is initially perpendicular to the quantization axis. The second contribution oscillates in time and results from the dependence on the quantization axis on λ. In fact, small variations of such axis result in small changes of the final orientations of the spin, which vanish (are maximal) for (half-) integer values of the rotation angle $\delta t/\pi$. The third term also oscillates in time but is only present if both δ and θ depend on the unknown parameter. Here, the spin orientation does not generally belong to a given plane for all values of λ (unless, for example, $\theta = \pi/2$), and therefore the optimal estimation is not universal.

We finally note that an improved performance in the parameter estimation may be achieved if the two-level Hamiltonian depends explicitly on time [50,53–55].

V. EXAMPLES

A. Level anticrossing induced by a constant term

Let us consider a two-level system with a longitudinal term that depends linearly on λ and a nonlongitudinal one that is

constant, i.e., parameter independent. It is generally assumed that, in order to maximize the sensitivity of the ground state with respect to the exact value of λ, the constant term should be transverse. To verify under which conditions the above assumption is correct, we consider the Hamiltonian given in Eq. (6), with the parameter dependence

$$\Delta = \alpha\lambda + \Delta_0. \tag{40}$$

The axis defined by the constant term is thus given by

$$\phi = -\arctan(\gamma/\Delta_0), \tag{41}$$

where $\phi = \pi/2$ ($\Delta_0 = 0$) corresponds to the standard case of transverse constant term. Inserting this expression in that of the quantum Fisher information, one can derive its dependence on the λ and on the other Hamiltonian parameters:

$$H(\lambda) = \frac{\alpha^2\gamma^2}{[(\alpha\lambda + \Delta_0)^2 + \gamma^2]^2}. \tag{42}$$

By differentiating with respect to Δ_0, one can show that the maximum of the QFI is found for $\Delta_0 = -\alpha\lambda$, which corresponds to a parameter-dependent optimum angle for the constant term of

$$\phi_{\rm opt} = \arctan(\gamma/\alpha\lambda). \tag{43}$$

This has a simple physical interpretation: the optimum constant term in the Hamiltonian is such that the diagonal gap Δ vanishes for the value of interest of the parameter λ. By plugging the optimal value of Δ_0 in the expression of the quantum Fisher information, one obtains

$$H_{\Delta_0=-\alpha\lambda}(\lambda) = (\alpha/\gamma)^2, \tag{44}$$

which is higher than the value obtained in the case of the transverse constant term, given by

$$H_{\Delta_0=0}(\lambda) = \frac{\alpha^2\gamma^2}{(\alpha^2\lambda^2 + \gamma^2)^2}. \tag{45}$$

We stress, however, that, while the value of Δ_0 that maximizes the quantum Fisher information is parameter dependent, the optimal observable is not. In fact, in view of the previous analysis, any spin component that lies in the xz plane ($r_2 = 0$) leads to a value of the Fisher information that equals that of $H(\lambda)$.

B. Driven double-well systems

It is often the case in condensed matter that double-well systems exhibit two lowest-energy levels well separated from the next pair by a large gap, i.e., larger than the other relevant energies, e.g., the tunneling energy and the frequency of the driving field. In those cases, a two-level approximation describes rather well the physics of the system, and the dynamics may be understood in terms of the celebrated periodic *Rabi Hamiltonian*,

$$H = \tfrac{1}{2}\omega_0\sigma_3 + \lambda\sigma_1\cos\omega t,$$

where the coupling λ is the quantity to be estimated and ω is the frequency of the driving field, which we assume to be known to the experimenter. The model cannot be solved exactly [56], since the Hamiltonian is not commuting with itself at different times. On the other hand, upon going to the appropriate interaction picture and neglecting the counter-rotating terms, the system may be described by a two-level time-independent Hamiltonian [57] which, in the relevant subspace, reads

$$\mathcal{H}_{\rm eff} = \begin{pmatrix} \frac{1}{2}\Omega & \gamma \\ \gamma & -\frac{1}{2}\Omega + 2\omega \end{pmatrix}, \tag{46}$$

where

$$\gamma = -\frac{\lambda}{4\Omega}[\Omega - (\omega_0 - \omega)], \tag{47}$$

$$\Omega = \sqrt{\lambda^2 + (\omega - \omega_0)^2}. \tag{48}$$

The reduction to a static problem is obtained by decomposing the σ_1 term of the Rabi Hamiltonian in two fields *circularly polarized* along the z axis, and then by transforming the Hamiltonian to a frame of reference rotating with one of these and neglecting the counter-rotating terms of the original Hamiltonian. The physics underlying this approximated Hamiltonian is that of a system with avoided level crossing and a gap $\simeq 2\gamma$ separating the otherwise crossing unperturbed levels. The quantity Δ introduced in the previous sections is here given by $\Delta = \Omega - 2\omega$. Inserting this expression in Eq. (10) we arrive at the QFI

$$H(\lambda) \overset{\omega\simeq\omega_0}{\simeq} \frac{1}{64\omega_0^2}\frac{1}{\left(1 - y + \frac{17}{64}y^2\right)^2}, \quad y = \frac{\lambda}{\omega_0}, \tag{49}$$

where, for the sake of simplicity, we have reported only the expression close to resonance $\omega \simeq \omega_0$. The QFI is maximized for $\lambda = \frac{32}{17}\omega_0$, indicating that, for any value of λ, optimization may be achieved by tuning the natural frequency of the well. As proved in the previous sections, those ultimate limits to precision may be achieved by measuring any observable of the form $\sigma_\theta = \sigma_1\sin\theta + \sigma_3\cos\theta$, where σ_3 is here the population of the unperturbed levels and σ_1 the corresponding polarization. More general driven systems with level anticrossing [58] may be also addressed in the same way.

C. Effective description of three-level systems

Level anticrossing may also occur in systems with more than two levels. In this case, the additional levels may influence the form of the eigenstates and, in turn, the behavior of the QFI when the value of the parameter λ is perturbed. Let us consider a three-level system with two close energy levels and a third level being well separated in energy and weakly coupled to the first two levels. The Hamiltonian for such a system reads

$$\mathcal{H}^{(3)} = \begin{pmatrix} \omega_1(\lambda) & \gamma(\lambda) & g \\ \gamma^*(\lambda) & \omega_2(\lambda) & g \\ g & g & \epsilon \end{pmatrix}, \tag{50}$$

where we assume a large gap between the third level and the others, i.e., $\epsilon \gg \omega_k$ and a weak coupling $g \ll 1$. In this regime, the system is amenable to an effective two-level description [46], with an effective Hamiltonian given by

$$\mathcal{H}^{(2)}_{\rm eff} = \begin{pmatrix} \omega_1(\lambda) + g^2/\epsilon & \gamma(\lambda) + g^2/\epsilon \\ \gamma(\lambda) + g^2/\epsilon & \omega_2(\lambda) + g^2/\epsilon \end{pmatrix}, \tag{51}$$

where we have also assumed $\gamma \in \mathbb{R}$. Using this effective description we may now exploit the approach of the previous

sections in order to assess the performances of this system as a scheme to estimate the value of the λ. The QFI may be evaluated by using Eq. (10). Up to first order in the quantity $\kappa = g^2/\epsilon$ we have

$$H_\kappa(\lambda) = H_0(\lambda) - 2\kappa\sqrt{H_0(\lambda)}\frac{2\gamma\Delta\partial_\lambda\gamma + \partial_\lambda\Delta(\Delta^2 - \gamma^2)}{(\gamma^2+\Delta^2)^2},$$

where $H_0(\lambda)$ is the QFI of Eq. (10), corresponding to $\kappa = 0$, i.e., a genuine two-level system. The possibility of enhancing estimation by coupling with additional levels thus depends on the explicit dependence on λ of the quantities γ and Δ.

VI. CONCLUSIONS

We have addressed in detail metrological applications of two-level systems, which can be applied to higher-dimensional systems in the vicinity of level anticrossing. We have shown that universally optimal strategies for parameter estimation may often be designed, independently, on the nature and the value of the parameter of interest, and thus without requiring any *a priori* knowledge of its approximate value. In particular, this is the case if the parameter-dependent state coincides with the system ground state, but also to a thermal state in the high-temperature limit, if either the transverse or the diagonal part of the Hamiltonian are parameter-independent. Parameter estimation, although nonuniversal, is possible in the case where the parameter dependence of the state results from a time evolution, driven by the qubit Hamiltonian. Finally, we have also analyzed few examples, which confirm the generality of our approach and pave the way for further applications.

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