Nonlinear quantum Langevin equations for bosonic modes in solid-state systems

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Based on the experimental evidence that impurities contribute to the dissipation properties of solid-state open quantum systems, we provide here a description in terms of nonlinear quantum Langevin equations of the role played by two-level systems in the dynamics of a bosonic degree of freedom. Our starting point is represented by the description of the system-environment coupling in terms of coupling to two separate reservoirs, modeling the interaction with external bosonic modes and two-level systems, respectively. Furthermore, we show how this model represents a specific example of a class of open quantum systems that can be described by nonlinear quantum Langevin equations. Our analysis offers a potential explanation of the parametric effects recently observed in circuit-QED cavity optomechanics experiments.

I. INTRODUCTION

The dynamics of open quantum systems, i.e., quantum systems that can be described as separate entities from their surrounding environment while being somehow coupled to it, is arguably one of the most fundamental problems in quantum mechanics, encompassing concepts such as the *measurement paradox* [1] and the boundary between quantum and classical physics [2]. On general grounds, the interaction between a quantum system and its environment represents an important aspect of the physics of condensed matter and complex systems, which has been the focus of extensive analysis [3–5], with repercussions in contexts ranging from the energy transport in photosynthetic complexes [6] to the physics of ultracold gases [7–9].

In the description of these systems, the inclusion of the role played by coupling to an external environment is necessary, if only because the system has to be coupled to an external measurement apparatus which, from the quantum-dynamical perspective of the system, represents a source of noise and dissipation. At the same time the manipulation of open quantum systems has recently led to the possibility of preparing and detecting quantum states of matter and radiation [10,11], paving the way for the definition of a new paradigm of *quantum technology*, which represents an important field for applications ranging from secure (quantum) communication [12] to sensing of electromagnetic fields [13] and to the detection of gravitational waves [14]. This prospect of technological application of quantum mechanics is rooted in the relatively recent development of fabrication techniques at the nanoscale, in particular, nanomechanical resonators, superconducting qubits, and, more in general, circuit quantum electrodynamics (QED) setups [15–18], where the characteristic scales involved in the dynamics of these systems naturally lead to the study of the quantum properties in the presence of coupling to an environment.

Within this framework, it has recently been observed that this coupling can represent an important resource leading to the notion of *reservoir engineering* [19]. This concept corresponds to the idea that, by manipulating the properties of the environment coupled to a given quantum system or even the nature of the system-environment coupling itself, it is possible to generate specific (quantum) states for the system. Prominent examples are represented by the recent achievements in the field of cavity optomechanics, where ground-state cooling [20] and squeezing below the standard quantum limit (SQL) [21–23], along with nearly quantum limited amplification [24,25] and nonreciprocal photon transmission [26], have been achieved by introducing a specific (Gaussian) state for the reservoir. While these examples correspond to inducing a specific state for the system by manipulating the state of the reservoir, in Refs. [27] and [28] it is shown that by designing a specific nonlinear coupling between system and environment, it is possible to protect certain quantum states (cat states) against decoherence.

If the coupling between the system and the environment is described by a linear Hamiltonian, the effects of noise and dissipation on the dynamics of the system can be described in terms of linear quantum Langevin equations (QLEs) [11]. These equations represent an extension to the quantum regime of the classical Langevin equations and, in analogy to their classical counterpart, include in the description of the dynamics of the system the role played by the environment, including dissipative and noise effects. However, the case of a linear system-environment coupling is not the most general situation that can arise. For instance, for nanomechanical resonators [29–33] and for circuit-QED setups [34–39], the experimental evidence of nonlinear phenomena related to the coupling between system and environment has emerged and, more importantly for our analysis, the relevance of impurities in this phenomenon has been discussed. For both setups, it has been shown (see, e.g., [29,34]) that the impurities naturally arising in the material composing the devices, its supports, and/or substrate represent a source of dissipation. These defects can be modeled in terms of two-level systems (TLSs).

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The reason behind the possibility of modeling impurities in these terms is represented by the fact that each impurity can be construed as quantum systems which exhibit two local energy minima. For instance, as a charged impurity that can hop between two defects in the crystal structure, or a dangling bond with two possible configurations.

More specifically, these TLSs exist primarily due to the disordered potential landscape of amorphous materials, e.g., in surface oxides of thin-film circuit electrodes [38], in the tunnel barrier of Josephson junctions [34], and at disordered interfaces [40,41], coupling with the bosonic degrees of freedom of the system, either through a purely electromagnetic interaction (optical and circuit-QED setups) or a phononic one in the context of nanomechanical systems [42].

II. MODEL

In this article we show under what conditions, considering a nonlinear coupling between system and a bath of TLSs, it is possible to derive a nonlinear QLE for the dynamics of the degrees of freedom of the system, having in mind a circuit-QED setup. In addition, we show how the nonlinear QLEs derived here can represent an explanation to some of the phenomena recently observed in the context of microwave quantum optomechanics [22].

The starting point for our analysis is represented by a bosonic system ($\mathcal{S}$) coupled to an environment ($\mathcal{E}$). The total Hamiltonian of the bipartite system ($\mathcal{S} + \mathcal{E}$) is given by

$$H = H_\mathcal{S} + H_\mathcal{E} + H_{\mathcal{S}-\mathcal{E}}, \tag{1}$$

where $H_\mathcal{S} = H_\mathcal{S}(c, c^\dagger)$ is the Hamiltonian of the isolated system, exhibiting a generic dependence on the annihilation (creation) operators c ($c^\dagger$) associated with the system, and $H_\mathcal{E}$ is the Hamiltonian for the bath.

We assume here that the environment Hamiltonian can be decomposed into two terms, $H_\mathcal{E}^\mathrm{B} = \sum_k \omega_k b_k^\dagger b_k$ and $H_\mathcal{E}^\mathrm{TLS}$, corresponding to a bath of free bosonic modes and to a bath of TLSs, respectively (see Fig. 1). The bosonic bath describes, for instance, the modes of the electromagnetic field of the environment. In our analysis we assume that these
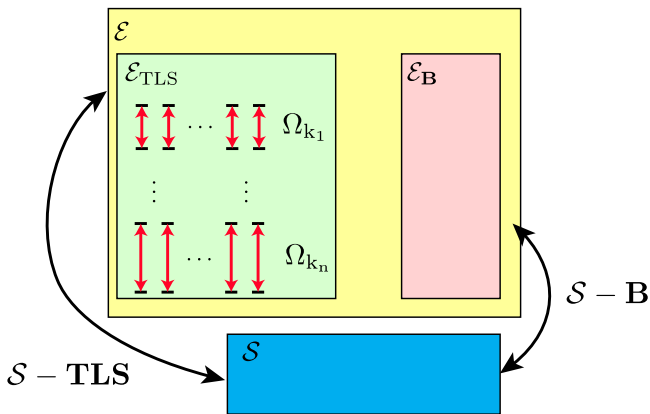

FIG. 1. Cartoon picture of the setup. The system $\mathcal{S}$ is coupled to an environment $\mathcal{E}$, which is constituted by a bosonic bath $\mathcal{E}_\mathrm{B}$ and a bath of TLSs $\mathcal{E}_\mathrm{TLS}$. The coupling between the two baths and the system is mediated by the Hamiltonians $H_{\mathcal{S}-\mathrm{B}}$ and $H_{\mathcal{S}-\mathrm{TLS}}$, respectively.

modes, while being associated with the noise properties and dissipation of the system, encompass also the external coherent fields driving the system whose properties are encoded in the state of the bath for the modes b_k (see, e.g., [11]). Our choice is equivalent to considering a coherent driving term for the system Hamiltonian and a purely thermal bath.

In this scenario, we describe the coupling between these modes and the degrees of freedom of the system by the following Hamiltonian:

$$H_{\mathcal{S}-\mathrm{B}} = \sum_k g_k^\mathrm{B}(c^\dagger b_k + c b_k^\dagger). \tag{2}$$

In addition, we model the bath of TLSs as a collection of spins $\mathbf{J}_k$. In this scenario we have that $H_\mathcal{E}^\mathrm{TLS} = \sum_k \Omega_k J_z^k$. This choice for the modeling of TLSs corresponds to the idea that, for each Ω_k multiple TLSs are present that collectively couple with the system $\mathcal{S}$. While for $\Omega_k \simeq \omega_\mathcal{S}$, where $\omega_\mathcal{S}$ corresponds to a characteristic frequency for the system, the presence of impurities leads to a renormalization of the linewidth associated with the linear response of the system induced by the coupling given in Eq. (2) (see Appendix D); for $\Omega_k \simeq n \omega_\mathcal{S}$, nonlinear contributions appear. In our analysis, also in light of the recent investigations concerning the relevance of two-photon emission processes by TLSs [43,44] when coupled to bosonic modes, we consider the case $n = 2$, representing the lowest-order approximation beyond linear coupling. This assumption appears to be compatible with the usual experimental conditions encountered in the context of circuit QED, where microwave cavities operate at frequencies corresponding to few GHz [15,16,45] while the energy separation of a TLS relevant for the physics of either of these systems is of the order of 10 GHz [45,46]. In this case, it is possible to write the system-TLS coupling Hamiltonian as

$$H_{\mathcal{S}-\mathrm{TLS}} = \sum_k g_k^\mathrm{TLS}(J_+^k c^2 + J_-^k c^{\dagger 2}). \tag{3}$$

If we assume that $|\mathbf{J}_k| \gg 1$, corresponding to the idea that for each value of k multiple TLSs couple to the system $\mathcal{S}$, by resorting to the Holstein-Primakoff (HP) realization of spin operators in terms of bosonic modes, we can replace the spin operators with bosonic ones. This mapping can be performed in two different ways, corresponding to complementary experimental conditions (see Appendix A). If it is assumed that the TLSs mostly reside in their ground state, we have that $J_3^k \simeq -j_k$, where j_k is the index of the representation associated with the spin $\mathbf{J}_k$ and the HP mapping reads $J_3^k \to d_k^\dagger d_k - j_k$, $J_-^k \to d_k$, $J_+^k \to d_k^\dagger$. In this case, the coupling between the system and the TLS bath can be approximated by

$$H_{\mathcal{S}-\mathrm{HP}_-} = \sum_k g_k^\mathrm{HP}(d_k^\dagger c^2 + d_k c^{\dagger 2}), \tag{4}$$

with $g_k^\mathrm{HP} = \sqrt{2 j_k} g_k^\mathrm{TLS}$. On the other hand, if the TLSs mainly reside in their excited state ($J_3^k \simeq +j_k$) the mapping can be written as $J_3^k \to j_k - d_k^\dagger d_k$, $J_-^k \to d_k^\dagger$, $J_+^k \to d_k$, leading to the following approximation for $H_{\mathcal{S}-\mathrm{TLS}}$:

$$H_{\mathcal{S}-\mathrm{HP}_+} = \sum_k g_k^\mathrm{TLS}(d_k c^2 + d_k^\dagger c^{\dagger 2}). \tag{5}$$

These two different forms of the HP mapping correspond to two different physical situations. In the former case, the TLSs prevalently reside in their ground state, corresponding to the idea that the impurities mainly reside in their ground state, implying a low-temperature regime. In this case, the bosonic excitations described by the operators d_k represent (weak) excitations around the ground state. On the other hand, the latter case corresponds to the situation in which the highest excited (metastable) state of the TLSs is weakly (de-)excited, corresponding, for instance, to the case in which an external drive induces excitations in the TLSs bath, leading to a possible interpretation of the linewidth narrowing observed in circuit-QED setups under strong driving conditions [35] in terms of nonlinear QLEs associated with the saturation of the TLSs. In this picture, the external drive effectively heats the impurities to their excited state, inducing the population inversion for the ensemble of TLSs and a consequent saturation, justifying the HP_+ transformation in terms of (weak) deexcitations of the highest excited state.

As we show in Appendix B, it is possible to derive QLEs for the system, provided that the environment Hamiltonian is described by a set of bosonic operators coupled linearly to the system degrees of freedom. It is important to note that the requirement of linearity concerning the system-environment Hamiltonian is limited to the bath degrees of freedom, meaning that its most general form can be expressed as

$$H_{S-\mathcal{E}} = \sum_k g_k [F^\dagger(c,c^\dagger)e_k + F(c,c^\dagger)e_k^\dagger], \qquad (6)$$

where e_k and $e_k^\dagger$ represent generic bosonic operators associated with the environment degrees of freedom. The form the system-environment coupling represents a sufficient condition for the derivation of a nonlinear QLE, along with the assumption that the modes of the bath are noninteracting. In other terms, it is necessary to assume a linear dependence of the coupling Hamiltonian on the environmental degrees of freedom, since in order to derive the QLEs for the system, the solution of the Heisenberg equation of motion for the environment degrees of freedom has to assume a specific form in which the contribution of the system and the environment operators can be represented as two separate additive terms (see Appendix B).

III. EQUATIONS OF MOTION

It is clear that since the form of H_{S-B} and $H_{S-HP_\pm}$ can be expressed in the form given by Eq. (6), with $F(c,c^\dagger)$ given by c, c^2, and $c^{\dagger 2}$, and with $e_k = b_k$ and $e_k = d_k$ for $S - B$, $S - HP_-$, and $S - HP_+$, respectively, we can write the dynamics of the system in terms of a (nonlinear) QLE as

$$\dot{c} = -i[c, H_S] - \left(\frac{\kappa}{2} + \kappa_N c^\dagger c\right)c + \sqrt{\kappa}\, c_{in} + 2\sqrt{\kappa_N}\, c^\dagger c_{in}^{TLS},$$

$$\qquad (7a)$$

$$\dot{c} = -i[c, H_S] - \left(\frac{\kappa}{2} - \kappa_N c^\dagger c\right)c + \sqrt{\kappa}\, c_{in} + 2\sqrt{\kappa_N}\, c^\dagger c_{in}^{TLS\dagger}.$$

$$\qquad (7b)$$

Equations (7a) and (7b), obtained considering the system-environment coupling given by H_{S-HP_-} and H_{S-HP_+}, respectively, are the main result of our analysis. The presence of a TLS bath leads to the appearance of nonlinear dissipative terms $(\pm\kappa_N c^\dagger c\, c)$ and to purely imaginary parametric noise terms $(2\sqrt{\kappa_N}\, c^\dagger c_{in}^{TLS(\dagger)})$. We stress here that these terms are the direct result of the modeling of the bath in terms of two separate environments (H_{S-B} and $H_S - HP_\pm$) and do not represent an *ad hoc* modification of the linear QLEs that can be derived in the absence of coupling to TLSs. In particular, while the nonlinear dissipation term possibly represents a natural extension to the nonlinear regime of linear QLEs, the parametric noise term is a nontrivial contribution associated with the presence of the TLS bath.

In addition, we observe here that, analogously to their linear counterpart, Eqs. (7a) and (7b) are time local, i.e., the dynamics is Markovian. As detailed in Appendix B, this property is related to the assumption that within the range of frequencies of interest, the coupling strength between system and environment is independent of the mode considered (wide-band-limit approximation) [47].

If we further consider a pump probe representative of a circuit-QED setup (e.g., a circuit optomechanical experiment), we can assume that the dynamics given by Eq. (7) is linearized around a strong coherent tone:

$$\alpha_p = \alpha_{in} \exp[-i\omega_p t].$$

The frequency ω_p is detuned by $\Delta = \omega_p - \omega_c$ from the cavity resonant frequency. As a result of the linearization scheme, we have that the amplitude of the cavity field oscillating at ω_p is given by the solution of a nonlinear algebraic equation. In Fig. 2 we have plotted the stationary value of the cavity field for the two choices of the HP mapping ($HP_\pm$). As expected, for small values of the driving field α_{in}, the stationary solution corresponds to the solution in the absence of nonlinear dissipation. However, for larger values of α_{in} the stationary solution substantially deviates from the solution of the linear system, with, for the parameters discussed here, a negligible difference between $HP_\pm$ cases.

Furthermore, the (first-order) dynamics of the fluctuations $c = \alpha + a$ around the stationary value induced by the pump (in a frame rotating at ω_p) is given by

$$\dot{a} = \left[i\Delta - \left(\frac{\kappa}{2} + 2\kappa_N|\alpha|^2\right)\right]a - \kappa_N\alpha^2 a^\dagger$$
$$+ \sqrt{\kappa}\, a_{in} + 2\sqrt{\kappa_N}\,\alpha^* a_{in}^{TLS}, \qquad (8a)$$

$$\dot{a} = \left[i\Delta - \left(\frac{\kappa}{2} - 2\kappa_N|\alpha|^2\right)\right]a + \kappa_N\alpha^2 a^\dagger$$
$$+ \sqrt{\kappa}\, a_{in} + 2\sqrt{\kappa_N}\,\alpha^* a_{in}^{TLS\dagger}, \qquad (8b)$$

the HP_- and HP_+ case, respectively (see Appendix C). It is possible to see that Eqs. (8a) and (8b) include a purely imaginary parametric term on top of a nonlinear dissipation term, implying linewidth broadening or narrowing, depending on the state of the TLSs bath. Recently, in Ref. [22] a term of the same form was introduced as an *ad hoc* parameter in order to match the experimental results of a cavity optomechanical
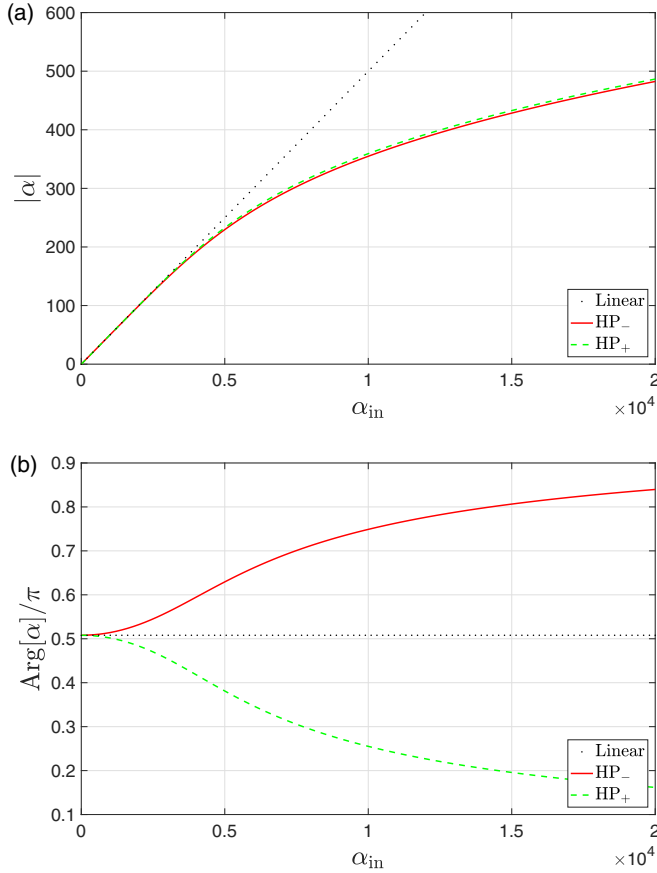
FIG. 2. Amplitude (a) and phase (b) for the stationary value (in a frame rotating at ω_p, see text) of the cavity field α in the presence of a driving α_in. Parameters: $\kappa_\mathrm{N} = 1.5 \times 10^{-4}$, $\Delta = 20$ (all quantities expressed in units of κ).

FIG. 3. Noise spectrum for the cavity field in the presence of an external drive $\alpha_\mathrm{in} = 700$, for (a) HP$_-$ and (b) HP$_+$ for $\langle a_\mathrm{in}^\dagger a_\mathrm{in} \rangle = \langle a_\mathrm{in}^{\mathrm{TLS}\dagger} a_\mathrm{in}^{\mathrm{TLS}} \rangle = 1$ (all other parameters as in Fig. 2).

experiment aimed at establishing squeezing below the SQL of a nanomechanical resonator.

Our description, therefore provides a potential explanation of such parametric effects in terms of nonlinear dissipation phenomena associated with the nonlinear coupling to a bath of TLSs. In order to characterize the effect induced by the presence of the nonlinear coupling to TLSs, we evaluate the fluctuation spectrum of the cavity field $S_\omega^\theta = 1/2 \langle \{X_\omega^\theta, X_{-\omega}^\theta\} \rangle$, with $X_\omega^\theta = 1/\sqrt{2}(a_{-\omega}^\dagger e^{i\theta} + a_\omega e^{-i\theta})$, assuming thermal fluctuations both for the bosonic and the TLS bath. As hinted by the structure of Eqs. (8a) and (8b), the presence of a parametric term induces squeezing, which can be experimentally observed by homodyne detection of the output field, in the cavity spectrum for both cases, as seen in Fig. 3, where the cavity fluctuation spectrum exhibits a clear dependence on the phase θ.

IV. CONCLUSIONS

We have reported here how it is possible to deduce nonlinear QLEs for the dynamics of an open quantum system from a nonlinear system-environment coupling Hamiltonian. Moreover, we have discussed how an effective nonlinear system-environment coupling can emerge in the presence of impurities modeled as TLSs. Ultimately, we have shown that the TLS-induced nonlinearities can represent a potential
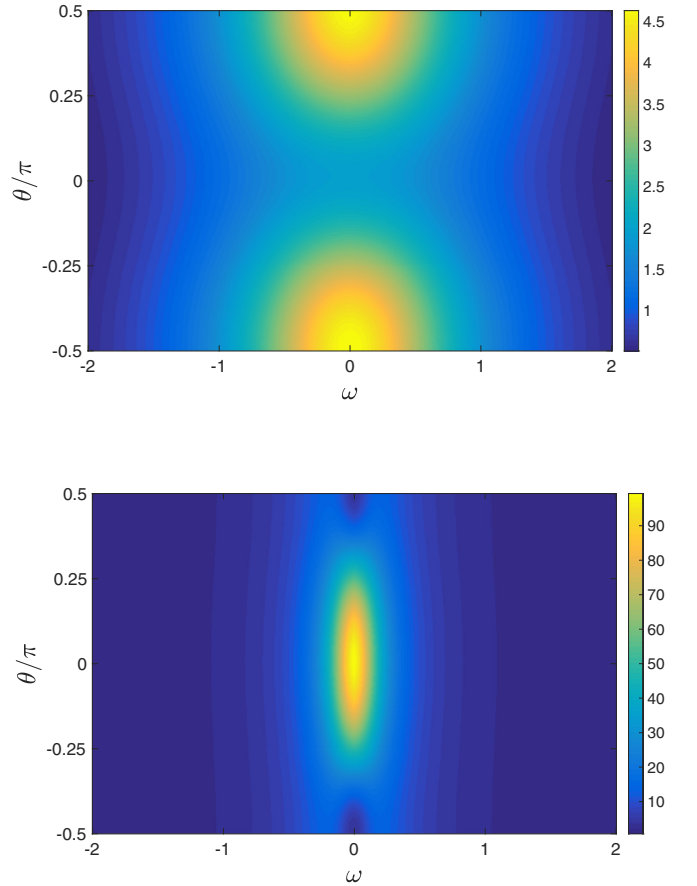
explanation for the imaginary parametric terms reported in Ref. [22].

APPENDIX A: HOLSTEIN-PRIMAKOFF TRANSFORMATION

We discuss here the Holstein-Primakoff realization allowing us to replace the spin operators J_z, $J_\pm$ obeying the usual SU(2) commutation relations

$$\left[J_z^\mathrm{k}, J_\pm^\mathrm{k} \right] = \pm J_\pm^\mathrm{k}, \quad [J_+^\mathrm{k}, J_-^\mathrm{k}] = 2 J_z^\mathrm{k}, \tag{A1}$$

with bosonic operators d_k, $d_\mathrm{k}^\dagger$, for which

$$[d_\mathrm{k}, d_\mathrm{k}^\dagger] = 1. \tag{A2}$$

As discussed in the main text, in order to map the spin operators obeying Eq. (A1) with the bosonic operators d_k, $d_\mathrm{k}^\dagger$, we have two possibilities, depending on the physical situation we want to describe. If we assume that $J_z^\mathrm{k} \simeq -j_\mathrm{k}$, this choice is indicated in the main text as HP$_-$, we can consider the

following transformation:

$$J_z^k = \hat{n}_k - j_k, \quad J_+^k = d_k^\dagger \sqrt{2j_k - \hat{n}_k}, \quad J_-^k = \sqrt{2j_k - \hat{n}_k}\, d_k,$$
(A3)

where $\hat{n}_k = d_k^\dagger d_k$. The operators J_z^k, $J_\pm^k$ can be shown to fulfill the SU(2) commutation relations

$$\left[J_z^k, J_+^k\right] = [\hat{n}_k, d_k^\dagger]\sqrt{2j_k - \hat{n}_k} = J_+^k,$$

$$\left[J_z^k, J_-^k\right] = \sqrt{2j_k - \hat{n}_k}[\hat{n}_k, d_k] = -J_-^k,$$
(A4a)

$$\left[J_+^k, J_-^k\right] = d_k^\dagger(\sqrt{2j_k - \hat{n}_k})^2 d_k - \sqrt{2j_k - \hat{n}_k}\,\hat{n}_k\,\sqrt{2j_k - \hat{n}_k}$$

$$= \hat{n}_k(2j_k - \hat{n}_k + 1) - 2j_k + \hat{n}_k - \hat{n}_k(2j_k - \hat{n}_k) = 2J_z^k.$$
(A4b)

In the limit $j_k \to \infty$, we have that

$$\frac{J_+^k}{\sqrt{2j_k}} = d_k^\dagger\sqrt{\frac{2j_k - \hat{n}_k}{2j_k}} = d_k^\dagger\left(1 - \frac{\hat{n}_k}{4j_k} + \cdots\right) \simeq d_k^\dagger,$$

$$\frac{J_-^k}{\sqrt{2j_k}} \simeq d_k, \quad \frac{J_z^k}{j_k} = \frac{\hat{n}_k}{j_k} - 1 \simeq -1.$$
(A5)

Therefore the bosonic excitations described by d_k and $d_k^\dagger$ correspond to (small) excitations around the $J_z^k = -j_k$ state. Conversely, we can write

$$J_z^k = j_k - \hat{n}_k, \quad J_-^k = d_k^\dagger\sqrt{2j_k - \hat{n}_k}, \quad J_+^k = \sqrt{2j_k - \hat{n}_k}\, d_k,$$
(A6)

so that when $j_k \to \infty$,

$$\frac{J_+^k}{\sqrt{2j_k}} \simeq d_k, \quad \frac{J_-^k}{\sqrt{2j_k}} \simeq d_k^\dagger, \quad \frac{J_z^k}{j_k} = 1 - \frac{\hat{n}_k}{j_k} \simeq 1,$$
(A7)

which correspond to the description of small fluctuations around the $J_z^k = j$ state, indicated as HP$_+$ in main text.

APPENDIX B: QLE FOR $F(c,c^\dagger)$

We discuss here the form of the QLEs generated by a model for which, following the notation introduced in Eq. (1) of the main text, H_S is left unspecified. The environment is given by a set of noninteracting bosonic modes described by $H_\mathcal{E} = \sum_k \omega_k e_k^\dagger e_k$, where e_k ($e_k^\dagger$) are the annihilation (creation) operators associated with mode k and the system-environment coupling is given by the following Hamiltonian:

$$H_{S-\mathcal{E}} = \sum_k g_k[F(c,c^\dagger)e_k^\dagger + F^\dagger(c,c^\dagger)e_k],$$
(B1)

where $F(c,c^\dagger)$ is a generic function of the creation and annihilation operators of the system. Since $H_{S-\mathcal{E}}$ is a linear operator with respect to the degrees of freedom of the bath and $e_k^{(\dagger)}$ commutes with H_S, we can follow the same strategy employed for the derivation of the linear QLEs [11] and write the equations of motion (EOM) for the bath field operators in the Heisenberg picture as

$$\dot{e}_k(t) = -i\omega_k e_k(t) - ig_k F(c,c^\dagger).$$
(B2)

Similarly, the EOM for the system can be written as

$$\dot{c}(t) = i[H_S, c(t)] + i\sum_k g_k([F,c]e_k^\dagger + [F^\dagger,c]e_k).$$
(B3)

Equation (B2) can be solved in terms of an initial condition t_0, yielding

$$e_k(t) = e^{-i\omega_k(t-t_0)}e_k(t_0) - ig_k\int_{t_0}^t e^{-i\omega_k(t-t')}F[c(t'),c^\dagger(t')]\,dt'.$$
(B4)

By substituting Eq. (B4) and its Hermitian conjugate into Eq. (B3) we obtain

$$\dot{c}(t) = i[H_S, c(t)] + i\sum_k g_k\left\{[F,c]\left[e^{i\omega_k(t-t_0)}e_k^\dagger(t_0) + ig_k\int_{t_0}^t e^{i\omega_k(t-t')}F^\dagger(t')\,dt'\right]\right.$$

$$\left. + [F^\dagger,c]\left[e^{-i\omega_k(t-t_0)}e_k(t_0) - ig_k\int_{t_0}^t e^{-i\omega_k(t-t')}F(t')\,dt'\right]\right\}.$$
(B5)

Like for the purely linear case, we introduce the density of states $D = \partial k/\partial\omega_k$ (supposing a continuum of states for the bath) and assume that, in the relevant frequency regime, g_k does not depend on the mode index k. If we define

$$g_k = \sqrt{\frac{\kappa}{2\pi D}},$$
(B6)

where κ is the mode-independent constant, we can write Eq. (B5) as

$$\dot{c}(t) = i[H_S, c(t)] + i\sum_k\sqrt{\frac{\kappa}{2\pi D}}\left\{[F,c]\left(e^{i\omega_k(t-t_0)}e_k^\dagger(t_0) + i\sqrt{\frac{\kappa}{2\pi D}}\int_{t_0}^t e^{i\omega_k(t-t')}F^\dagger(t')\,dt'\right)\right.$$

$$\left. + [F^\dagger,c]\left(e^{-i\omega_k(t-t_0)}e_k(t_0) - i\sqrt{\frac{\kappa}{2\pi D}}\int_{t_0}^t e^{-i\omega_k(t-t')}F(t')\,dt'\right)\right\}$$

$$= i[H_S, c(t)] + \sqrt{\kappa}\left\{[F,c]\left(-c_{in}^\dagger(t) - \frac{\sqrt{\kappa}}{2}F^\dagger(t)\right) + [F^\dagger,c]\left(-c_{in}(t) + \frac{\sqrt{\kappa}}{2}F(t)\right)\right\},$$
(B7)

where we have defined $c_{\text{in}}(t)$ as

$$c_{\text{in}}(t) = -\frac{i}{\sqrt{2\pi D}}\sum_k e^{-i\omega_k(t-t_0)} e_k(t_0). \tag{B8}$$

The definition introduced in Eq. (B6) corresponds to what in the context of electronic transport is defined as "a wide-band-limit approximation" and, allowing us to write the QLE given in Eq. (B7) in time-local form, can be considered equivalent to the Markov approximation [47].

Let us focus on the case, discussed in the text, of two separate baths: a bosonic bath with operators b_k and a bath of TLSs with HP-transformed modes d_k. We define two functions F_b and F_{TLS} of the system operators that couple to the bosonic and TLS baths, respectively. The QLE (B7) then reads

$$\dot{c}(t) = i[H_S, c(t)] + \sqrt{\kappa}\left\{[F_b, c]\left(-c_{\text{in}}^{\dagger} - \frac{\sqrt{\kappa}}{2}F_b^{\dagger}\right) + [F_b^{\dagger}, c]\left(-c_{\text{in}} + \frac{\sqrt{\kappa}}{2}F_b\right)\right\}$$
$$+ \sqrt{\kappa_N}\left\{[F_{\text{TLS}}, c]\left(-c_{\text{in}}^{\text{TLS}\dagger} - \frac{\sqrt{\kappa_N}}{2}F_{\text{TLS}}^{\dagger}\right) + [F_{\text{TLS}}^{\dagger}, c]\left(-c_{\text{in}}^{\text{TLS}} + \frac{\sqrt{\kappa_N}}{2}F_{\text{TLS}}\right)\right\}. \tag{B9}$$

Assuming a linear coupling between the system and the bosonic bath and choosing the HP$_-$ mapping for the TLSs, one obtains $F_b = c$ and $F_{\text{TLS}} = c^2$. Substituting these into Eq. (B9) gives

$$\dot{c} = i[H_S, c(t)] - \left(\frac{\kappa}{2} + \kappa_N c^{\dagger}c\right)c + \sqrt{\kappa}c_{\text{in}} + 2\sqrt{\kappa_N}c^{\dagger}c_{\text{in}}^{\text{TLS}}, \tag{B10}$$

which corresponds to Eq. (7a) of the main text. On the contrary, if the HP$_+$ mapping is chosen, one obtains Eq. (7b) with $F_{\text{TLS}} = c^{\dagger 2}$.

APPENDIX C: LINEARIZATION OF THE QUANTUM LANGEVIN EQUATIONS

Here we outline the linearization strategy that allows us, in the presence of a strong coherent tone $\alpha_p = \alpha_{\text{in}}e^{-i\omega_p t}$, to recast Eqs. (7b) of the main text in terms of equations describing the stationary state (in a frame rotating at ω_p) and the fluctuations around this stationary state, given by Eqs. (8a) and (8b) of the main text.

Focusing on Eq. (7a),

$$\dot{c} = -i[c, H_S] - \left(\frac{\kappa}{2} + \kappa_N c^{\dagger}c\right)c + \sqrt{\kappa}c_{\text{in}} + 2\sqrt{\kappa_N}c^{\dagger}c_{\text{in}}^{\text{TLS}}. \tag{C1}$$

In the presence of a strong coherent pump $\alpha_p = \alpha_{\text{in}}e^{-i\omega_p t}$, we seek a solution of the form $c = \alpha + a$,

$$-i\omega_p \alpha + \dot{a} = -i\omega_c(\alpha + a) - \left[\frac{\kappa}{2} + \kappa_N(\alpha^* + a^{\dagger})(\alpha + a)\right]$$
$$\times (\alpha + a) + \sqrt{\kappa}(\alpha_{\text{in}} + a_{\text{in}})$$
$$+ 2\sqrt{\kappa_N}(\alpha^* + a^{\dagger})a_{\text{in}}^{\text{TLS}}, \tag{C2}$$

where without loss of generality, we have assumed that $H_S = \omega_c c^{\dagger}c$.

Neglecting the fluctuation terms, we obtain the equation for the steady-state solution

$$0 = i\Delta\alpha - \frac{\kappa}{2}\alpha - \kappa_N\alpha|\alpha|^2 + \sqrt{\kappa}\alpha_{\text{in}}, \tag{C3}$$

where $\Delta = \omega_p - \omega_c$. From Eq. (C2) the equation for the fluctuation around the steady-state solution value of α given above is thus expressed as

$$\dot{a} = \left[i\Delta - \left(\frac{\kappa}{2} + 2\kappa_N|\alpha|^2\right)\right]a - \kappa_N\alpha^2 a^{\dagger} + \sqrt{\kappa}a_{\text{in}}$$
$$+ 2\sqrt{\kappa_N}\alpha^* a_{\text{in}}^{\text{TLS}}. \tag{C4}$$

With a similar procedure one can also show that Eq. (7b) leads to Eq. (8b). Notice that the nonlinear dissipative terms $\mp 2\kappa_N|\alpha|^2 a$ in Eqs. (8a) and (8b) lead to the broadening or narrowing of the linewidth associated with the linearized response of the cavity field fluctuations, respectively (see Fig. 4).

APPENDIX D: FLUCTUATION SPECTRUM OF THE NONLINEAR MODEL

Assuming that, in addition to the strong coherent tone, the dynamics of the system is affected by thermal fluctuations of both the bosonic and the TLS baths degrees of freedom,
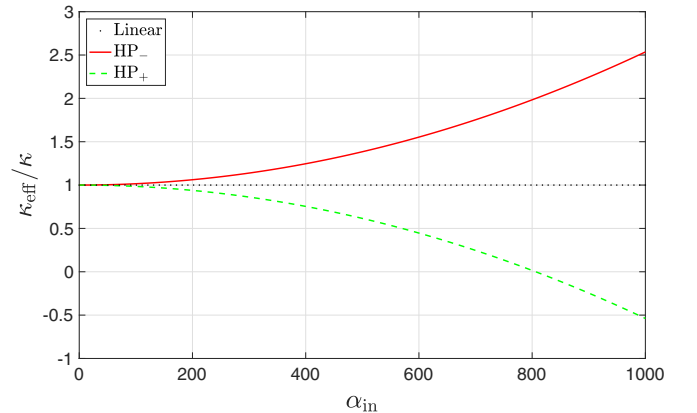

FIG. 4. The total effective dissipation of the linearized models Eq. (8a) (solid red) and Eq. (8b) (dashed green) that correspond to the cases where the majority of the TLSs are in the ground state and excited state, respectively. They are compared to the case of pure linear dissipation (black dots). Here we assume the system to be a simple cavity with $H_S = \omega_c c^{\dagger}c$. In units of κ, the parameters are $\Delta = \omega_p - \omega_c = 20$ and $\kappa_N = 1.5\times10^{-4}$.
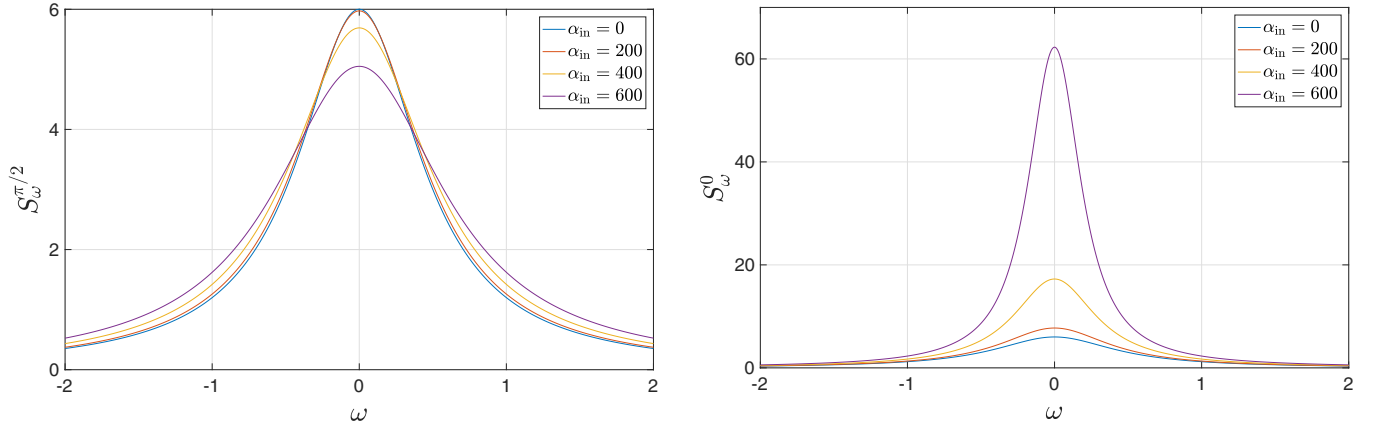
FIG. 5. The cavity spectra related to the Holstein-Primakoff couplings (a) HP$_-$ and (b) HP$_+$ for the largest uncertainty quadrature ($\theta = \pi/2$ and $\theta = 0$, respectively). In (a) the linewidth widens as $\alpha_{\rm in}$ becomes larger (larger values of $\alpha_{\rm in}$ correspond to smaller values of the maximum at $\omega = 0$), whereas in (b) the linewidth becomes narrower (larger values of $\alpha_{\rm in}$ correspond to larger values of the maximum at $\omega = 0$). Here the thermal populations of the bosonic and TLS baths are $n_{\rm th} = n_{\rm th}^{\rm TLS} = 1$, and in the units of κ, the other parameters are $\Delta = 20$ and $\kappa_N = 1.5 \times 10^{-4}$.

we evaluate here the spectrum of these fluctuations focusing on the HP$_-$ case (an analogous derivation holds for the HP$_+$ mapping). The fluctuation spectrum

$$S_\omega^\theta = \tfrac{1}{2}\langle\{X_\omega^\theta, X_{-\omega}^\theta\}\rangle, \tag{D1}$$

with $X_\omega^\theta = 1/\sqrt{2}(a_{-\omega}^\dagger e^{i\theta} + a_\omega e^{-i\theta})$, can be obtained by Fourier transforming the QLE given by Eq. (8a) and its Hermitian conjugate

$$\left[-i(\omega + \Delta) + \frac{\kappa}{2} + 2\kappa_N|\alpha|^2\right]a_\omega + \kappa_N\alpha^2 a_{-\omega}^\dagger$$
$$= \sqrt{\kappa}a_{{\rm in},\omega} + 2\sqrt{\kappa_N}\alpha^* a_{{\rm in},\omega}^{\rm TLS}, \tag{D2a}$$

$$\left[-i(\omega - \Delta) + \frac{\kappa}{2} + 2\kappa_N|\alpha|^2\right]a_{-\omega}^\dagger + \kappa_N\alpha^{*2} a_\omega$$
$$= \sqrt{\kappa}a_{{\rm in},-\omega}^\dagger + 2\sqrt{\kappa_N}\alpha a_{{\rm in},-\omega}^{{\rm TLS}\dagger}, \tag{D2b}$$

with the usual convention for the Fourier transform, according to which $a_t \overset{\rm FT}{\longmapsto} a_\omega$ and $a_t^\dagger \overset{\rm FT}{\longmapsto} a_{-\omega}^\dagger$.

Defining

$$A = -i(\omega + \Delta) + \frac{\kappa}{2} + 2\kappa_N|\alpha|^2, \tag{D3a}$$

$$B = \kappa_N\alpha^2, \tag{D3b}$$

$$C = -i(\omega - \Delta) + \frac{\kappa}{2} + 2\kappa_N|\alpha|^2, \tag{D3c}$$

the QLE for the system can be expressed as

$$\begin{pmatrix} a_\omega \\ a_{-\omega}^\dagger \end{pmatrix} = \frac{1}{AC - |B|^2} \begin{pmatrix} C & -B \\ -B^* & A \end{pmatrix}$$
$$\times \begin{pmatrix} \sqrt{\kappa}a_{{\rm in},\omega} + 2\sqrt{\kappa_N}\alpha^* a_{{\rm in},\omega}^{\rm TLS} \\ \sqrt{\kappa}a_{{\rm in},-\omega}^\dagger + 2\sqrt{\kappa_N}\alpha a_{{\rm in},-\omega}^{{\rm TLS}\dagger} \end{pmatrix}, \tag{D4}$$

and

$$a_\omega = \chi_d(\omega)a_{{\rm in},\omega} + \chi_{\rm x}(\omega)a_{{\rm in},-\omega}^\dagger + \chi_d^{\rm TLS}(\omega)a_{{\rm in},\omega}^{\rm TLS}$$
$$+ \chi_{\rm x}^{\rm TLS}(\omega)a_{{\rm in},-\omega}^{{\rm TLS}\dagger}, \tag{D5a}$$

$$a_{-\omega}^\dagger = \chi_{\rm x}^*(-\omega)a_{{\rm in},\omega} + \chi_d^*(-\omega)a_{{\rm in},-\omega}^\dagger + \chi_{\rm x}^{{\rm TLS}*}(-\omega)a_{{\rm in},\omega}^{\rm TLS}$$
$$+ \chi_d^{{\rm TLS}*}(-\omega)a_{{\rm in},-\omega}^{{\rm TLS}\dagger}, \tag{D5b}$$

where

$$\chi_d(\omega) = \sqrt{\kappa}C(AC - |B|^2)^{-1}, \tag{D6a}$$
$$\chi_{\rm x}(\omega) = -\sqrt{\kappa}B(AC - |B|^2)^{-1}, \tag{D6b}$$
$$\chi_d^{\rm TLS}(\omega) = 2\sqrt{\kappa_N}\alpha^* C(AC - |B|^2)^{-1}, \tag{D6c}$$
$$\chi_{\rm x}^{\rm TLS}(\omega) = -2\sqrt{\kappa_N}\alpha B(AC - |B|^2)^{-1}. \tag{D6d}$$

If we assume that the thermal populations of the baths are given by $\langle a_{{\rm in},\omega}a_{{\rm in},\omega'}^\dagger\rangle = (n_{\rm th} + 1)\delta(\omega - \omega')$ and $\langle a_{{\rm in},\omega}^{\rm TLS}a_{{\rm in},\omega'}^{{\rm TLS}\dagger}\rangle = (n_{\rm th}^{\rm TLS} + 1)\delta(\omega - \omega')$, the cavity spectrum can be written as

$$S_\omega^\theta = \tfrac{1}{4}[(|\chi_d(\omega)|^2 + |\chi_{\rm x}(-\omega)|^2)\langle\{a_{{\rm in},\omega}, a_{{\rm in},\omega}^\dagger\}\rangle + (|\chi_d(-\omega)|^2 + |\chi_{\rm x}(\omega)|^2)\langle\{a_{{\rm in},-\omega}^\dagger, a_{{\rm in},-\omega}\}\rangle]$$

$$+ \tfrac{1}{4}[(\chi_d(\omega)\chi_{\rm x}(-\omega)e^{-i2\theta} + \chi_d^*(\omega)\chi_{\rm x}^*(-\omega)e^{i2\theta})\langle\{a_{{\rm in},\omega}, a_{{\rm in},\omega}^\dagger\}\rangle$$

$$+ (\chi_d(-\omega)\chi_{\rm x}(\omega)e^{-i2\theta} + \chi_d^*(-\omega)\chi_{\rm x}^*(\omega)e^{i2\theta})\langle\{a_{{\rm in},-\omega}^\dagger, a_{{\rm in},-\omega}\}\rangle]$$

$$+ \tfrac{1}{4}[(|\chi_d^{\rm TLS}(\omega)|^2 + |\chi_{\rm x}^{\rm TLS}(-\omega)|^2)\langle\{a_{{\rm in},\omega}^{\rm TLS}, a_{{\rm in},\omega}^{{\rm TLS}\dagger}\}\rangle$$

$$+ (|\chi_d^{\rm TLS}(-\omega)|^2 + |\chi_{\rm x}^{\rm TLS}(\omega)|^2)\langle\{a_{{\rm in},-\omega}^{{\rm TLS}\dagger}, a_{{\rm in},-\omega}^{\rm TLS}\}\rangle]$$

$$+ \tfrac{1}{4}[(\chi_d^{\rm TLS}(\omega)\chi_{\rm x}^{\rm TLS}(-\omega)e^{-i2\theta} + \chi_d^{{\rm TLS}*}(\omega)\chi_{\rm x}^{{\rm TLS}*}(-\omega)e^{i2\theta})\langle\{a_{{\rm in},\omega}^{\rm TLS}, a_{{\rm in},\omega}^{{\rm TLS}\dagger}\}\rangle$$

$$+ \left(\chi_d^{\text{TLS}}(-\omega) \chi_x^{\text{TLS}}(\omega) e^{-i2\theta} + \chi_d^{\text{TLS*}}(-\omega) \chi_x^{\text{TLS*}}(\omega) e^{i2\theta} \right) \langle \{ a_{\text{in},-\omega}^{\text{TLS}\dagger}, a_{\text{in},-\omega}^{\text{TLS}} \} \rangle \big]$$

$$= \tfrac{1}{2} \big[|\chi_d(\omega)|^2 + |\chi_d(-\omega)|^2 + |\chi_x(\omega)|^2 + |\chi_x(-\omega)|^2$$

$$+ 2 \cos(\theta + \phi) |\chi_d(\omega) \chi_x(-\omega) + \chi_d(-\omega) \chi_x(\omega)| \big] \big(n_{\text{th}} + \tfrac{1}{2} \big)$$

$$+ \tfrac{1}{2} \big[\big| \chi_d^{\text{TLS}}(\omega) \big|^2 + \big| \chi_d^{\text{TLS}}(-\omega) \big|^2 + \big| \chi_x^{\text{TLS}}(\omega) \big|^2 + \big| \chi_x^{\text{TLS}}(-\omega) \big|^2$$

$$+ 2 \cos(\theta + \phi^{\text{TLS}}) \big| \chi_d^{\text{TLS}}(\omega) \chi_x^{\text{TLS}}(-\omega) + \chi_d^{\text{TLS}}(-\omega) \chi_x^{\text{TLS}}(\omega) \big| \big] \big(n_{\text{th}}^{\text{TLS}} + \tfrac{1}{2} \big), \tag{D7}$$

where $\phi^{(\text{TLS})} = \text{Arg}[\chi_d^{(\text{TLS})}(\omega) \chi_x^{(\text{TLS})}(-\omega) + \chi_d^{(\text{TLS})}(-\omega) \chi_x^{(\text{TLS})}(\omega)]$. In Fig. 5(a) we have plotted the cavity spectrum for the HP$_-$, and the spectrum related to HP$_+$ coupling derived from Eq. (8b) is presented in Fig. 5(b).

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