Nonclassicality of coherent states: Entanglement of joint statistics

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Simple joint measurements of pairs of observables reveal that states considered universally as classical-like, such as SU(2) spin coherent states, Glauber coherent states, and thermal states, are actually nonclassical. We show that this holds because we can find a joint measurement the statistics of which is not separable. Eventually this may be extended to all states different from the maximally mixed state.

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I. INTRODUCTION

Quantumness is the *raison d'être* of quantum theory as well as the resource behind the quantum-technology revolution, as exemplified via entanglement and Bell's tests. In the conventional approach, quantumness holds for a limited class of states, difficult to generate and preserve in practice.

Nonclassical effects always emerge as the impossibility of confining randomness of two or more variables within probability distributions [1]. For example, this is actually the case of the celebrated quantum tests of the Bell type [2–4]. This includes as a particular case mainstream tests such as the failure of the Glauber-Sudarshan P function to be a true probability density [5].

In this paper we provide a different perspective by showing that states considered universally as classical-like, such as SU(2) spin coherent states, Glauber coherent states, and thermal states, are actually nonclassical. Eventually this can be extended to all states except the maximally mixed state. We show that this holds via entanglement of joint statistics.

To show this we consider the simultaneous measurement of two compatible observables in an enlarged system-apparatus space, that provides complete information about the statistics of two incompatible system observables. This is to say that we can recover their exact individual statistics after a suitable data inversion applied to the corresponding observed marginal distributions [6]. Then we apply the data inversion to the joint statistics. In classical physics this always leads to the joint statistics of the corresponding system observables, a *bona fide* probability distribution. We show that this holds because in classical physics all joint distributions are separable, so the inversion of the joint distribution works equally well as the inversion of the marginals.

However, in quantum physics this is no longer the case, and the inversion can lead to pathological joint distributions that are not probabilities. In such a case we say that the state is nonclassical. We describe the general procedure in Sec. II. We apply it to the qubit case in Sec. III and to Glauber quadrature coherent states and thermal states in Sec. IV via quadrature measurements in double homodyne detection.

II. BASIC SETTINGS

Nonclassicality cannot be a single-observable property since within classical physics it is always possible to reproduce exactly the statistics of any quantum observable. Nonclassical effects can only emerge when addressing the joint statistics of multiple observables, especially if they are incompatible. Let us show how, from two different perspectives: probability distributions and characteristic functions.

A. Probability distributions

In the most general case, joint measurements require the coupling of the system space $\mathcal{H}_s$ with auxiliary degrees of freedom $\mathcal{H}_a$. We consider the simultaneous measurement of two compatible observables, $\tilde{X}$ and $\tilde{Y}$, in the enlarged space $\mathcal{H}_s \otimes \mathcal{H}_a$ with outcomes x and y, respectively, and joint probability $\tilde{p}_{X,Y}(x,y)$. Since this corresponds to the statistics or a real measurement we have that $\tilde{p}_{X,Y}(x,y)$ is a well-behaved probability distribution. The corresponding marginal distributions are

$$\tilde{p}_X(x) = \sum_y \tilde{p}_{X,Y}(x,y), \quad \tilde{p}_Y(y) = \sum_x \tilde{p}_{X,Y}(x,y), \tag{2.1}$$

where we are assuming a discrete range for x and y without loss of generality. We assume that these marginals provide complete information about two system observables in the system space $\mathcal{H}_s$, say X and Y, respectively, which may be incompatible. This is to say that their probability distributions $p_A(a)$ for $A = X,Y$ and $a,a' = x,y$ can be retrieved from the observed marginals $\tilde{p}_A(a)$ as

$$p_A(a) = \sum_{a'} \mu_A(a,a')\, \tilde{p}_A(a'), \tag{2.2}$$

where the functions $\mu_A(a,a')$ are completely known as far as we know the measurement being performed and the initial state of the auxiliary degrees of freedom $\mathcal{H}_a$. We stress that relation (2.2) is an assumption that holds or not depending on the observable A, the measurement performed, and the initial state of the ancilla. Whenever the inversion is possible, the functions $\mu_A(a,a')$ can be easily determined by imposing Eq. (2.2) for arbitrary states of the system being observed.

The key idea is to extend this inversion (2.2) from the marginals to the complete joint distribution to obtain a joint distribution $p_{X,Y}(x,y)$ for the X and Y variables in the system

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state [6]:

$$p_{X,Y}(x,y) = \sum_{x',y'} \mu_X(x,x')\, \mu_Y(y,y')\, \tilde{p}_{X,Y}(x',y'). \tag{2.3}$$

This is actually a definition of $p_{X,Y}(x,y)$ motivated by the classical case, where after Eq. (2.2) the kernels $\mu_A(a,a')$ are actually independent conditional probabilities of getting a given a', with

$$\sum_a \mu_A(a,a') = 1. \tag{2.4}$$

In particular this implies that the kernels in Eq. (2.3) must be the same as those introduced in Eqs. (2.2) since $p_{X,Y}(x,y)$ must give the correct marginals

$$p_X(x) = \sum_y p_{X,Y}(x,y), \quad p_Y(y) = \sum_x p_{X,Y}(x,y). \tag{2.5}$$

Parallels can be drawn with the construction joint probability distributions via the inversion of moments [7].

B. Characteristic functions

An alternative approach can be formulated in terms of characteristic functions defined as usual as the Fourier transform of the probability distributions, assuming now a continuous range for a,

$$C_A(u) = \int da\, e^{iau}\, p_A(a) = \langle e^{iuA} \rangle, \tag{2.6}$$

which can be inverted in the form

$$p_A(a) = \frac{1}{2\pi} \int du\, e^{-iau}\, C_A(u). \tag{2.7}$$

Since both $C_A(u)$ and $p_A(a)$ contain full information about the statistics of A the characteristic function can equally well serve for our purposes.

The simultaneous measurement of $\tilde{X}$ and $\tilde{Y}$ leads to a joint characteristic function

$$\tilde{C}_{X,Y}(u,v) = \langle e^{i(u\tilde{X}+v\tilde{Y})} \rangle = \int dx dy e^{i(ux+vy)} \tilde{p}_{X,Y}(x,y), \tag{2.8}$$

from which two marginal characteristics can be derived for each observable rather simply as

$$\tilde{C}_X(u) = \tilde{C}_{X,Y}(u,0), \quad \tilde{C}_Y(v) = \tilde{C}_{X,Y}(0,v). \tag{2.9}$$

In many interesting practical settings, such as the one to be examined in Sec. IV, the observed $\tilde{C}_A(u)$ and true $C_A(u)$ characteristics are simply related in the form

$$\tilde{C}_A(u) = H_A(u) C_A(u), \tag{2.10}$$

where $H_A(u)$ is an instrumental function, which is assumed to be known as far as we know the details of the measurement being performed. This is the case of linear shift invariant systems where H is the frequency response of the system, or the optical transfer function in classical imaging optics. That is to say that $\tilde{p}_A(a)$ is the result of convolving $p_A(a)$ with the impulse response function, which is the Fourier transform of H. Note that $H_A(0) = 1$ by normalization of probability distributions.

Assuming that $H_A(u)$ has no zeros, as it will be our case here, the analog of the inversion (2.2) is after Eq. (2.10) simply

$$C_A(u) = \tilde{C}_A(u)/H_A(u). \tag{2.11}$$

Applying the inversion to the joint statistics we get

$$C_{X,Y}(u,v) = \frac{\tilde{C}_{X,Y}(u,v)}{H_X(u) H_Y(v)}, \tag{2.12}$$

as a particular counterpart of Eq. (2.3). The question is whether the so-inferred characteristic $C_{X,Y}(u,v)$ leads to a true probability distribution $p_{X,Y}(x,y)$ via Fourier inversion:

$$p_{X,Y}(x,y) = \frac{1}{(2\pi)^2} \int du\, dv\, e^{-i(ux+vy)}\, C_{X,Y}(u,v), \tag{2.13}$$

i.e., whether the integral exists and $p_{X,Y}(x,y)$ is non-negative, as it is always the case in classical physics as shown next.

C. Classical physics

Let us show that in classical physics these inversion procedures (2.3) and (2.12) always lead to a *bona fide* probability distribution $p_{X,Y}(x,y)$. Classically the state of the system can be completely described by a legitimate probability distribution p_j, where index j runs over all admissible states $\boldsymbol{\lambda}_j$ for the system. This is the corresponding phase space, assumed to form a discrete set for simplicity and without loss of generality. There is no limit to the number of points $\boldsymbol{\lambda}_j$ so it may approach a continuum if necessary.

So the observed joint statistics can be always expressed as

$$\tilde{p}_{X,Y}(x,y) = \sum_j p_j\, \tilde{X}(x|\boldsymbol{\lambda}_j)\, \tilde{Y}(y|\boldsymbol{\lambda}_j), \tag{2.14}$$

where $\tilde{A}(a|\boldsymbol{\lambda}_j)$ is the conditional probability that the observable $\tilde{A}$ takes the value a when the system state is $\boldsymbol{\lambda}_j$. By definition, phase-space points $\boldsymbol{\lambda}_j$ have definite values for every observable so the factorization $\tilde{X}(x|\boldsymbol{\lambda}_j)\tilde{Y}(y|\boldsymbol{\lambda}_j)$ holds. Strictly speaking they are the product of delta functions. Applying Eq. (2.2) we get the conditional probabilities for the system variables:

$$A(a|\lambda_j) = \sum_{a'} \mu_A(a,a')\, \tilde{A}(a'|\boldsymbol{\lambda}_j). \tag{2.15}$$

Thus, because of the separable form Eq. (2.14) we readily get from Eqs. (2.3) and (2.15) that the result of the inversion is the actual joint distribution for X and Y,

$$p_{X,Y}(x,y) = \sum_j p_j\, X(x|\boldsymbol{\lambda}_j)\, Y(y|\boldsymbol{\lambda}_j), \tag{2.16}$$

and therefore a legitimate statistics. Thus, lack of positivity or any other pathology of the retrieved joint distribution $p_{X,Y}(x,y)$ is then a signature of nonclassical behavior.

Similarly, the procedure outlined above in terms of characteristic functions leads always in classical physics to a *bona fide* distribution. This is because the observed characteristics is always separable as the Fourier transform of Eq. (2.14),

$$\tilde{C}_{X,Y}(u,v) = \sum_j p_j\, \tilde{C}_X(u|\boldsymbol{\lambda}_j)\, \tilde{C}_Y(v|\boldsymbol{\lambda}_j), \tag{2.17}$$

where $\tilde{C}_A(u|\boldsymbol{\lambda}_j)$ are the corresponding conditional characteristics. Then, after Eq. (2.12) we get also a separable joint characteristics for system variables X and Y,

$$C_{X,Y}(u,v) = \sum_j p_j\, C_X(u|\boldsymbol{\lambda}_j)\, C_Y(v|\boldsymbol{\lambda}_j), \tag{2.18}$$

that leads via Fourier transform to the same legitimate distribution in Eq. (2.16).

III. QUBIT EXAMPLE

Let us focus on the qubit as the simplest quantum system $\mathcal{H}_s$. The most general state of the qubit is

$$\rho = \tfrac{1}{2}(\sigma_0 + \boldsymbol{s}\cdot\boldsymbol{\sigma}), \quad |\boldsymbol{s}| \leqslant 1, \tag{3.1}$$

where $\boldsymbol{s}$ is a three-dimensional real vector with $|\boldsymbol{s}| \leqslant 1$, σ_0 is the 2×2 identity matrix, and $\boldsymbol{\sigma}$ are the Pauli matrices. The task is finding for every ρ a suitable measurement so that the inversion (2.3) of the observed statistics $\tilde{p}_{X,Y}(x,y)$ cannot be a probability distribution. To this end, we will use that any measurement performed in the enlarged space $\mathcal{H}_s \otimes \mathcal{H}_a$ can be conveniently described by a positive operator-valued measure (POVM) in $\mathcal{H}_s$:

$$\tilde{\Delta}_{X,Y}(x,y) = \tfrac{1}{4}[\sigma_0 + \boldsymbol{\eta}(x,y)\cdot\boldsymbol{\sigma}]. \tag{3.2}$$

Positivity and normalization require that

$$\tilde{\Delta}_{X,Y}(x,y) \geqslant 0, \quad \sum_{x,y} \tilde{\Delta}_{X,Y}(x,y) = \sigma_0, \tag{3.3}$$

so that

$$|\boldsymbol{\eta}(x,y)| \leqslant 1, \quad \sum_{x,y} \boldsymbol{\eta}(x,y) = \boldsymbol{0}. \tag{3.4}$$

The corresponding statistics is

$$\tilde{p}_{X,Y}(x,y) = \mathrm{tr}[\rho\tilde{\Delta}_{X,Y}(x,y)] = \tfrac{1}{4}[1 + \boldsymbol{\eta}(x,y)\cdot\boldsymbol{s}], \tag{3.5}$$

and naturally

$$\tilde{p}_{X,Y}(x,y) \geqslant 0, \quad \sum_{x,y} \tilde{p}_{X,Y}(x,y) = 1. \tag{3.6}$$

For definiteness, let us consider the case

$$\boldsymbol{\eta}(x,y) = \frac{\eta}{\sqrt{3}}(x,y,xy), \tag{3.7}$$

where $x,y = \pm 1$ and η is a real parameter we will assume positive without loss of generality $1 \geqslant \eta > 0$. Actually, for $\eta = 1$ we have that $\tilde{p}_{X,Y}(x,y)$ is a discrete and complete sampling of the SU(2) Husimi function for two-dimensional systems [8]. The observed marginals are

$$\tilde{p}_X(x) = \frac{1}{2}\left(1 + x\frac{\eta}{\sqrt{3}}s_x\right), \quad \tilde{p}_Y(y) = \frac{1}{2}\left(1 + y\frac{\eta}{\sqrt{3}}s_y\right), \tag{3.8}$$

that provide complete information about the system observables $X = \sigma_x$ and $Y = \sigma_y$ with exact statistics:

$$p_X(x) = \tfrac{1}{2}(1 + xs_x), \quad p_Y(y) = \tfrac{1}{2}(1 + ys_y). \tag{3.9}$$

The inversion of the marginals is carried out by the functions

$$\mu_A(a,a') = \frac{1}{2}\left(1 + \frac{\sqrt{3}}{\eta}aa'\right), \tag{3.10}$$

so that the inversion of the joint distribution in Eq. (2.3) leads to

$$p_{X,Y}(x,y) = \frac{1}{4}\left(1 + xs_x + ys_y + xys_z\frac{\sqrt{3}}{\eta}\right). \tag{3.11}$$

A. Nonclassicality of all states different from the maximally mixed state

Throughout we are free to chose the axes and the observables measured. In this spirit, using SU(2) symmetry, and without loss of generality, we can choose axes so that $s_x = s_y = 0, s_z = |\boldsymbol{s}|$, so that

$$p_{X,Y}(x,y) = \frac{1}{4}\left(1 + xy\frac{\sqrt{3}}{\eta}|\boldsymbol{s}|\right). \tag{3.12}$$

This can take negative values for $x = -y = \pm 1$,

$$p_{X,Y}(\pm 1, \mp 1) = \frac{1}{4}\left(1 - \frac{\sqrt{3}}{\eta}|\boldsymbol{s}|\right) < 0, \tag{3.13}$$

provided that $\eta < \sqrt{3}|\boldsymbol{s}|$. Clearly for all $\boldsymbol{s} \neq \boldsymbol{0}$ we can always chose η satisfying this relation. So every state different from the maximally mixed state $\boldsymbol{s} = \boldsymbol{0}$ is nonclassical.

In this regard it is worth noting that all pure states of the qubit are SU(2) coherent states [9]. Because of their definition and properties they are often regarded as the closets analogs of the Glauber coherent states that can exist in finite-dimensional spaces. Accordingly, since Glauber coherent states are universally regarded as classical, the SU(2) coherent states are reported as the most classical allowed in finite-dimensional systems. This is because their joint angular-momentum statistics can be described by a *bona fide* classical-like distribution on the corresponding phase space, which is the sphere. This is discussed in great detail in Ref. [10], for example, regarding their Glauber-Sudarshan SU(2) P function. This is to say that their classical-like resemblance refers to their angular-momentum statistical properties, although they would be nonclassical by their finite-dimensional nature. However, we have just shown that even if we just focus on the angular-momentum statistics they are actually as nonclassical as any other spin state when we look beyond the P function.

B. Entanglement of statistics

Let us provide an explicit demonstration that if $p_{X,Y} < 0$ the observed statistics (3.12) cannot be expressed in a separable form. *Separable* means that there is a *bona fide* probability distribution p_j so that

$$\tilde{p}_{X,Y}(x,y) = \sum_j \frac{p_j}{4}\left(1 + x\frac{\eta}{\sqrt{3}}\lambda_{j,x}\right)\left(1 + y\frac{\eta}{\sqrt{3}}\lambda_{j,y}\right), \tag{3.14}$$

leading to

$$p_{X,Y}(x,y) = \sum_j \frac{p_j}{4}(1 + x\lambda_{j,x})(1 + y\lambda_{j,y}), \tag{3.15}$$

where, since the phase space is a sphere, $\boldsymbol{\lambda}_j$ are three-dimensional real vectors with unit modulus $|\boldsymbol{\lambda}_j| \leqslant 1$, being

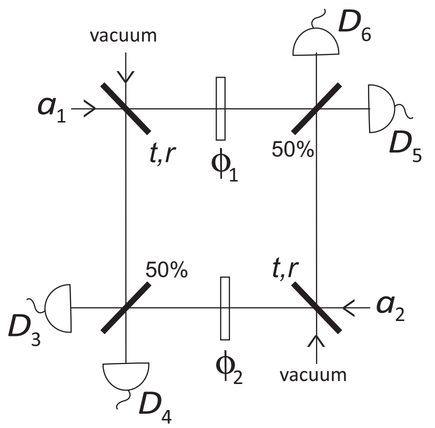

FIG. 1. Eight-port homodyne detector.

$\lambda_{j,x}$ and $\lambda_{j,y}$ the corresponding components. We recall that there is no limit to the number of vectors $\boldsymbol{\lambda}_j$. Then, if the separable form Eq. (3.15) holds we have after Eq. (3.12) that

$$\sum_j p_j \lambda_{j,x}\lambda_{j,y} = \frac{\sqrt{3}}{\eta}|\boldsymbol{s}|. \tag{3.16}$$

We can readily show that separability (3.15) and negativity (3.13) are contradictory. This is because $|\boldsymbol{\lambda}_j| \leqslant 1$ so that $\sum_j p_j \lambda_{j,x}\lambda_{j,y} \leqslant 1$. Thus separability implies $\sqrt{3}|\boldsymbol{s}|/\eta \leqslant 1$ while negativity implies just the opposite $\sqrt{3}|\boldsymbol{s}|/\eta > 1$. Therefore, negativity of the inferred distribution $p_{X,Y}(x,y)$ is equivalent to entanglement of the observed statistics $\tilde{p}_{X,Y}(x,y)$.

C. Practical implementation

Comparing Eqs. (3.1) and (3.2) with Eq. (3.7) it can be readily seen that for $\eta = 1$ the elements of the POVM (3.2) are proportional to projectors on pure states with

$$\langle\sigma_x\rangle = x/\sqrt{3}, \quad \langle\sigma_y\rangle = y/\sqrt{3}, \quad \langle\sigma_z\rangle = xy/\sqrt{3}. \tag{3.17}$$

If we write the most general pure state in the basis of eigenvectors of σ_z as

$$|\psi\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \end{pmatrix}, \tag{3.18}$$

we get

$$\langle\sigma_x\rangle = \sin\theta\cos\phi, \quad \langle\sigma_y\rangle = \sin\theta\sin\phi, \quad \langle\sigma_z\rangle = \cos\theta, \tag{3.19}$$

so that the states satisfying the conditions (3.17) can be easily found by suitably combining θ and ϕ values with

$$\theta = \pm\theta_0 \bmod \pi, \quad \phi = \pm\phi_0 \bmod \pi, \tag{3.20}$$

being

$$\tan\theta_0 = \sqrt{2}, \quad \phi_0 = \pi/4. \tag{3.21}$$

The projection on these states can be easily implemented in practice in a one-photon realization of the qubit via the version of the eight-port homodyne detector schematized in Fig. 1 [11–13]. Let the qubit be spanned by the one-photon states $|1,0\rangle$ and $|0,1\rangle$, where $|n_1,n_2\rangle$ denote photon-number states with $n_{1,2}$ photons in two field modes $a_{1,2}$. We consider these states as the eigenstates of σ_z with eigenvalues 1 and -1, respectively. The modes $a_{1,2}$ are mixed with two further modes in vacuum as schematized in Fig. 1. The two input beam splitters are identical and unbalanced, with real transmission and reflection coefficients t and r,

$$t = \sin\frac{\theta_0}{2}, \quad r = \cos\frac{\theta_0}{2}, \tag{3.22}$$

with a relative π phase change in the lower-side reflections. There are also two phase plates introducing phase shifts $\phi_{1,2}$ with

$$\phi_1 = -\phi_2 = \pi/4. \tag{3.23}$$

The output beam splitters are balanced, also with real transmission and reflection coefficients and a π phase change in the lower-side reflections. Detectors placed at the four output beams detect the exit port of the photon, so there are only four possible outcomes. The input-output relations for the complex amplitudes are, omitting for simplicity the vacuum modes that will not contribute to the final result,

$$\begin{aligned} a_3 &= \frac{1}{\sqrt{2}}(-ra_1 + te^{i\phi_2}a_2), \quad a_4 = \frac{1}{\sqrt{2}}(-ra_1 - te^{i\phi_2}a_2), \\ a_5 &= \frac{1}{\sqrt{2}}(te^{i\phi_1}a_1 - ra_2), \quad a_6 = \frac{1}{\sqrt{2}}(te^{i\phi_1}a_1 + ra_2), \end{aligned} \tag{3.24}$$

where a_j is the amplitude of the field mode impinging on detector D_j. Following the analyses in Ref. [13] for a one-photon case, the probability that the detector D_j clicks is $p(j) = |\langle j|\psi\rangle|^2$ where the un-normalized vectors $|j\rangle$ are, following the same criterion as in Eq. (3.18),

$$\begin{aligned} |3\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} r \\ -te^{-i\phi_2} \end{pmatrix}, \quad |4\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} r \\ te^{-i\phi_2} \end{pmatrix}, \\ |5\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} t \\ -re^{-i\phi_1} \end{pmatrix}, \quad |6\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} t \\ re^{-i\phi_1} \end{pmatrix}. \end{aligned} \tag{3.25}$$

Therefore, using all preceding equations in this section, it can be easily seen that the detectors click with the probabilities in Eqs. (3.5) and (3.7) for $\eta = 1$. More specifically D_3 clicks with probability $\tilde{p}_{X,Y}(-1,-1)$, detector D_4 clicks with probability $\tilde{p}_{X,Y}(1,1)$, D_5 clicks with probability $\tilde{p}_{X,Y}(-1,1)$, and D_6 clicks with probability $\tilde{p}_{X,Y}(1,-1)$.

D. Extension to larger dimension

This analysis may be extended to systems in Hilbert spaces of arbitrary dimension. For pure states $|\psi\rangle$ this can be readily done by focusing on the two-dimensional subspace spanned by the pair $|\psi\rangle$ and $|\psi_\perp\rangle$, where $|\psi_\perp\rangle$ is any state orthogonal to $|\psi\rangle$. We may then define $\sigma_z = |\psi\rangle\langle\psi| - |\psi_\perp\rangle\langle\psi_\perp|$ and accordingly for the other Pauli matrices. For mixed states we may focus on their projection on any two-dimensional space that can be regarded as the marginal distribution of a larger statistics. Alternatively, we may deal with dichotomic observables, such as parity or any other on/off detectors [14,15].

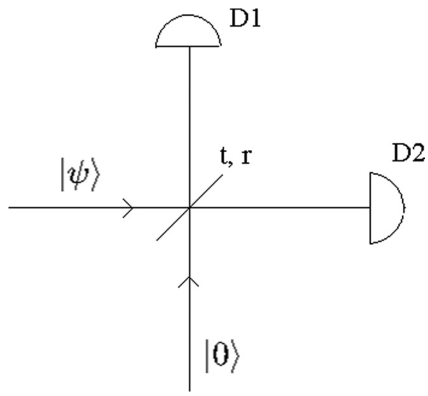

FIG. 2. Diagram of the experimental realization. Detectors D1 and D2 are homodyne detectors measuring quadratures $X_{1,\theta}$ and $Y_{2,\theta}$, respectively.

IV. UNBALANCED DOUBLE HOMODYNE DETECTION

Next we will demonstrate that there is a simple practical procedure leading to pathological quadrature joint statistics for Glauber coherent states and thermal states.

A. Procedure

The experiment consists of a double homodyne detector (see Fig. 2), where the observed state $|\psi\rangle$ is mixed with vacuum in an unbalanced beam splitter, with transmission and reflection coefficients t and r, respectively. At the output of the beam splitter, two homodyne detectors perform the simultaneous measurement of the commuting rotated quadratures $X_{1,\theta}$ and $Y_{2,\theta}$ in the corresponding modes, where θ is the phase of the local oscillator. We understand this as a noisy simultaneous measurement of the noncommuting quadratures X and Y in the signal mode in state $|\psi\rangle$. This is the relation between the corresponding observables:

$$\tilde{X} = X_{1,\theta} = rX_\theta + tX_{0,\theta},$$
$$\tilde{Y} = Y_{2,\theta} = tY_\theta - rY_{0,\theta}, \tag{4.1}$$

where $X_{0,\theta}$ and $Y_{0,\theta}$ are the corresponding rotated quadratures for the input mode in vacuum, while X_θ and Y_θ are the rotated quadratures in the signal mode, with

$$X_\theta = X\cos\theta + Y\sin\theta,$$
$$Y_\theta = -X\sin\theta + Y\cos\theta. \tag{4.2}$$

The quadratures are defined as the real and imaginary parts of the corresponding complex-amplitude operator $a = X + iY$ and when necessary we will take advantage of the fact that the vacuum is invariant under quadrature rotations.

Focusing on the characteristics-based approach in Sec. II B we begin with the observed joint characteristics for the observables $X_{1,\theta}$ and $Y_{2,\theta}$,

$$\tilde{C}'_{X,Y}(u',v') = \langle e^{i(u'\tilde{X}+v'\tilde{Y})}\rangle, \tag{4.3}$$

and we proceed to retrieve the joint characteristics for the observables X and Y. Using relations (4.1) and (4.2) in Eq. (4.3) we consider that the characteristic function already adapted for our target variables X and Y is $\tilde{C}_{X,Y}(u,v) = \tilde{C}'_{X,Y}(u',v')$ where

$$u = u'r\cos\theta - v't\sin\theta,$$
$$v = u'r\sin\theta + v't\cos\theta. \tag{4.4}$$

With this we get that the observed joint characteristic function can be expressed as

$$\tilde{C}_{X,Y}(u,v) = C^{(S)}_{X,Y}(u,v)\,H_{X,Y}(u,v), \tag{4.5}$$

where

$$C^{(S)}_{X,Y}(u,v) = \langle\psi|e^{i(uX+vY)}|\psi\rangle, \tag{4.6}$$

$$H_{X,Y}(u,v) = \langle 0|e^{i(zX_{0,\theta}+wY_{0,\theta})}|0\rangle = e^{-(z^2+w^2)/8}, \tag{4.7}$$

and

$$z = u't = \frac{t}{r}(u\cos\theta + v\sin\theta),$$
$$w = -v'r = -\frac{r}{t}(-u\sin\theta + v\cos\theta). \tag{4.8}$$

It turns out that $C^{(S)}_{X,Y}(u,v)$ is the symmetrically ordered characteristic function for X and Y, while H is the two-dimensional frequency response with

$$H_X(u) = H_{X,Y}(u,0), \quad H_Y(v) = H_{X,Y}(0,v), \tag{4.9}$$

and after Eqs. (4.7) and (4.8)

$$H_{X,Y}(u,v) = e^{-(fu^2+gv^2+2\gamma uv)/8}, \tag{4.10}$$

with

$$f = \frac{t^2}{r^2}\cos^2\theta + \frac{r^2}{t^2}\sin^2\theta,$$
$$g = \frac{t^2}{r^2}\sin^2\theta + \frac{r^2}{t^2}\cos^2\theta,$$
$$\gamma = \frac{t^2 - r^2}{2t^2r^2}\sin(2\theta). \tag{4.11}$$

Finally we arrive at the general relation for arbitrary input $|\psi\rangle$ using Eqs. (2.12), (4.5), and (4.9),

$$C_{X,Y}(u,v) = C^{(S)}_{X,Y}(u,v)\frac{H_{X,Y}(u,v)}{H_{X,Y}(u,0)H_{X,Y}(0,v)}, \tag{4.12}$$

with

$$\frac{H_{X,Y}(u,v)}{H_{X,Y}(u,0)H_{X,Y}(0,v)} = \exp(-\gamma uv/4). \tag{4.13}$$

This is the factor which makes the whole difference between classical and quantum physics (see discussion below). It holds provided that the input beam splitter is unbalanced, $t \neq r$, and that there is a rotation between the measured and inferred variables, $\theta \neq 0$ and $\pi/2$. These are the key ingredients allowing the entanglement of statistics required to disclose nonclassical properties, as discussed in Sec. II.

B. Glauber coherent states

To illustrate this procedure with a meaningful case let us assume that $|\psi\rangle$ is a coherent state $|\psi\rangle = |\alpha\rangle$, with

$a|\alpha\rangle = \alpha|\alpha\rangle$ and $\alpha = x_0 + iy_0$, so that

$$C^{(S)}_{X,Y}(u,v) = \langle\alpha|e^{i(uX+vY)}|\alpha\rangle = e^{i(ux_0+vy_0)}e^{-(u^2+v^2)/8}. \tag{4.14}$$

The final $C_{X,Y}(u,v)$ in Eq. (4.12) can be expressed in matrix form as

$$C_{X,Y}(u,v) = e^{i\boldsymbol{\xi}^*\cdot\boldsymbol{s}}e^{-\boldsymbol{\xi}^* M \boldsymbol{\xi}}, \tag{4.15}$$

with $\boldsymbol{\xi} = (u,v)^T$ and $\boldsymbol{s} = (x_0,y_0)^T$, where the subscript T denotes transposition and M is the 2×2 real symmetric matrix

$$M = \frac{1}{8}\begin{pmatrix} 1 & \gamma \\ \gamma & 1 \end{pmatrix}, \tag{4.16}$$

that does not depend on α. The condition for the existence of the integral (2.13) leading to the $p_{X,Y}(x,y)$ distribution is that M should be non-negative, that is, with positive eigenvalues, which holds provided $|\gamma| \leqslant 1$. Otherwise, there is no joint distribution $p_{X,Y}(x,y)$, contrary to the classical case shown in Sec. II C. Since M does not depend on α the condition $|\gamma| > 1$ leading to a nonclassical result can be satisfied at once for every coherent state by a suitable choice of beam splitter and phase θ. For example, for $\theta = \pi/4$, $t^2 > 1/\sqrt{2}$.

C. Thermal states

This result can be extended to mixed thermal states using their expansion in the coherent-state basis as

$$\rho = \frac{1}{\pi\bar{n}}\int d^2\alpha e^{-|\alpha|^2/\bar{n}}|\alpha\rangle\langle\alpha|, \tag{4.17}$$

where $\bar{n}$ is the mean number. After the result (4.14) we get the following symmetrical-order joint characteristic function for thermal states:

$$C^{(S)}_{X,Y}(u,v) = \mathrm{tr}[\rho e^{i(uX+vY)}] = e^{-(1+2\bar{n})(u^2+v^2)/8}, \tag{4.18}$$

leading to a final $C_{X,Y}(u,v)$ of the form (4.15) with $\boldsymbol{s} = \boldsymbol{0}$ and

$$M = \frac{1}{8}\begin{pmatrix} 1+2\bar{n} & \gamma \\ \gamma & 1+2\bar{n} \end{pmatrix}, \tag{4.19}$$

that depends on the particular thermal state being considered. In this case M fails to be non-negative when $|\gamma| > 1 + 2\bar{n}$. This is a more stringent condition as $\bar{n}$ grows, that is, as ρ becomes proportional to the identity matrix. So for every t and r there are thermal states with large enough $\bar{n}$ that behave as classical-like. Vice versa, for every $\bar{n}$ we can find t and r values so that the thermal state behaves as nonclassical.

D. Discussion

1. Squeezed Q function

Although the above analysis focuses on characteristic functions, it may be worth showing that the observed joint statistics $\tilde{p}_{X,Y}(x,y)$ results from projection of the observed state $|\psi\rangle$ on quadrature squeezed states $|\xi_{x,y}\rangle$,

$$\tilde{p}_{X,Y}(x,y) = |\langle\xi_{x,y}|\psi\rangle|^2, \tag{4.20}$$

where the states $|\xi_{x,y}\rangle$ are defined by the eigenvalue equation

$$[(r^2 - t^2)a^\dagger e^{i\theta} + ae^{-i\theta}]|\xi_{x,y}\rangle = 2(rx + ity)|\xi_{x,y}\rangle. \tag{4.21}$$

This can be easily shown from the defining eigenvalue equations

$$\tilde{X}|\phi\rangle = x|\phi\rangle, \quad \tilde{Y}|\phi\rangle = y|\phi\rangle, \tag{4.22}$$

combining them as

$$\big(r\tilde{X} + it\tilde{Y}\big)|\phi\rangle = (rx + ity)|\phi\rangle, \tag{4.23}$$

using Eqs. (4.1) and (4.2), and then finally projecting on the vacuum on the mode a_0, being $|\xi_{x,y}\rangle = \langle 0|\phi\rangle$.

Thus the measuring states $|\xi_{x,y}\rangle$ are quadrature squeezed states provided that the input beam splitter is unbalanced, $t \neq r$. The squeezing direction in the X-Y plane is specified by the phase θ. This is to say that the statistics $\tilde{p}_{X,Y}(x,y)$ is actually a squeezed Q function. This reduces to the standard Q in the balanced scheme $t = r$ so that $|\xi_{x,y}\rangle$ become coherent states $|\alpha\rangle$ with $\alpha = (x + iy)/\sqrt{2}$ [16].

2. Nonclassical measurement

The fact that the statistics is given by projection on nonclassical states does not spoil the interest of the result. Actually, this is the same case of the most paradigmatic nonclassical tests, such as sub-Poissonian statistics and quadrature squeezing. They also crucially rely on the projection on highly nonclassical measuring states: number states and infinitely squeezed states, respectively. Moreover, it has been shown that such nonclassical effects vanish if the measuring states become classical-like [17].

3. The vacuum

For the proper comparison with the classical model in Eq. (2.14) it must be understood that in this case we refer to classical models where the vacuum means a field of definite zero amplitude. Since our result relies on the frequency response (4.7) it may be regarded as a quantum-vacuum effect, as other relevant nonclassical effects in quantum optics such as spontaneous emission [5].

4. Entanglement of statistics

We think it is worth pointing out that the nonclassical test found here, that is, M lacking positive semidefiniteness, has a very close resemblance with the inseparability criterion for Gaussian states [18]. This might be expected since we have already commented on the fact that nonclassicality is equivalent to the lack of factorization for the observed statistics.

V. CONCLUSION

We have used a simple and general protocol to disclose nonclassical effects for states customarily regarded as the most classical states. These are the Glauber coherent states, thermal states, and SU(2) coherent states for spin variables. Moreover, we have shown that for all states there is always a measurement setting where the inferred joint distribution cannot represent probabilities, with the only exception being the totally incoherent mixed state in finite-dimensional spaces.

So there is no state that would always allow us to infer true probability distributions. These results are consistent with previous works that have also reported nonclassical properties for these states following different approaches [1,19], and with some more recent works extending nonclassical correlations and entanglement to all quantum states [20].

We have shown that nonclassicality holds because the observed joint probability distribution is not separable. We have to stress that this does not refer to actual particles, but just to the dependence of the statistics on the two observed variables.

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