

Ultimate precision of joint quadrature parameter estimation with a Gaussian probe

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The Holevo Cramér-Rao bound is a lower bound on the sum of the mean-square error of estimates for parameters of a state. We provide a method for calculating the Holevo Cramér-Rao bound for estimation of quadrature mean parameters of a Gaussian state by formulating the problem as a semidefinite program. In this case, the bound is tight; it is attained by purely Gaussian measurements. We consider the example of a symmetric two-mode squeezed thermal state undergoing an unknown displacement on one mode. We calculate the Holevo Cramér-Rao bound for joint estimation of the conjugate parameters for this displacement. The optimal measurement is different depending on whether the state is entangled or separable.

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I. INTRODUCTION

Quantum mechanics sets a limit on how accurately one can measure two noncommuting observables. This is exemplified by the Heisenberg uncertainty relation for position and momentum, which can be generalized to arbitrary observables [1]. This relation sets a precision limit to state estimation problem of the noncommuting observables. For example, if we were to simultaneously measure two quadrature operators \mathcal{Q} and \mathcal{P} with the canonical commutation relation $[\mathcal{Q}, \mathcal{P}] = i$ [2,3] of a quantum state ρ , then the precision is limited by $\Delta\mathcal{Q}\Delta\mathcal{P} \geq \frac{1}{2}$. However, if we are interested in estimating channel parameters instead, this restriction does not apply. In this case, entanglement can be used to enhance the precision of channel parameter estimates [4–10], for example, estimating the squeezing applied to a probe [11]. Light-matter interferometry can be used to improve the estimate of a Gaussian process applied to a matter system [12]. The precision can also be improved with a cleverly chosen single-mode state, for the estimation of a small displacement, for example [13].

We will consider in detail the example of estimation of the parameters θ_1 and θ_2 of the displacement operation

$$D(\theta_1, \theta_2) = \exp(i\theta_2\mathcal{Q} - i\theta_1\mathcal{P}), \quad (1)$$

acting on a probe state. It was shown in Refs. [5,14] that by using a two-mode entangled probe, one can estimate the displacement to arbitrary high accuracy. The probe is a symmetric two-mode squeezed thermal state. If the state is pure, it is known as a two-mode squeezed vacuum state or an Einstein-Podolski-Rosen (EPR) state [2]. By symmetric we mean that the state has equal squeezing and noise in all quadratures.

A measurement was proposed that can give an arbitrarily precise estimate of both θ_1 and θ_2 simultaneously. This measurement, which resembles continuous variable superdense coding [15], involves passing one mode on an entangled probe to sense the displacement operation and then jointly measuring it with an entangled ancilla. We call this measurement the double-homodyne joint measurement [see Fig. 1(b)]. This extremely precise estimation scheme was experimentally demonstrated in an optical system [16].

Genoni *et al.* [14] showed that for a symmetric two-mode squeezed state probe, in the limit of large entanglement, the double-homodyne joint measurement approaches the ultimate precision bounds calculated using the symmetric logarithmic derivative (SLD) quantum Fisher information. However, for a general finite squeezing level, there is a gap between the precision of the estimation from dual homodyne measurement and the limit set by the right logarithmic derivative (RLD) and SLD quantum Fisher information. This is not surprising since in general we know that the RLD and SLD bounds are not tight [17]. This raises two questions: (i) Can we derive tight bounds for the precision? and (ii) Is there a better measurement that will give a higher precision than the dual homodyne measurement?

We address these questions for a general two-mode Gaussian probe. In this work, we calculate the Holevo Cramér-Rao (CR) bound [18,19], which is an asymptotically achievable bound under some conditions [20–23]. However, unlike the RLD and SLD bounds, computing the Holevo bound is in general a hard problem because it involves an optimization of a nonlinear function over a space of Hermitian matrices. To date, it has been solved in only a few simple cases. Providing the states satisfy certain conditions, an explicit formula can be found for Gaussian states [18,24] or pure states [25,26]. Suzuki found a formula in terms of the RLD and SLD CR bounds, for a qubit state parameterized by two parameters [27].

Previously, we performed this optimization for the special case when the probe was a pure two-mode entangled state and one mode experiences an unknown displacement [28]. When the probe is mixed or if the channel is dissipative, then the space of the optimization problem is over infinite dimensional Hermitian matrices. However, for Gaussian states, the probe and measurement can be completely characterized by its first and second moment [18,19]. This reduces the optimization space to four-dimensional positive semidefinite matrices which can be solved efficiently using semidefinite programming (SDP) [29]. Furthermore, the SDP and its dual program provide a necessary and sufficient condition for optimality of the solution, which can be verified analytically. Holevo solved the problem for mean estimation of Gaussian states 40 years ago [18,19]. Our contribution is to recognize this as an SDP that can be solved efficiently.

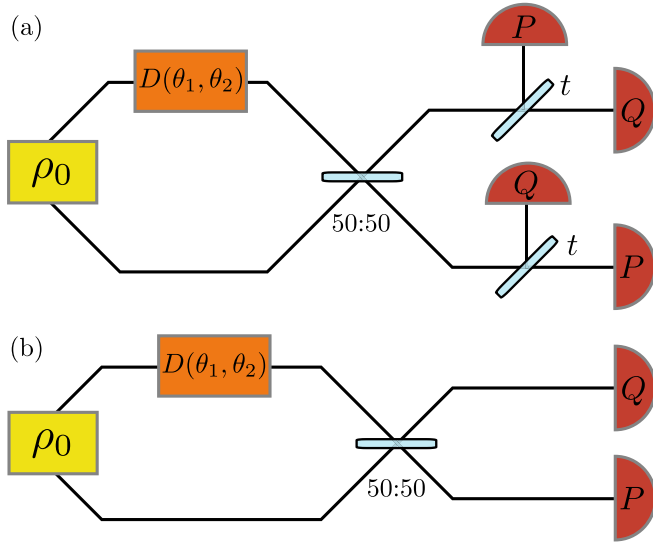


FIG. 1. The probe state ρ_0 , a two-mode squeezed state, undergoes an unknown displacement $D(\theta_1, \theta_2)$. This figure shows the optimal measurement for estimating the displacement. (a) The optimal measurement to perform when $r < r_0$ is a double-unbalanced-heterodyne joint measurement. The two modes are mixed with a 50:50 beam splitter. Each output of the beam splitter then passes through another beam splitter with transmission t given by (52). Homodyne measurements of the \mathcal{P} and \mathcal{Q} quadratures are performed on the outputs of the beam splitters. (b) The optimal measurement to perform when $r \geq r_0$ is a double-homodyne joint measurement. The two modes are mixed with a 50:50 beam splitter. A homodyne measurement of the \mathcal{P} quadrature is performed on one output of the beam splitter, and a homodyne measurement of the \mathcal{Q} quadrature is performed on the other.

For the specific case of a symmetric two-mode squeezed state, we find that the double-homodyne joint measurement is an optimal measurement when the squeezing level is high enough such that the probe is entangled. When the probe is separable, we find that the double-homodyne joint measurement is suboptimal. We propose a different measurement scheme which is optimal.

In this paper, we provide a recipe for calculating the ultimate precision of an unbiased estimate of displacement using a two-mode Gaussian probe. We start with an introduction to multiparameter local quantum estimation in Sec. II. In Sec. III we formulate the problem of displacement estimation for two-mode Gaussian states in terms of an SDP. Section IV gives an application of this formalism to the symmetric two-mode squeezed state. Finally, we end with some concluding remarks in Sec. V.

II. MULTIPARAMETER LOCAL ESTIMATION

In classical parameter estimation theory, one starts with a random variable X that depends on some unknown parameter vector $\theta = (\theta_1, \theta_2, \dots, \theta_N)$ through a conditional probability density function $f(x; \theta)$. The random variable X arises from the measurement of some state $\rho(\theta)$. From X , one can form a vector function $\hat{\theta} = \hat{\theta}(X)$ that gives an unbiased estimate of θ . The goal is to find a precise estimate of θ . The bound on how precise these unbiased estimators can be is determined by the CR bound [30,31],

$$V_\theta[\hat{\theta}] \geq \frac{1}{I}, \quad (2)$$

which relates the mean-square error (MSE) matrix

$$V_\theta[\hat{\theta}]_{jk} := \mathbb{E}[(\hat{\theta}_j - \theta_j)(\hat{\theta}_k - \theta_k)] \quad (3)$$

to the classical Fisher information matrix

$$I_{jk} := -\mathbb{E}\left[\frac{\partial^2}{\partial\theta_j\partial\theta_k} \ln f\right]. \quad (4)$$

Under certain conditions, this bound can be asymptotically achieved by the maximum likelihood estimator. We are interested in the sum of the MSE, obtained by taking the trace of the MSE matrix $\Sigma := \text{Tr}\{V_\theta[\hat{\theta}]\}$.

Quantum parameter estimation theory [32–35] aims to determine the ultimate precision with which certain parameters θ can be determined from a quantum state ρ_θ that depends on those parameters. This was developed by Helstrom [33,36,37], Holevo [18,19], and others [38,39] in the 1970s. There exists a whole family of quantum Fisher information matrices, each of which gives rise to its own CR bounds to the mean-square error matrix [40]. However, none of these bounds are generally tight. Two commonly used CR bounds are based on the SLD [33,36] and RLD [38,39] Fisher information matrix.

The SLD operators $L_j^{(S)}$ and RLD operators $L_j^{(R)}$ are obtained as solutions to the implicit operator equations

$$\frac{\partial\rho}{\partial\theta_j} = \frac{1}{2}(\rho L_j^{(S)} + L_j^{(S)}\rho) \quad (\text{SLD}), \quad (5)$$

$$\frac{\partial\rho}{\partial\theta_j} = \rho L_j^{(R)} \quad (\text{RLD}). \quad (6)$$

The SLD operators are Hermitian, but the RLD operators might not be Hermitian. From the log-derivative operators, the SLD and RLD Fisher information matrices are defined by

$$G_{jk}^{(S)} := \text{tr}\left[\rho \frac{1}{2}(L_j^{(S)} L_k^{(S)} + L_k^{(S)} L_j^{(S)})\right] \quad (\text{SLD}), \quad (7)$$

$$G_{jk}^{(R)} := \text{tr}(\rho L_j^{(R)} L_k^{(R)\dagger}) \quad (\text{RLD}), \quad (8)$$

from which we get the two CR bounds

$$\Sigma \geq \text{Tr}\{(G^{(S)})^{-1}\} =: C^{(S)}, \quad (9)$$

$$\Sigma \geq \text{Tr}\{\text{Re}(G^{(R)})^{-1}\} + \text{Tr}\{\text{Abs}\{\text{Im}(G^{(R)})^{-1}\}\} =: C^{(R)}, \quad (10)$$

where $\text{Tr}\{\text{Abs}\{X\}\}$ is the sum of the absolute values of the eigenvalues of a matrix X . The SLD CR bound, $C^{(S)}$ gives the optimal precision in estimating each parameter separately. However, for multiparameter estimation, if optimal measurements for measuring each parameter separately do not commute (which is usually the case), then the SLD bound is not attainable. The RLD bound, $C^{(R)}$ is also in general not attainable. However, when $L^{(R)}$ is Hermitian, $C^{(R)}$ provides an achievable bound for the joint estimates [41–43]. In general, there is no hierarchy between $C^{(S)}$ and $C^{(R)}$.

Holevo unified these two bounds through the Holevo CR bound [18,19]. This bound is achieved in the asymptotic limit of a joint measurement over infinite copies of the state [20]. The Holevo CR bound is always greater or equal to $C^{(S)}$ and $C^{(R)}$. The bound involves a minimization over $X = (X_1, X_2, \dots, X_N)$ where X_j are Hermitian operators that satisfy

the unbiased conditions

$$\text{tr}(\rho X_j) = 0, \quad (11)$$

$$\text{tr}\left(\frac{\partial \rho_\theta}{\partial \theta_j} X_k\right) = \delta_{jk}. \quad (12)$$

The Holevo CR bound is

$$\mathcal{V} \geq \min_X \text{Tr}\{Z_\theta[X]\} + \text{TrAbs}\{\text{Im}Z_\theta[X]\} := C^{(\text{H})}, \quad (13)$$

where

$$Z_\theta[X]_{jk} := \text{tr}(\rho X_j X_k). \quad (14)$$

Holevo derived this bound in his original work [18,19], but the bound in this form was introduced by Nagaoka [44]. A major obstacle preventing the more widespread use of the Holevo CR bound is that unlike the RLD and SLD bounds, which can be calculated directly, the Holevo bound involves a nontrivial optimization problem.

III. HOLEVO BOUND FOR MEAN VALUE ESTIMATION WITH GAUSSIAN PROBES

When the probe is Gaussian, Holevo's bound can be simplified. It can be formulated in terms of the first and second moments of the probe state only. In this section, we summarize Holevo's result on mean value estimation of Gaussian probes. For the proofs and technicalities of these results, we recommend the interested reader to consult Holevo's original work [18,19].

A. Holevo's bound

We want to estimate two parameters θ_1 and θ_2 that are imprinted on the displacement of a two-mode Gaussian state. Extension to more parameters or mode are straightforward (see Appendix C). To arrive at Holevo's result we need to introduce some notations.

For any $z = [y_1 \ x_1 \ y_2 \ x_2]^T$ in a four-dimensional real vector space Z , let

$$\mathcal{R}(z) = x_1 \mathcal{P}_1 + y_1 \mathcal{Q}_1 + x_2 \mathcal{P}_2 + y_2 \mathcal{Q}_2, \quad (15)$$

where \mathcal{P}_j and \mathcal{Q}_j are the usual quadrature operators for the j th mode in quantum optics. $\mathcal{R}(z)$ are called canonical observables, and the canonical commutation relation becomes

$$[\mathcal{R}(z), \mathcal{R}(z')] = i \Delta(z, z'), \quad (16)$$

where

$$\Delta(z, z') = x'_1 y_1 - x_1 y'_1 + x'_2 y_2 - x_2 y'_2 \quad (17)$$

is a skew-symmetric bilinear form. By the Baker-Campbell-Hausdorff formula, we have an equivalent representation of the canonical commutation relation as

$$\mathcal{V}(z)\mathcal{V}(z') = \exp\left[\frac{i}{2}\Delta(z, z')\right]\mathcal{V}(z+z'), \quad (18)$$

where $\mathcal{V}(z) = e^{i\mathcal{R}(z)}$ is the Weyl operator. The characteristic function of a state \mathcal{S} is then defined through $\mathcal{V}(z)$ as $\chi_z[\mathcal{S}] = \text{tr}\{\mathcal{S}\mathcal{V}(z)\}$. This is the inverse-Weyl or Wigner transform that maps an operator in the Hilbert space to some

square-integrable function in Z . We say \mathcal{S} is Gaussian if the state is completely characterized by its first and second moments [2]:

$$\chi_z[\mathcal{S}] = \exp\left[i m(z) - \frac{1}{2}\alpha(z, z)\right], \quad (19)$$

where

$$m(z) = \text{tr}[\mathcal{S}\mathcal{R}(z)], \quad (20)$$

$$\alpha(z, z') = \frac{1}{2}\text{tr}[\mathcal{S}\{\mathcal{R}(z) - m(z), \mathcal{R}(z') - m(z')\}], \quad (21)$$

and $\{\mathcal{A}, \mathcal{B}\} = \mathcal{A}\mathcal{B} + \mathcal{B}\mathcal{A}$. The mean value function m is a function of the unknown parameters through

$$m(z) = \theta_1 m_1(z) + \theta_2 m_2(z). \quad (22)$$

The correlation function α is an inner product on Z , which defines a Euclidean space (Z, α) . Now let \mathcal{D} be the associated operator of the form Δ in (Z, α) ,

$$\Delta(z, z') = \alpha(z, \mathcal{D}z') \quad \forall z, z' \in Z. \quad (23)$$

Define $m_j \in Z$ by $m_j(z) = \alpha(m_j, z)$.

Holevo's CR bound is

$$\Sigma \geq \inf_{\mathcal{F}} \text{Tr}F^{-1} =: \Sigma_*, \quad (24)$$

where F is a 2×2 matrix with components

$$F_{jk} = \alpha(m_j, \mathcal{F}m_k), \quad (25)$$

and the infimum is taken over all real symmetric operators \mathcal{F} in Z , such that the complex extension of \mathcal{F} satisfies

$$0 \leq (1 + \frac{1}{2}i\mathcal{D})\mathcal{F}(1 + \frac{1}{2}i\mathcal{D}) \leq (1 + \frac{1}{2}i\mathcal{D}) \quad (26)$$

in the complexification of the Euclidean space (Z, α) . $\mathcal{A} \leq \mathcal{B}$ denotes $\alpha(z, \mathcal{A}z) \leq \alpha(z, \mathcal{B}z)$ for all $z \in Z$. Since $1 + \frac{1}{2}i\mathcal{D}$ is positive definite, constraint (26) is equivalent to

$$0 \leq \mathcal{F} \leq (1 + \frac{1}{2}i\mathcal{D})^{-1}. \quad (27)$$

B. Optimal measurement

For estimating the mean of Gaussian probes, Holevo showed that the bound can be attained by a Gaussian measurement. Let \mathcal{F}_* be the operator in Z that furnishes the minimum in (24) and F_* be the corresponding matrix in (25). The optimal estimator are given by the observables $\mathcal{R}(z_j^*)$ where

$$\begin{bmatrix} z_1^* \\ z_2^* \end{bmatrix} = F_*^{-1} \begin{bmatrix} \mathcal{F}_* m_1 \\ \mathcal{F}_* m_2 \end{bmatrix}. \quad (28)$$

$\mathcal{R}(z_1^*)$ and $\mathcal{R}(z_2^*)$ can be measured simultaneously to attain precision Σ_* .

C. Matrix representation

The optimization problem for computing Holevo's bound can be expressed as a semidefinite program. This can be clearly seen if we introduce four vectors $\{e_1, e_2, e_3, e_4\}$ that forms an orthonormal basis in the Euclidean space (Z, α) such that $\alpha(e_j, e_k) = \delta_{jk}$ and introduce

$$\mathbb{D}_{jk} := \alpha(e_j, \mathcal{D}e_k) \quad (29)$$

and

$$\mathbb{M}_{jk} := \alpha(m_j, e_k), \quad (30)$$

$$\mathbb{F}_{jk} := \alpha(e_j, \mathcal{F}e_k), \quad (31)$$

so that

$$F_{jk} = \alpha(m_j, \mathcal{F}m_k) \quad (32)$$

$$= \sum_{mn} \alpha(m_j, e_m) \alpha(e_m, \mathcal{F}e_n) \alpha(e_n, m_k) \quad (33)$$

$$= (\mathbb{M}\mathbb{F}\mathbb{M}^\top)_{jk}. \quad (34)$$

Let \mathbb{S}^n be the set of all $n \times n$ real symmetric matrices. Holevo's bound is obtained as a solution to the following program:

Program 1. Holevo's bound:

$$\Sigma_* = \min_{\mathbb{F} \in \mathbb{S}^4} \text{Tr} F^{-1} \quad (35)$$

$$\text{subject to } 0 \leq \mathbb{F} \leq \mathbb{C}, \quad (36)$$

where $F = \mathbb{M}\mathbb{F}\mathbb{M}^\top$ and $\mathbb{C} := (1 + \frac{1}{2}i\mathbb{D})^{-1}$. This is recognized as an SDP (see Appendix A) that can be solved efficiently using standard numerical techniques.

IV. WORKED EXAMPLE: SYMMETRIC TWO-MODE SQUEEZED STATE

We illustrate the computation of Holevo's bound through a specific example. We start with a mixed two-mode squeezed state $\rho_0 = S_2(r)[\rho_{\text{th}}(v) \otimes \rho_{\text{th}}(v)]S_2^\dagger(r)$ as our probe where

$$\rho_{\text{th}}(v) = \frac{2}{(1+2v)} \sum_n \binom{2v-1}{2v+1}^n |n\rangle\langle n| \quad (37)$$

is a thermal state with mean photon number $v - \frac{1}{2}$ and quadrature variance $\alpha(z, z) = v$. The vacuum state corresponds to $v = \frac{1}{2}$. The ket $|n\rangle$ is the Fock state with n photons, and

$$S_2(r) := \exp(ra_1a_2 - ra_1^\dagger a_2^\dagger) \quad (38)$$

is the two-mode squeezing operator where a_j and a_j^\dagger are the j th mode annihilation and creation operators with commutation relation $[a, a^\dagger] = 1$. Having prepared the probe ρ_0 , we send one mode through a displacement

$$D(\theta_1, \theta_2) := \exp(i\theta_2 \mathcal{Q}_1 - i\theta_1 \mathcal{P}_1) \quad (39)$$

to get ρ_θ , where θ_1 and θ_2 are the two unknown parameters that we wish to determine. In what follows, we shall compute the Holevo bound and present a measurement that achieves this bound. We then compare this bound with the RLD and SLD bounds.

A. Problem formulation

Having the state ρ_θ , we can already write its characteristic function and find Holevo's bound directly. But, instead, we choose to perform a unitary transformation to decouple the two modes of the probe. The transformation we perform is

$$U = \exp\left[\frac{\pi}{4}(a_1^\dagger a_2 - a_1 a_2^\dagger)\right], \quad (40)$$

which corresponds to interfering the two modes on a 50:50 beam splitter. This extra step is not necessary but is done for convenience so that the intermediate expressions in computing the bound become less cumbersome. This of course will not change the final result since the unitary operation can be considered part of the measurement. The correlation function is

$$\alpha(z, z') = v \begin{bmatrix} y_1 \\ x_1 \\ y_2 \\ x_2 \end{bmatrix}^\top \begin{bmatrix} e^{-2r} & 0 & 0 & 0 \\ 0 & e^{2r} & 0 & 0 \\ 0 & 0 & e^{2r} & 0 \\ 0 & 0 & 0 & e^{-2r} \end{bmatrix} \begin{bmatrix} y'_1 \\ x'_1 \\ y'_2 \\ x'_2 \end{bmatrix} \quad (41)$$

and mean

$$m(z) = \frac{1}{\sqrt{2}} \begin{bmatrix} \theta_1 \\ \theta_2 \\ -\theta_1 \\ -\theta_2 \end{bmatrix}^\top \begin{bmatrix} y_1 \\ x_1 \\ y_2 \\ x_2 \end{bmatrix}. \quad (42)$$

From this, the two vectors m_1 and m_2 in Z are

$$m_1 = \frac{1}{v\sqrt{2}} [e^{2r} 0 \ -e^{-2r} \ 0]^\top, \quad (43)$$

$$m_2 = \frac{1}{v\sqrt{2}} [0 \ e^{-2r} \ 0 \ -e^{2r}]^\top. \quad (44)$$

We now pick four orthonormal bases in (α, Z) . Holevo's bound does not depend on our choice of basis, any basis would do, and one such basis is

$$[e_1 \ e_2 \ e_3 \ e_4] = \frac{1}{\sqrt{v}} \begin{bmatrix} e^r & 0 & 0 & 0 \\ 0 & e^{-r} & 0 & 0 \\ 0 & 0 & e^{-r} & 0 \\ 0 & 0 & 0 & e^r \end{bmatrix}. \quad (45)$$

In this basis,

$$\mathbb{M} = \frac{1}{\sqrt{2v}} \begin{bmatrix} e^r & 0 & -e^{-r} & 0 \\ 0 & e^{-r} & 0 & -e^r \end{bmatrix} \quad (46)$$

and

$$\mathbb{D} = \frac{1}{v} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}. \quad (47)$$

We show in Appendix B that the solution to the SDP program (35) is

$$\Sigma_* = \begin{cases} \frac{4v^2-1}{2v \cosh 2r-1} & \text{if } r < r_0 \\ 4ve^{-2r} & \text{if } r \geq r_0 \end{cases}, \quad (48)$$

where $r_0 = \frac{1}{2} \ln(2v)$ and an optimal \mathbb{F}_* attaining this is

$$\mathbb{F}_* = \frac{2v}{4v^2-1} \begin{bmatrix} 2v-e^{-2r} & 0 & 0 & 0 \\ 0 & 2v-e^{2r} & 0 & 0 \\ 0 & 0 & 2v-e^{2r} & 0 \\ 0 & 0 & 0 & 2v-e^{-2r} \end{bmatrix}$$

for $r < r_0$ and

$$\mathbb{F}_* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (49)$$

for $r \geq r_0$.

B. Optimal measurements that attains the bound

To find the optimal measurement achieving Σ_* , we substitute the solution for \mathbb{F}_* into (28) to obtain z_j^* . For $r < r_0$

$$z_1^* = \sqrt{2}[t \ 0 \ t-1 \ 0]^T, \quad (50)$$

$$z_2^* = \sqrt{2}[0 \ 1-t \ 0 \ -t]^T, \quad (51)$$

where

$$t = \frac{2ve^{2r} - 1}{4v \cosh 2r - 2}. \quad (52)$$

The observable corresponding to this is

$$\begin{aligned} \mathcal{R}(z_1^*) &= \sqrt{2}t\mathcal{Q}_1 - \sqrt{2}(1-t)\mathcal{Q}_2, \\ \mathcal{R}(z_2^*) &= \sqrt{2}(1-t)\mathcal{P}_1 - \sqrt{2}t\mathcal{P}_2, \end{aligned} \quad (53)$$

whose physical realization is shown in Fig. 1(a).

For $r \geq r_0$, we have

$$z_1^* = \sqrt{2}[1 \ 0 \ 0 \ 0]^T, \quad (54)$$

$$z_2^* = \sqrt{2}[0 \ 0 \ 0 \ -1]^T, \quad (55)$$

which is a special case of Eq. (53) with $t = 1$. The observables corresponding to these vectors are then

$$\mathcal{R}(z_1^*) = \sqrt{2}\mathcal{Q}_1, \quad (56)$$

$$\mathcal{R}(z_2^*) = -\sqrt{2}\mathcal{P}_2, \quad (57)$$

which is realized by the setup in Fig. 1(b). The two vectors z_1^* and z_2^* provide an unbiased estimator as can be checked by noticing that $m(z_j^*) = \text{tr}[\rho\mathcal{R}(z_j^*)] = \theta_j$.

C. Discussions

Figure 2 shows the SLD and RLD CR bounds from Refs. [14,45], our Holevo CR bound Eq. (48), and the sum of MSE for a double-homodyne joint measurement. The Holevo CR bound is greater than or equal to the RLD and SLD CR bounds. When $r \geq r_0$, the sum of MSE for the double-homodyne joint measurement is equal to the Holevo CR bound. When $r \leq r_0$, the Holevo CR bound is equal to the RLD CR bound. The double-unbalanced-heterodyne joint measurement outperforms the double-homodyne joint measurement in this case, giving a sum of MSE equal to the RLD and Holevo CR bounds. When $r > r_0$, the double-unbalanced-heterodyne joint measurement is impossible, requiring a beam splitter transmission greater than 1 [from Eq. (52)].

Interestingly, we note that r_0 is the threshold beyond which the probe becomes entangled as can be checked using Duan's inseparability criterion [46]. At $r = r_0$, the sum of MSE is exactly 2, which turns out to be the same as one get by doing a heterodyne measurement on a single-mode coherent state probe. This is the best one can do when restricted to single-mode Gaussian probes. Regardless of whether the probe is entangled or not, the optimal measurement scheme requires mixing the two modes on a 50:50 beam splitter, after which we end up with two uncorrelated states. If the probe state was

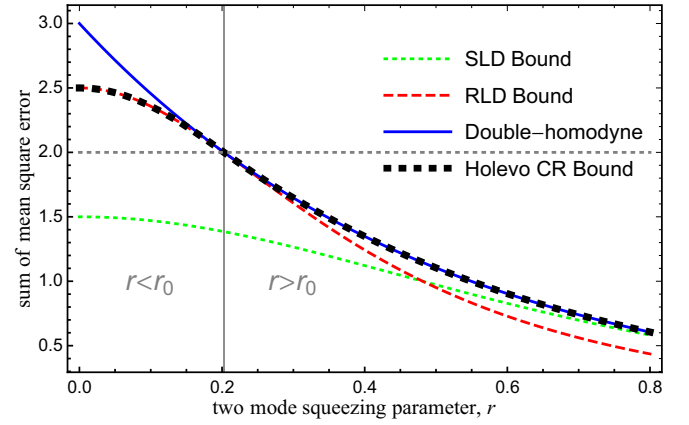


FIG. 2. Plot of symmetric logarithmic derivative (SLD), right logarithmic derivative (RLD), and Holevo Cramér-Rao (CR) bounds, and double-homodyne joint measurement sum of mean-square error (MSE) for $v = 0.75$. The vertical line corresponds to r_0 . For $r \geq r_0$, the Holevo CR bound is equal to the sum of MSE of the double-homodyne joint measurement. For $r \leq r_0$, the Holevo CR bound is equal to the RLD bound and the sum of MSE for the double-unbalanced-heterodyne joint measurement. The horizontal line corresponds to the best sum of MSE when using a single-mode Gaussian probe.

originally entangled, the states after the 50:50 beam splitter will have a quadrature variance below the vacuum noise, while if the original state is separable, all quadrature variances will always be greater than the vacuum noise.

The double-unbalanced-heterodyne measurement can be seen as obtaining two independent estimates for each displacement parameter and then making an optimal estimate from these. As t varies, the precision of one estimate decreases at the expense of a better precision for the second estimate. Suppose the system is entirely classical, and we have a classical state with covariances of \mathcal{P} and \mathcal{Q} the same as the quantum state. Because the system is classical, \mathcal{P} and \mathcal{Q} can be measured simultaneously without an additional noise penalty imposed by quantum mechanics. In this case, the double-unbalanced-heterodyne would outperform the dual-homodyne measurement as we get two independent estimates for θ_1 and two independent estimates for θ_2 . However, for the quantum system, the double-unbalanced-heterodyne measurement incurs a noise penalty due to the vacuum noise coupling through the unused ports of the beam splitters. There is a trade-off between a decreased precision due to the vacuum noise, and an increased precision obtained from the availability of an independent second estimate. When the measurement noise is greater than the vacuum noise, the increase in precision we get from the second estimate outweighs the loss of precision due to the vacuum noise contaminating the first estimate. This is no longer true when the measurement noise is smaller than the vacuum noise.

Even when the probe is separable, the optimal measurement still requires a joint measurement of the two modes. Hence, perhaps counterintuitively, the optimal measurement is not separable despite the probe being separable. Nevertheless, this is consistent with previous work [47], where a joint measurement was found to provide a higher mutual information than a separable measurement. The performance advantage

is attributed to the state having a nonzero quantum discord, despite having no entanglement.

V. CONCLUSION

In conclusion, we provided a method to calculate the Holevo CR bound for the estimation of the mean quadrature parameters of a two-mode Gaussian state, by converting a problem to an SDP. An SDP can be efficiently solved numerically. Additionally, conditions proving optimality of an SDP solution exist, allowing for an analytical solution to be verified. Our method can be easily extended to Gaussian states with any number of modes.

Using this method we were able to find an analytical solution for the Holevo CR bound of the displacement on one mode of a symmetric two-mode squeezed thermal state. A double-homodyne joint measurement is optimal if the state is entangled, and a double-unbalanced-heterodyne joint measurement is optimal if the state is separable.

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APPENDIX A: CONVERSION OF PROBLEM TO SEMIDEFINITE PROGRAM (SDP)

We show that the problem of computing Holevo's bound for mean value estimation of Gaussian states is a semidefinite program. We formulate the original problem of finding Σ_* into a dual form SDP. Holevo's bound is the following:

Program 2. Holevo's bound:

$$\Sigma_* = \min_{\mathbb{F} \in \mathbb{S}^4} \text{Tr} F^{-1} \quad (\text{A1})$$

$$\text{subject to } 0 \leq \mathbb{F} \leq \mathbb{C}, \quad (\text{A2})$$

where \mathbb{S}^n is the set of $n \times n$ real symmetric matrices, $F = \mathbb{M}\mathbb{F}\mathbb{M}^\dagger$, and \mathbb{M} is a fixed real two-by-four matrix. Also $\mathbb{C} := (1 + \frac{1}{2}i\mathbb{D})^{-1}$ is a fixed Hermitian four-by-four matrix.

To cast this nonlinear optimization problem to an SDP, we use the standard trick of introducing an auxiliary two-by-two real matrix H that serves as an upper bound to F^{-1} . So Holevo's bound becomes

Program 3.

$$\Sigma_* = \min_{\mathbb{F} \in \mathbb{S}^4, H \in \mathbb{S}^2} \text{Tr} H \quad (\text{A3})$$

$$\text{subject to } 0 \leq \mathbb{F} \leq \mathbb{C} \quad (\text{A4})$$

$$H \geq F^{-1}. \quad (\text{A5})$$

Consider

$$W(\mathbb{F}, H) = \begin{bmatrix} H & I_2 \\ I_2 & F \end{bmatrix} \geq 0 \quad (\text{A6})$$

$$\Leftrightarrow W/F = H - F^{-1} \geq 0 \quad (\text{A7})$$

$$\Leftrightarrow H \geq F^{-1}, \quad (\text{A8})$$

where W/F is the Schur's complement of F in W , and I_n is the $n \times n$ identity matrix. We can formulate the SDP for Σ_* as

$$\Sigma_* = \min_{\mathbb{F} \in \mathbb{S}^4, H \in \mathbb{S}^2} \text{Tr}\{H\} \quad (\text{A9})$$

subject to

$$\begin{aligned} & \begin{bmatrix} H & I_2 \\ I_2 & \mathbb{M}\mathbb{F}\mathbb{M}^\dagger \end{bmatrix} \oplus \mathbb{F} \oplus -\mathbb{F} \geq 0_4 \oplus 0_4 \oplus -\mathbb{C} \quad (\text{A10}) \\ \Leftrightarrow & \underbrace{\begin{bmatrix} H & 0_2 \\ 0_2 & \mathbb{M}\mathbb{F}\mathbb{M}^\dagger \end{bmatrix} \oplus \mathbb{F} \oplus -\mathbb{F}}_{\sum_j y_j B_j} \geq \underbrace{\begin{bmatrix} 0_2 & -I_2 \\ -I_2 & 0_2 \end{bmatrix}}_{\mathbb{C}} \oplus 0_4 \oplus -\mathbb{C}, \quad (\text{A11}) \end{aligned}$$

where 0_n is the $n \times n$ zero matrix. We can decompose the LHS into a sum $\sum_j y_j B_j$ where $y = [y_1 \dots y_{13}]^\top$ is a vector of real numbers and B_j are the 13 matrices given by

$$B_j = \begin{bmatrix} \mathbb{A}_j & 0_2 \\ 0_2 & \mathbb{M}\mathbb{A}_j\mathbb{M}^\dagger \end{bmatrix} \oplus \mathbb{A}_j \oplus -\mathbb{A}_j \quad \text{for } j = 1, \dots, 13. \quad (\text{A12})$$

$\{\mathbb{A}_j\}$ are 10 real symmetric matrices that form a basis for the set of 4×4 real symmetric matrices. Similarly, $\{\mathbb{B}_j\}$ are three real symmetric matrices that form a basis for the set of 2×2 real symmetric matrices. They are given by the following:

$$\begin{aligned} \mathbb{A}_1 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \mathbb{A}_2 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \mathbb{A}_3 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \mathbb{A}_4 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \mathbb{A}_5 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \mathbb{A}_6 &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \mathbb{A}_7 &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} & \mathbb{A}_8 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \mathbb{A}_9 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} & \mathbb{A}_{10} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{aligned}$$

and $\mathbb{A}_j = 0$ for $j = 11, 12, 13$;

$$\mathbb{B}_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \mathbb{B}_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbb{B}_{13} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \mathbb{B}_j = 0 \quad \text{for } j = 1, \dots, 10. \quad (\text{A13})$$

The objective function can be written as $\text{tr}\{H\} = y^\top b$ where $b = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0]^\top$. Finally, we have the problem statement as the following:

Program 4. Standard SDP dual problem formulation of Holevo's bound:

$$\Sigma_* = \min_y y^\top b \tag{A14}$$

$$\text{subject to } \sum_j y_j B_j \geq C. \tag{A15}$$

This is traditionally called the dual problem.

The primal problem statement is:

Program 5. Standard SDP primal problem formulation of Holevo's bound:

$$\Sigma^* = \max_X \text{Tr} C X \tag{A16}$$

$$\text{subject to } \text{Tr} B_j X = b_j \text{ for } j = 1, \dots, 13, \tag{A17}$$

where X is a positive Hermitian matrix. This problem is bounded above and strictly feasible, which means that it satisfies strong duality: $\Sigma_* = \Sigma^*$.

APPENDIX B: SOLUTION TO THE WORKED EXAMPLE

In this appendix we provide the solution to the worked example. We present X^* and y^* that we claim is optimal. We first verify that X^* and y^* satisfy the primal and dual constraint. Next we show that the primal and dual value they provide are the same, indicating that the solution is optimal.

We consider the solutions for $r \geq r_0$ and $r < r_0$ separately.

1. Solution for $r < r_0$

For $r < r_0$, we claim that a solution is achieved by y^* and X^* having the form

$$y^* = \left[c_1 \ c_2 \ c_2 \ c_1 \ 0 \ -c_0 \ 0 \ 0 \ -c_0 \ 0 \ \frac{4v^2-1}{4v \cosh 2r-2} \ \frac{4v^2-1}{4v \cosh 2r-2} \ 0 \right]^\top, \tag{B1}$$

$$X^* = X_1^* \oplus 0_4 \oplus X_3^*, \tag{B2}$$

where we are free to choose $c_0 : 0 \leq c_0 \leq \frac{2v}{1-4v^2} + \frac{v}{2v \cosh 2r-1}$ and

$$c_1 = \frac{2v(2v - e^{-2r})}{4v^2 - 1} - e^{-2r} c_0, \tag{B3}$$

$$c_2 = \frac{2v(2v - e^{2r})}{4v^2 - 1} - e^{2r} c_0, \tag{B4}$$

$$X_1^* = \begin{bmatrix} 1 & 0 & -\frac{4v^2-1}{4v \cosh 2r-2} & 0 \\ 0 & 1 & 0 & -\frac{4v^2-1}{4v \cosh 2r-2} \\ -\frac{4v^2-1}{4v \cosh 2r-2} & 0 & \frac{(4v^2-1)^2}{(4v \cosh 2r-2)^2} & 0 \\ 0 & -\frac{4v^2-1}{4v \cosh 2r-2} & 0 & \frac{(4v^2-1)^2}{(4v \cosh 2r-2)^2} \end{bmatrix}, \tag{B5}$$

$$X_3^* = \frac{(4v^2 - 1)^2}{2v(4v \cosh 2r - 2)^2} \begin{bmatrix} e^{2r} & i & -1 & -i e^{2r} \\ -i & e^{-2r} & i e^{-2r} & -1 \\ -1 & -i e^{-2r} & e^{-2r} & i \\ i e^{2r} & -1 & -i & e^{2r} \end{bmatrix}. \tag{B6}$$

Simple algebra confirms that X^* satisfies $\text{Tr} B_j X^* = b_j$, and the nonzero eigenvalues of X^* are

$$\left(\frac{(4v^2 - 1)^2 \cosh 2r}{2v(2v \cosh 2r - 1)^2}, 1 + \frac{(4v^2 - 1)^2}{4(2v \cosh 2r - 1)^2} (\text{deg } 2) \right), \tag{B7}$$

where (deg 2) indicates that the eigenvalue has degeneracy 2. The eigenvalues are nonnegative, so X^* is a valid solution to the primal problem.

Now let us verify that y^* satisfies the dual problem constraint (A15):

$$\sum_j y_j^* B_j = \begin{bmatrix} \frac{4v^2-1}{4v \cosh 2r-2} & 0 & 0 & 0 \\ 0 & \frac{4v^2-1}{4v \cosh 2r-2} & 0 & 0 \\ 0 & 0 & \frac{4v \cosh 2r-2}{4v^2-1} & 0 \\ 0 & 0 & 0 & \frac{4v \cosh 2r-2}{4v^2-1} \end{bmatrix} \oplus \begin{bmatrix} c_1 & 0 & -c_0 & 0 \\ 0 & c_2 & 0 & -c_0 \\ -c_0 & 0 & c_2 & 0 \\ 0 & -c_0 & 0 & c_1 \end{bmatrix} \oplus \begin{bmatrix} -c_1 & 0 & c_0 & 0 \\ 0 & -c_2 & 0 & c_0 \\ c_0 & 0 & -c_2 & 0 \\ 0 & c_0 & 0 & -c_1 \end{bmatrix}, \tag{B8}$$

where

$$\mathbb{C} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \oplus 0_4 \oplus \frac{2v}{4v^2 - 1} \begin{bmatrix} -2v & i & 0 & 0 \\ -i & -2v & 0 & 0 \\ 0 & 0 & -2v & i \\ 0 & 0 & -i & -2v \end{bmatrix}. \tag{B9}$$

The nonzero eigenvalues of $\sum_j y_j^* \mathbb{B}_j - \mathbb{C}$ are then

$$\left[\begin{array}{c} \frac{4v^2-1}{4v \cosh 2r-2} + \frac{4v \cosh 2r-2}{4v^2-1} \text{ (deg 2)} \\ 2(c_0 + \frac{2v}{4v^2-1}) \cosh 2r \\ \frac{\cosh 2r}{4v^2-1} \left\{ 2v + (4v^2 - 1)c_0 + \sqrt{[2v + (4v^2 - 1)c_0]^2 - \frac{8v(4v^2-1)c_0}{\cosh^2 2r}} \right\} \\ \frac{\cosh 2r}{4v^2-1} \left\{ 2v + (4v^2 - 1)c_0 - \sqrt{[2v + (4v^2 - 1)c_0]^2 - \frac{8v(4v^2-1)c_0}{\cosh^2 2r}} \right\} \\ \frac{1}{4v^2-1} \{ 4v^2 - [2v + (4v^2 - 1)c_0] \cosh 2r + \sqrt{[2v + (4v^2 - 1)c_0]^2 \cosh^2 2r - 4v^2 - 4v(4v^2 - 1)c_0} \} \text{ (deg 2)} \\ \frac{1}{4v^2-1} \{ 4v^2 - [2v + (4v^2 - 1)c_0] \cosh 2r - \sqrt{[2v + (4v^2 - 1)c_0]^2 \cosh^2 2r - 4v^2 - 4v(4v^2 - 1)c_0} \} \text{ (deg 2)} \end{array} \right]. \tag{B10}$$

The first five eigenvalues are positive when $v \geq \frac{1}{2}$ and $c_0 \geq 0$, while the last is positive when $c_0 \leq \frac{2v}{1-4v^2} + \frac{v}{2v \cosh 2r-1}$.

The value of the dual is $y^\top b = \frac{4v^2-1}{2v \cosh 2r-1}$. It can be verified using simple algebra that the primal value $\text{Tr}CX^*$ is also equal to $\frac{4v^2-1}{2v \cosh 2r-1}$. Since the primal is equal to the dual, we know that the solution is optimal.

One might wonder why the optimal measurement does not depend on c_0 . Any c_0 would give rise to an \mathbb{F}_* that is optimal,

$$\mathbb{F}_* = \begin{bmatrix} c_1 & 0 & -c_0 & 0 \\ 0 & c_2 & 0 & -c_0 \\ -c_0 & 0 & c_2 & 0 \\ 0 & -c_0 & 0 & c_1 \end{bmatrix}, \tag{B11}$$

and hence different \mathbb{F}_* ; however, the vectors $\mathcal{F}_* m_j$ does not depend on c_0 . By direct computation

$$\mathcal{F}_* m_1 = \frac{\sqrt{2}}{4v^2 - 1} \begin{bmatrix} 2ve^{2r} - 1 \\ 0 \\ 1 - 2ve^{-2r} \\ 0 \end{bmatrix}, \tag{B12}$$

$$\mathcal{F}_* m_2 = \frac{\sqrt{2}}{4v^2 - 1} \begin{bmatrix} 2ve^{-2r} - 1 \\ 0 \\ 1 - 2ve^{2r} \\ 0 \end{bmatrix} \tag{B13}$$

is independent of c_0 .

2. Solution for $r \geq r_0$

When $r \geq r_0$, we claim that the optimal values of X and y that attains h_* and g_* in the SDP program (4) and (5) are given by

$$y^* = [1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 2ve^{-2r} \ 2ve^{-2r} \ 0]^\top, \tag{B14}$$

$$X^* = X_1^* \oplus X_2^* \oplus X_3^*, \tag{B15}$$

where

$$X_1^* = \begin{bmatrix} 1 & 0 & -2ve^{-2r} & 0 \\ 0 & 1 & 0 & -2ve^{-2r} \\ -2ve^{-2r} & 0 & 4v^2e^{-4r} & 0 \\ 0 & -2ve^{-2r} & 0 & 4v^2e^{-4r} \end{bmatrix}, \quad (\text{B16})$$

$$X_2^* = \frac{e^{-2r}(1 - 4v^2e^{-4r})}{2v} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (\text{B17})$$

$$X_3^* = e^{-2r} \begin{bmatrix} 2v & i & -2ve^{-2r} & -4v^2e^{-2r}i \\ -i & \frac{1}{2v} & e^{-2r}i & -2ve^{-2r} \\ -2ve^{-2r} & -e^{-2r}i & \frac{1}{2v} & i \\ 4v^2e^{-2r}i & -2ve^{-2r} & -i & 2v \end{bmatrix}. \quad (\text{B18})$$

To justify this claim, we need to show that X^* and y^* satisfies constraints (A15) and (A17) and that the value of the dual solution is equal to the primal solution, $\Sigma_* = \Sigma^*$.

To check the constraint for the dual (A15), we compute the eigenvalues of $\sum_j y_j^* B_j - \mathbb{C}$ where

$$\sum_j y_j^* B_j = \begin{bmatrix} 2ve^{-2r} & 0 & 0 & 0 \\ 0 & 2ve^{-2r} & 0 & 0 \\ 0 & 0 & \frac{e^{2r}}{2v} & 0 \\ 0 & 0 & 0 & \frac{e^{2r}}{2v} \end{bmatrix} \oplus \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad (\text{B19})$$

and \mathbb{C} is given in (B9). The nonzero eigenvalues of $\sum_j y_j^* B_j - C$ are

$$\left(1, \frac{1 + 4v^2e^{-4r}}{2ve^{-2r}}, \frac{4v^2 + 1}{4v^2 - 1} \right), \quad (\text{B20})$$

each occurring with degeneracy two. Since $v > \frac{1}{2}$, so all of the eigenvalues are nonnegative. Hence y^* is a valid solution.

Simple algebra confirms that the primal constraint $\text{Tr}\{B_j X^*\} = b_j$ is also satisfied. The nonzero eigenvalues of X^* are

$$\left(1 + 4v^2e^{-4r} (\text{deg } 2), \frac{(1 - 4v^2e^{-4r})e^{-2r}}{2v} (\text{deg } 2), \frac{(1 + 4v^2)(1 - 2ve^{-2r})e^{-2r}}{2v}, \frac{(1 + 4v^2)(1 + 2ve^{-2r})e^{-2r}}{2v} \right). \quad (\text{B21})$$

All of these eigenvalues are nonnegative provided $e^{2r} - 2v \geq 0$, which is just the condition for $r \geq r_0$. Therefore X^* is positive definite when $r \geq r_0$, and the constraints for the primal problem are satisfied. Therefore we have shown that y^* and X^* specified above are a valid solution.

Next, by direct computation, $y_*^\top b = 4ve^{-2r}$ and also $\text{Tr}\{CX^*\} = 4ve^{-2r}$. Since the primal is equal to the dual, the solution is optimal.

APPENDIX C: GENERALIZATION TO n -MODE STATES

To generalize the results in Sec. III to an n -mode Gaussian state, we extend the definition of z to $z = [y_1 \ x_1 \ y_2 \ x_2 \ \dots \ y_n \ x_n]^\top$ in a $2n$ -dimensional real vector space Z and the canonical observables

$$\mathcal{R}(z) = \sum_j x_j \mathcal{P}_j + y_j \mathcal{Q}_j, \quad (\text{C1})$$

where \mathcal{P}_j and \mathcal{Q}_j are the quadrature operators for the j th mode. The skew-symmetric bilinear form generalizes to

$$\Delta(z, z') = \sum_j x'_j y_j - x_j y'_j \quad (\text{C2})$$

such that the commutation relation Eq. (16) still holds. Equation (20) defining the mean value function and Eq. (21) defining the correlation function of the Gaussian state remain unchanged. To estimate l displacement parameters θ_j for $j = 1, \dots, l$, we introduce $m_j(z)$ for $j = 1, \dots, l$ such that

$$m(z) = \sum_j \theta_j m_j(z). \quad (\text{C3})$$

The results of Sec. IIIA then follow with only minor modification to the size of the matrix F , which is now l -by- l . The definitions and results of Secs. IIIB, IIIC, and Appendix A are still valid after appropriately extending the matrix and vector dimensions.

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