

# Hydrodynamic tails and a fluctuation bound on the bulk viscosity

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We study the small frequency behavior of the bulk viscosity spectral function using stochastic fluid dynamics. We obtain a number of model independent results, including the long-time tail of the bulk stress correlation function and the leading nonanalyticity of the spectral function at small frequency. We also establish a lower bound on the bulk viscosity which is weakly dependent on assumptions regarding the range of applicability of fluid dynamics. The bound on the bulk viscosity  $\zeta$  scales as  $\zeta_{\min} \sim (P - \frac{2}{3}\mathcal{E})^2 \sum_i D_i^{-2}$ , where  $D_i$  are the diffusion constants for energy and momentum and  $P - \frac{2}{3}\mathcal{E}$ , where  $P$  is the pressure and  $\mathcal{E}$  is the energy density, is a measure of scale breaking. Applied to the cold Fermi gas near unitarity,  $|\lambda/a_s| \gtrsim 1$ , where  $\lambda$  is the thermal de Broglie wavelength and  $a_s$  is the  $s$ -wave scattering length, this bound implies that the ratio of bulk viscosity to entropy density satisfies  $\zeta/s \gtrsim 0.1\hbar/k_B$ . Here,  $\hbar$  is Planck's constant and  $k_B$  is Boltzmann's constant.

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## I. INTRODUCTION

Hydrodynamic tails reflect the fact that fluid dynamics is an effective theory, in which the classical equations of motions are the lowest order approximation to a more complete theory involving averages over fluctuations of the fundamental variables. The classical equations of motion in fluid dynamics describe the evolution of conserved quantities such as mass, energy, and momentum. These equations depend on the form of the associated currents [1]. In fluid dynamics the currents are expanded in gradients of hydrodynamic variables, and the corresponding expansion coefficients are known as transport coefficients. Transport coefficients control dissipative effects and fluctuation-dissipation relations imply that dissipative terms must be accompanied by stochastic forces. The presence of stochastic terms manifests itself in the form of long-time, nonanalytic, tails in correlation functions [2–4].

Long-time tails have been observed in computer simulations of fluids [5,6], but they are more difficult to detect experimentally. In the present work we will study correlation function of the bulk stress, with an emphasis on dilute quantum fluids, such as the dilute Fermi gas near unitarity. Bulk stresses are interesting because the bulk viscosity can be strongly enhanced near a phase transition [7], and quantum fluids provide attractive applications because hydrodynamic fluctuations are enhanced in systems in which the microscopic transport coefficients are small. The existing literature contains only very limited information on the bulk stress correlation function. The only calculation of the bulk tail in a nonrelativistic theory away from the critical point that we have been able to find appears to be wrong [4]. There are a number of studies of hydrodynamic tails near the liquid-gas endpoint and the superfluid transition [8], and there is a calculation of the bulk tail in a relativistic nonconformal fluid at zero mean charge density in [9].

In this work we compute the long-time tail of the bulk stress correlation function in a nonrelativistic fluid. We apply the result to the dilute Fermi gas near unitarity, and derive a bound on the bulk viscosity of a nonconformal fluid. This bound only depends on the shear viscosity and thermal conductivity of the fluid, combined with a measure of conformal symmetry breaking in the equation of state. The bound is similar

to lower bounds on the shear viscosity in relativistic and nonrelativistic fluids that have been derived in [10–12]. Finally, we discuss constraints on the bulk viscosity spectral function of a nonrelativistic fluid.

## II. KUBO FORMULA

In this section we will determine the relation between the bulk viscosity and the low frequency behavior of the retarded correlation function of the stress tensor. This relation, known as the Kubo formula, can be determined by matching the linear response relation for the stress induced by an external strain to the low-frequency behavior of the response predicted by fluid dynamics. The Kubo formula for the shear and bulk viscosity of a nonrelativistic fluid has been rederived many times [1,13], but there are a number of subtleties that we would like to emphasize. We will make use of a formalism developed in [14–17], which is based on studying the response of the fluid to a nontrivial background metric  $g_{ij}(t, \vec{x})$ . Correlation functions of the stress tensor are determined using linear response theory, and the constraints of Galilean symmetry can be incorporated by requiring the equations of fluid dynamics to satisfy diffeomorphism invariance.

The retarded correlation function of the stress tensor  $\Pi^{ij}$  is defined by

$$G_R^{ijkl}(\omega, \mathbf{k}) = -i \int dt \int d\mathbf{x} e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}} \Theta(t) \times \langle [\Pi^{ij}(t, \mathbf{x}), \Pi^{kl}(0, \mathbf{0})] \rangle. \quad (1)$$

The retarded correlator determines the stress induced by a small strain  $g_{ij}(t, \mathbf{x}) = \delta_{ij} + h_{ij}(t, \mathbf{x})$ ,

$$\delta \Pi_{ij}(\omega, \mathbf{k}) = -\frac{1}{2} G_R^{ijkl}(\omega, \mathbf{k}) h_{kl}(\omega, \mathbf{k}). \quad (2)$$

In fluid dynamics the stress tensor is expanded in terms of gradients of the thermodynamic variables. We write  $\Pi_{ij} = \Pi_{ij}^0 + \Pi_{ij}^1 + \dots$ , where

$$\Pi_{ij}^0 = \rho v_i v_j + P g_{ij} \quad (3)$$

is the ideal fluid part and  $\Pi_{ij}^i$  with  $i \neq 0$  are viscous corrections. Here,  $\rho$  is the mass density of the fluid,  $v_i$  is the

velocity, and  $P$  is the pressure. At first order in the gradient expansion  $\Pi_{ij}^1 = -\eta\sigma_{ij} - \zeta g_{ij}\langle\sigma\rangle$  with

$$\sigma_{ij} = \nabla_i v_j + \nabla_j v_i + \dot{g}_{ij} - \frac{2}{3}g_{ij}\langle\sigma\rangle, \quad (4)$$

$$\langle\sigma\rangle = \nabla \cdot v + \frac{\dot{g}}{2g}, \quad (5)$$

where  $\sigma_{ij}$  is the shear stress tensor,  $\eta$  is the shear viscosity,  $\zeta$  is the bulk viscosity,  $g$  is the determinant of the metric, and  $\nabla_i$  is the covariant derivative associated with  $g_{ij}$ . The terms involving time derivatives of the metric are fixed by diffeomorphism invariance [15]. Roughly, we can think of these terms as arising from the nonrelativistic reduction of a generally covariant stress tensor,  $\sigma_{ij} \sim \nabla_i u_j \sim u_0 \Gamma_{ij}^0 \sim \dot{g}_{ij}$ , where  $u_0, u_i$  are the temporal and spatial components of the four-velocity and  $\Gamma_{\mu\nu}^\alpha$  is the Christoffel symbol.

We will consider a harmonic perturbation of the form  $h_{ij}(t, \mathbf{x}) = \delta_{ij} h e^{-i\omega t}$ . At the level of ideal fluid dynamics this perturbation induces two terms in the stress tensor. The first,  $\delta\Pi_{ij} = P h_{ij}$ , arises from the direct coupling of  $\Pi_{ij}^0$  to the background metric. The second term follows from the equations of ideal fluid dynamics in a nontrivial background. The continuity equation implies  $\delta\rho = \frac{i\omega}{2} h \rho_0$ , where  $\rho_0$  is the unperturbed mass density. This leads to a shift in the pressure  $\delta P = (\partial P)/(\partial\rho)_s \delta\rho$ .

At first order in gradients the response is carried by the coupling to the background metric in Eq. (4). As expected, the response to a bulk strain  $h_{ij} \sim \delta_{ij}$  is independent of the shear viscosity. At order  $O(\omega)$  we get

$$\frac{1}{9} G_R^{ijij}(\omega, \mathbf{0}) = - \left[ \frac{2}{3} P - \left( \frac{\partial P}{\partial\rho} \right)_s \rho \right] - i\omega\zeta, \quad (6)$$

where repeated indices are summed over. The Kubo relation is

$$\zeta = - \lim_{\omega \rightarrow 0} \frac{1}{9\omega} \text{Im} G_R^{ijij}(\omega, \mathbf{0}). \quad (7)$$

In the following we will derive a slightly more convenient version of this Kubo relation. Bulk viscosity is a measure of scale breaking, and we would like to find a version of the Kubo relation in which this property is manifest. In the local rest frame of the fluid the trace of the stress tensor is proportional to the pressure. In [17] we showed that in equilibrium scale breaking can be characterized by the quantity<sup>1</sup>

$$\Delta_{\text{Tr}} P = P - \frac{2}{3} \mathcal{E}^0. \quad (8)$$

Here, we use  $\mathcal{E}^0$  to denote the energy density in the rest frame of the fluid. In ideal fluid dynamics the total energy density is given by  $\mathcal{E} = \mathcal{E}^0 + \frac{1}{2}\rho\mathbf{v}^2$ . We can now make use of the fact that the energy density of the fluid is conserved,

$$\frac{\partial\mathcal{E}}{\partial t} + \nabla \cdot J^\epsilon = 0, \quad (9)$$

where  $J^\epsilon$  is the energy current. This relation implies that for  $\omega \neq 0$  the retarded Green function  $G_R^{\epsilon ii}(\omega, \mathbf{k})$  of the energy density and the trace of the stress tensor must vanish as  $\mathbf{k} \rightarrow 0$ .

<sup>1</sup>We use the subscript Tr to distinguish the trace anomaly  $\Delta_{\text{Tr}} P$  from the quantity  $\Delta P$ , which is a fluctuation in the pressure.

A more formal proof of this statement using Ward identities was given in [18]; see also [19,20]. We conclude that we can use any linear combination of the form  $\mathcal{O} = \frac{1}{3}\Pi^{ii} + c\mathcal{E}$  to define the Kubo relation for the bulk viscosity. Here, we will use  $c = -\frac{2}{3}$ . This choice has the nice property that the Kubo relation

$$\zeta = - \lim_{\omega \rightarrow 0} \frac{1}{\omega} \text{Im} G_R^{\mathcal{O}\mathcal{O}}(\omega, \mathbf{0}), \quad \mathcal{O} = \frac{1}{3}(\Pi^{ii} - 2\mathcal{E}) \quad (10)$$

involves an operator which is manifestly sensitive to the trace anomaly in the hydrodynamic limit,  $\mathcal{O} = \Delta_{\text{Tr}} P = P - \frac{2}{3}\mathcal{E}^0$ .

### III. HYDRODYNAMIC FLUCTUATIONS

There are many possible strategies for evaluating the retarded correlation function of  $\mathcal{O} = \Delta_{\text{Tr}} P$ . An example is the microscopic calculation in [21], where we compute the bulk viscosity in a dilute Fermi gas based on a perturbative calculation of quasiparticle properties. In this work we will employ a different strategy and compute the retarded correlation using a macroscopic theory of the long distance properties of the fluid. This theory is stochastic fluid dynamics [13]. As we will show this theory provides a universal prediction of the leading nonanalyticity in  $G_R^{\mathcal{O}\mathcal{O}}(\omega, \mathbf{0})$  as  $\omega \rightarrow 0$ . It also provides a lower bound on  $\zeta$ , but this bound is sensitive to microscopic physics.

In order to explore the role of hydrodynamic fluctuations we will expand  $\Delta P$  to second order in hydrodynamic variables. Higher-order terms can be computed, but they provide corrections that are subleading in  $\omega/\omega_{br}$ . Here  $\omega_{br}$  is the breakdown scale of hydrodynamics, which we will define more carefully below. The probability of a fluctuation of the hydrodynamic variable is proportional to  $\exp(\Delta S)$ , where  $\Delta S$  is the change in entropy of the fluid [22]. We can write

$$S = \int d^3\mathbf{x} s(\rho, \mathcal{E}^0), \quad (11)$$

so that

$$\begin{aligned} \Delta S = \int d^3\mathbf{x} \left\{ \left( \frac{\partial s}{\partial\rho} \right)_{\mathcal{E}^0} \Delta\rho + \left( \frac{\partial s}{\partial\mathcal{E}^0} \right)_\rho \Delta\mathcal{E}^0 \right. \\ \left. + \frac{1}{2} \left( \frac{\partial^2 s}{\partial\rho^2} \right)_{\mathcal{E}^0} (\Delta\rho)^2 + \frac{\partial^2 s}{\partial\rho\partial\mathcal{E}^0} \Delta\rho\Delta\mathcal{E}^0 \right. \\ \left. + \frac{1}{2} \left( \frac{\partial^2 s}{\partial(\mathcal{E}^0)^2} \right)_\rho (\Delta\mathcal{E}^0)^2 + \dots \right\}. \quad (12) \end{aligned}$$

We can use the conservation laws for the mass density  $\rho$  and the energy density  $\mathcal{E}$  to show that the linear terms vanish. The quadratic terms can be simplified by using a set of thermodynamic variables that diagonalizes the quadratic form. A suitable set of variables is provided by  $(\rho, T)$  [13,23]. The entropy functional that governs fluctuations in  $\rho$ ,  $T$ , and  $\mathbf{v}$  is

$$\begin{aligned} \Delta S = - \frac{1}{2T_0} \int d^3\mathbf{x} \left\{ \frac{1}{\rho_0} \left( \frac{\partial P}{\partial\rho} \right)_T (\Delta\rho)^2 \right. \\ \left. + \frac{c_V}{T_0} (\Delta T)^2 + \rho_0 \mathbf{v}^2 + \dots \right\}, \quad (13) \end{aligned}$$

where  $(T_0, \rho_0)$  denote the mean values of the temperature and density and  $(\Delta T, \Delta\rho, \mathbf{v})$  are local fluctuations. We can expand

$\mathcal{O} = \Delta_{\text{Tr}} P$  to second order in  $(\Delta T, \Delta \rho)$ ,

$$\begin{aligned} \mathcal{O} = & \mathcal{O}_0 + a_\rho \Delta \rho + a_T \Delta T + a_{\rho\rho} (\Delta \rho)^2 \\ & + a_{\rho T} \Delta \rho \Delta T + a_{TT} (\Delta T)^2 + \dots \end{aligned} \quad (14)$$

The hydrodynamic tails are determined by the second-order terms. The corresponding coefficients can be expressed in terms of thermodynamic quantities. We find

$$a_{\rho\rho} = \frac{1}{2} \frac{\partial}{\partial \rho} \left[ c_T^2 - \frac{2}{3} \left( \frac{h}{m} - \frac{T \alpha \kappa_T}{\rho} \right) \right]_T, \quad (15)$$

$$a_{\rho T} = \left. \frac{\partial c_T^2}{\partial T} \right|_\rho - \frac{2}{3} \left. \frac{\partial c_V}{\partial \rho} \right|_T, \quad (16)$$

$$a_{TT} = \frac{1}{2} \left[ \frac{1}{T} \left( 1 - \rho \frac{\partial}{\partial \rho} \right)_T - \frac{2}{3} \left. \frac{\partial}{\partial T} \right|_\rho \right] c_V. \quad (17)$$

Here,  $c_T$  is the isothermal speed of sound,  $h$  is the enthalpy per particle,  $\alpha$  is the thermal expansion coefficient,  $\kappa_T$  is the bulk modulus, and  $c_V$  is the specific heat at constant volume. We define these quantities in the Appendix. The coefficients  $a_{\alpha\beta}$  with  $\alpha, \beta = (\rho, T)$  are sensitive to conformal symmetry breaking and vanish in the ideal gas limit. A numerical estimate of  $a_{\alpha\beta}$  therefore requires a nontrivial equation of state. As an example we consider a dilute Fermi gas governed by an  $s$ -wave interacting with scattering length  $a_s$ . In the high-temperature limit the trace anomaly is given by [21]

$$\Delta_{\text{Tr}} P = \frac{2\pi}{3m^4 a_s} \frac{\rho^2}{T}, \quad (18)$$

where we employ units  $\hbar = k_B = 1$ . In the limit  $a_s \rightarrow \infty$  the dilute Fermi gas is scale invariant and the trace anomaly vanishes. Using Eq. (18) we find

$$(a_{\rho\rho}, a_{\rho T}, a_{TT}) = \frac{2\pi}{3m^4 T^3 a_s} (T^2, -2\rho T, \rho^2). \quad (19)$$

#### IV. HYDRODYNAMIC TAILS: FORMALISM

In order to study hydrodynamic tails we consider the correlation function of  $\Delta_{\text{Tr}} P$  expanded to second order in  $(\Delta \rho, \Delta T)$ . In statistical field theory it is convenient to start from the symmetrized correlation function

$$G_S^{\mathcal{O}\mathcal{O}}(\omega, \mathbf{k}) = \int d^3x \int dt e^{i(\omega t - \mathbf{k} \cdot \mathbf{x})} \langle \frac{1}{2} \{ \mathcal{O}(t, \mathbf{x}), \mathcal{O}(0, 0) \} \rangle. \quad (20)$$

This function is related to the retarded correlator by the fluctuation-dissipation theorem. For  $\omega \rightarrow 0$  we have

$$G_S(\omega, \mathbf{k}) \simeq -\frac{2T}{\omega} \text{Im} G_R(\omega, \mathbf{k}). \quad (21)$$

At second order in  $(\Delta \rho, \Delta T)$  and at the level of the Gaussian entropy functional the symmetrized correlation function factorizes into a set of two-point functions:

$$\begin{aligned} G_S^{\mathcal{O}\mathcal{O}}(\omega, 0) = & \int \frac{d\omega'}{2\pi} \int \frac{d^3\mathbf{k}}{(2\pi)^3} [2a_{\rho\rho}^2 \Delta_S^{\rho\rho}(\omega', \mathbf{k}) \Delta_S^{\rho\rho}(\omega - \omega', \mathbf{k}) \\ & + a_{\rho T}^2 \Delta_S^{\rho\rho}(\omega', \mathbf{k}) \Delta_S^{TT}(\omega - \omega', \mathbf{k}) \\ & + 2a_{TT}^2 \Delta_S^{TT}(\omega', \mathbf{k}) \Delta_S^{TT}(\omega - \omega', \mathbf{k})], \end{aligned} \quad (22)$$

where  $\Delta_S^{\rho\rho}$  is the symmetrized density correlation function

$$\Delta_S^{\rho\rho}(\omega, \mathbf{k}) = \int d^3x \int dt e^{i(\omega t - \mathbf{k} \cdot \mathbf{x})} \langle \frac{1}{2} \{ \rho(t, \mathbf{x}), \rho(0, 0) \} \rangle \quad (23)$$

and  $\Delta_S^{TT}$  is the temperature correlation function. Note that by working with  $(\Delta T, \Delta \rho)$  we avoid off-diagonal correlation functions such as  $\Delta_S^{\rho T}$ . Also note that in hydrodynamics the symmetrized function  $\Delta_S$  reduces to the statistical correlation function.

The Kubo relation involves the retarded, not the symmetrized, correlation function. We can reconstruct the retarded function using the fluctuation-dissipation relation (21). Consider the first term in Eq. (22). At low frequency the contribution to  $G_R$  can be written as [10,11]

$$\begin{aligned} G_R^{\mathcal{O}\mathcal{O}}(\omega, 0)|_{\rho\rho} = & 2a_{\rho\rho}^2 \int \frac{d\omega'}{2\pi} \int \frac{d^3\mathbf{k}}{(2\pi)^3} [\Delta_R^{\rho\rho}(\omega', \mathbf{k}) \Delta_S^{\rho\rho}(\omega - \omega', \mathbf{k}) \\ & + \Delta_S^{\rho\rho}(\omega', \mathbf{k}) \Delta_R^{\rho\rho}(\omega - \omega', \mathbf{k})]. \end{aligned} \quad (24)$$

This is an example of a more general relation that one can prove using hydrodynamic effective actions, which shows that the retarded correlation functions can be derived using a perturbative expansion based on a combination of retarded and symmetrized propagators [8,9,24–29].

The two-point functions of the temperature and density in first-order dissipative hydrodynamics are well known [1]. The temperature correlation function is dominated by a diffusive heat wave. The symmetric and retarded correlation functions are

$$\Delta_S^{TT}(\omega, \mathbf{k}) = \frac{2T^2}{c_P} \frac{D_T \mathbf{k}^2}{\omega^2 + (D_T \mathbf{k}^2)^2}, \quad (25)$$

$$\Delta_R^{TT}(\omega, \mathbf{k}) = \frac{T}{c_P} \frac{-D_T \mathbf{k}^2}{-i\omega + D_T \mathbf{k}^2}, \quad (26)$$

where  $c_P$  is the specific heat at constant pressure,  $D_T = \kappa/c_P$  is the thermal diffusion constant, and  $\kappa$  is the thermal conductivity. The two-point function of the density is more complicated, because the density couples to both propagating sound modes and diffusive heat modes. The symmetric correlation function is [1]

$$\begin{aligned} \Delta_S^{\rho\rho}(\omega, \mathbf{k}) = & 2\rho T \left\{ \frac{\Gamma k^4}{(\omega^2 - c_s^2 k^2)^2 + (\Gamma \omega k^2)^2} \right. \\ & + \frac{\Delta c_P}{c_s^2} \frac{D_T \mathbf{k}^2}{\omega^2 + (D_T k^2)^2} \\ & \left. - \frac{\Delta c_P}{c_s^2} \frac{(\omega^2 - c_s^2 k^2) D_T \mathbf{k}^2}{(\omega^2 - c_s^2 k^2)^2 + (\Gamma \omega k^2)^2} \right\}, \end{aligned} \quad (27)$$

where  $k^2 = \mathbf{k}^2$ ,  $c_s$  is the speed of sound, and  $\Delta c_P = (c_P - c_V)/c_V$ . We have also defined the sound attenuation constant

$$\Gamma = \frac{4}{3} \frac{\eta}{\rho} + \frac{\zeta}{\rho} + \kappa \left( \frac{1}{c_V} - \frac{1}{c_P} \right) = \frac{4}{3} \frac{\eta}{\rho} \left[ 1 + \frac{3}{4} \frac{\zeta}{\eta} + \frac{3}{4} \frac{\Delta c_P}{P_r} \right], \quad (28)$$

where  $P_r = (c_P \eta)/(\rho \kappa)$  is the Prandtl number, the ratio of the momentum and thermal diffusion constants. At high

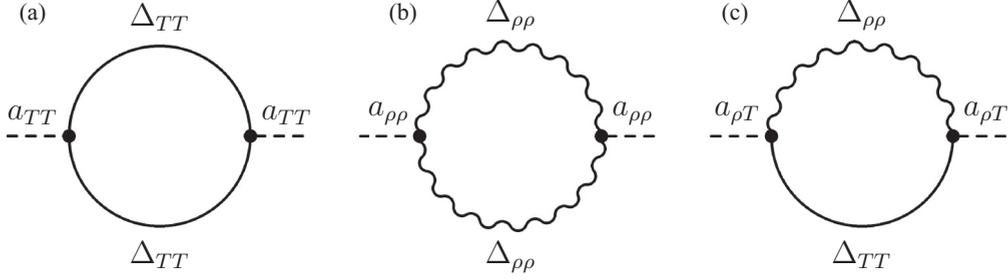


FIG. 1. Diagrammatic representation of the leading contribution of thermal fluctuations to the bulk stress correlation function. The dashed line corresponds to the operator  $\mathcal{O} = P - \frac{2}{3}\mathcal{E}$ . Solid lines denote the diffusive temperature correlator and wavy lines denote the density correlation function, determined by the sound pole and the diffusive heat mode.

temperature  $\Delta c_p = 2/3$  and  $\text{Pr} = 2/3$  [30], and at low temperature  $\Delta c_p/\text{Pr} \rightarrow 0$ .

The two-point function of the density has a complicated pole structure and the calculation of loop diagrams can be simplified by separating the different terms. We will also separate the contributions from sound and diffusive modes,

$$\Delta_{R,S}^{\rho\rho}(\omega, \mathbf{k}) = \Delta_{R,S}^{sd}(\omega, \mathbf{k}) + \Delta_{R,S}^{ht}(\omega, \mathbf{k}) + \Delta_{R,S}^m(\omega, \mathbf{k}). \quad (29)$$

In the long-wavelength limit the sound contribution can be written as

$$\Delta_S^{sd}(\omega, \mathbf{k}) = \rho T \frac{\Gamma k^3}{2\omega c_s} \left\{ \frac{1}{(\omega - c_s k)^2 + (\frac{\Gamma}{2}k^2)^2} - \frac{1}{(\omega + c_s k)^2 + (\frac{\Gamma}{2}k^2)^2} \right\}, \quad (30)$$

$$\Delta_R^{sd}(\omega, \mathbf{k}) = \rho \frac{\Gamma k}{2c_s} \left\{ \frac{1}{\omega - c_s k + i\frac{\Gamma}{2}k^2} - \frac{1}{\omega + c_s k + i\frac{\Gamma}{2}k^2} \right\} \quad (31)$$

and the diffusive heat mode is

$$\Delta_S^{ht}(\omega, \mathbf{k}) = 2\rho T \frac{\Delta c_p}{c_s^2} \frac{D_T \mathbf{k}^2}{\omega^2 + (D_T k^2)^2}, \quad (32)$$

$$\Delta_R^{ht}(\omega, \mathbf{k}) = \rho \frac{\Delta c_p}{c_s^2} \frac{-D_T \mathbf{k}^2}{-i\omega + D_T k^2}. \quad (33)$$

Finally, there is a term that is sensitive to both sound and diffusive modes

$$\Delta_S^m(\omega, \mathbf{k}) = -2\rho T \frac{\Delta c_p}{c_s^2} \frac{k D_T}{2c_s} \left\{ \frac{\omega - c_s k}{(\omega - c_s k)^2 + (\frac{\Gamma}{2}k^2)^2} - \frac{\omega + c_s k}{(\omega + c_s k)^2 + (\frac{\Gamma}{2}k^2)^2} \right\}, \quad (34)$$

$$\Delta_R^m(\omega, \mathbf{k}) = \rho \frac{\Delta c_p}{c_s^2} \frac{i\omega k D_T}{2c_s} \left\{ \frac{1}{\omega - c_s k + i\frac{\Gamma}{2}k^2} - \frac{1}{\omega + c_s k + i\frac{\Gamma}{2}k^2} \right\}. \quad (35)$$

## V. HYDRODYNAMIC TAILS: ONE-LOOP DIAGRAMS

In this section we will compute the leading infrared behavior of the three one-loop diagrams shown in Fig. 1. The two-point function of the density has three distinct contributions, see Eq. (29), and as a result there are ten one-loop diagrams total. As we will see, only four of them contribute to the low-frequency behavior of  $G_R(\omega, \mathbf{0})$ .

(1) The simplest diagram involves diffusive fluctuations of the temperature only. We consider Eq. (24) with  $(\rho\rho) \rightarrow (TT)$  and use the retarded and symmetrized functions given in Eqs. (25) and (26). We perform the frequency integral by closing the contour in the complex  $\omega$  plane. We find

$$G_R^{\mathcal{O}\mathcal{O}}(\omega, \mathbf{0})|_{TT}^{ht} = -\frac{2a_{TT}^2 T^3}{c_p^2} \int \frac{d^3 k}{(2\pi)^3} \frac{k^2}{k^2 - \frac{i\omega}{2D_T}}, \quad (36)$$

where  $TT$  refers to the presence of two temperature correlation functions and  $ht$  indicates that these modes are dominated by a diffusive heat mode. The integral in Eq. (36) is ultraviolet divergent. We will regularize the integral using a momentum cutoff  $\Lambda$ . We will see that there are two types of terms. Hydrodynamic tails are nonanalytic in  $\omega$  and independent of the cutoff. Fluctuation terms are sensitive to the cutoff and contribute to  $G_R$  in the same way as transport coefficients. This implies that the cutoff dependence can be absorbed into the bare transport parameters. However, we will see that this procedure implies bounds on the transport coefficients.

After introducing a cutoff we can compute the integral in Eq. (36) by expanding in  $\omega$ . The leading terms are

$$G_R^{\mathcal{O}\mathcal{O}}(\omega, \mathbf{0})|_{TT}^{ht} = -\frac{2a_{TT}^2 T^3}{c_p^2} L(\omega, \Lambda, 2D_T), \quad (37)$$

where we have defined

$$L(\omega, \Lambda, 2D_T) = \frac{1}{2\pi^2} \left\{ \frac{\Lambda^3}{3} + \frac{i\omega\Lambda}{2D_T} - \frac{\pi}{2\sqrt{2}}(1+i) \left( \frac{\omega}{2D_T} \right)^{3/2} + \dots \right\}. \quad (38)$$

Note that the small parameter in the low-frequency expansion is  $\epsilon \equiv \omega/(D_T \Lambda^2)$ . We observe that the  $\Lambda^3$  term can be viewed as a contribution to the compressibility term in Eq. (6) and the  $i\omega\Lambda$  term is a contribution to the bulk viscosity. This term is sensitive to scale breaking via the coefficient  $a_{TT}$  and it scales inversely with the thermal conductivity. The last term

is a hydrodynamic tail. The imaginary part can be viewed as a  $\sqrt{\omega}$  contribution to the frequency-dependent bulk viscosity  $\zeta(\omega)$  and the real part is a  $1/\sqrt{\omega}$  contribution the bulk viscosity relaxation time. This term signals the breakdown of second-order deterministic fluid dynamics in the low-frequency limit.

(2) A similar diffusive heat contribution appears in the two point function of the density. Comparing Eqs. (25) and (26) to Eqs. (32) and (33) we observe that this contribution is equal to the previous term up to an overall factor. We get

$$G_R^{OO}(\omega, \mathbf{0})|_{\rho\rho}^{ht} = -\frac{2a_{\rho\rho}^2 T \rho^2 (\Delta c_P)^2}{c_s^2} L(\omega, \Lambda, 2D_T). \quad (39)$$

In the case of a dilute gas Eq. (39) and Eq. (37) are comparable in magnitude, but in general the two contributions can be different.

(3) Another diffusive heat contribution is contained in the mixed  $\Delta_{\rho\rho} \Delta_{TT}$  term, shown as the third diagram in Fig. 1. We get

$$G_R^{OO}(\omega, \mathbf{0})|_{\rho T}^{ht} = -\frac{a_{\rho T}^2 \rho T^2 \Delta c_P}{c_P c_s^2} L(\omega, \Lambda, 2D_T). \quad (40)$$

(4) The two point function of the density also contains a sound contribution. This term is quite different, because sound is a propagating mode and sound attenuation is controlled by  $\Gamma$ , which is not only sensitive to  $\kappa$  but also to the shear viscosity  $\eta$  and a possible microscopic contribution to  $\zeta$ . We determine this term using the two point functions in Eqs. (30) and (31). We observe that there are two types of contributions, characterized by the relative sign of the real part of the pole position,  $\omega_{\pm}' = \pm c_s k + O(\omega, k^2)$ . We first consider diagrams where the poles are on opposite sides of the real axis. We get

$$G_R^{OO}(\omega, \mathbf{0})|_{\rho\rho}^{sd} = -\frac{a_{\rho\rho}^2 T \rho^2}{c_s^4} L(\omega, \Lambda, \Gamma), \quad (41)$$

where the index *sd* indicates the contribution from the sound mode. The diagram where the two poles are on the same side gives

$$\begin{aligned} G_R^{OO}(\omega, \mathbf{0})|_{\rho\rho}^{sd} \\ = -\frac{a_{\rho\rho}^2 T \rho^2}{4c_s^2} \int \frac{d^3 k}{(2\pi)^3} \frac{k^2}{(\omega - 2c_s k + i\Gamma k^2)(c_s k - i\frac{\Gamma k^2}{2})}. \end{aligned} \quad (42)$$

This integral is UV divergent, but it is less IR sensitive than Eq. (36). In particular, the low-frequency behavior is governed by  $c_s k \gg \Gamma k^2$ . As a result, the contribution to the  $i\omega$  term in  $G_R^{OO}(\omega, \mathbf{0})$  is suppressed by a factor  $(\Gamma \Lambda / c_s)$  relative to Eq. (37).

(5) The remaining diagrams fall into two categories. The first class involves mixed diagrams in which a diffusive heat mode is coupled to a propagating sound mode. These diagrams are suppressed because if one of the propagators is put on shell the other propagator is far off shell, and the diagram is not infrared sensitive. The other diagrams involve the mixed sound-heat propagator in Eq. (34). The on-shell residue of this propagator is suppressed. We finally collect the contributions from Eqs. (37)–(43). We get

$$G_R^{OO}(\omega, \mathbf{0}) = -A_T L(\omega, \Lambda, 2D_T) - A_{\Gamma} L(\omega, \Lambda, \Gamma), \quad (43)$$

where we have defined

$$\begin{aligned} A_T &= \frac{2a_{TT}^2 T^3}{c_p^2} + \frac{2a_{\rho\rho}^2 \rho^2 T (\Delta c_P)^2}{c_s^4} + \frac{a_{\rho T}^2 \rho T^2 \Delta c_P}{c_P c_s^2}, \\ A_{\Gamma} &= \frac{a_{\rho\rho}^2 \rho^2 T}{c_s^4}. \end{aligned} \quad (44)$$

## VI. PHENOMENOLOGICAL ESTIMATES

### A. Hydrodynamic tail

In the previous section we showed that the  $\omega^{3/2}$  term in the retarded correlation function is uniquely determined in terms of the equation of state and the transport parameters. This term has several physical effects: it determines the long-time tail of the correlation function, it governs the small frequency limit of the bulk viscosity spectral function, and it determines the  $\omega \rightarrow 0$  divergence in the relaxation time. We first consider the correlation function

$$C_{\zeta}(t) = \int \frac{d\omega}{2\pi} G_S^{OO}(\omega, \mathbf{0}) e^{-i\omega t}. \quad (45)$$

For  $t \rightarrow \infty$  we obtain a  $t^{-3/2}$  tail

$$C_{\zeta}(t) = \frac{T}{4\pi^{3/2}} \left( \frac{A_T}{(2D_T)^{3/2}} + \frac{A_{\Gamma}}{\Gamma^{3/2}} \right) \frac{1}{t^{3/2}}. \quad (46)$$

This contribution is computed most easily by starting from the momentum integral in Eq. (36), and then performing the frequency integral before the momentum integral. The hydrodynamic tail in the bulk stress correlator was first computed by Pomeau and Résibois [4], but their result does not appear to be correct. In particular, the expression for  $C_{\zeta}(t)$  given in [4] does not vanish for a scale-invariant fluid. In our work  $C_{\zeta}(t) \sim a_{\alpha\beta}^2 \sim (\Delta_{TP})^2$  automatically vanishes for a scale-invariant fluid.

The contribution of critical fluctuations to the tail in the bulk stress correlation function was computed by Onuki [8], both in model H (liquid-gas end point) and model F (superfluid transition). In principle the model F result for  $T > T_c$  is directly applicable to the Fermi gas near unitarity. Model F contains two hydrodynamic variables, a linear combination of the energy density  $\mathcal{E}$  and the density  $\rho$ , as well as the superfluid density. Above  $T_c$  only the energy density–like variable contributes. In this regime there are two differences compared to our analysis: (1) we keep both  $\mathcal{E}$  and  $\rho$ ; (2) the model F analysis uses a more complicated functional form of the thermal conductivity  $\kappa(k^2, t)$  with  $t = (T - T_c)/T_c$ , which reduces to a simple constant for  $t \gg 1$ . This implies that the model F tail should be similar to our tail for large  $t$ . This is difficult to verify, because the coupling between the energy density–like variable to the bulk stress does not manifestly respect scale invariance. The bulk tail in a relativistic nonconformal fluid was computed by Kovtun and Yaffe [9]. These authors assume that the mean density of the fluid vanishes, so that we cannot directly compare to the nonrelativistic limit.

### B. Spectral function

A second quantity of interest is the spectral function

$$\zeta(\omega) = -\frac{1}{9\omega} \text{Im} G_R^{OO}(\omega, \mathbf{0}). \quad (47)$$

The existence of a hydrodynamic tail implies that

$$\zeta(\omega) = \zeta(0) - \left( \frac{A_T}{(2D_T)^{3/2}} + \frac{A_\Gamma}{\Gamma^{3/2}} \right) \frac{\sqrt{\omega}}{36\sqrt{2}\pi}. \quad (48)$$

This result can be combined with other model independent information about the spectral function. The high-frequency tail of the bulk viscosity was determined using the operator product expansion [31]

$$\zeta(\omega) = \frac{\mathcal{C}}{36\pi\sqrt{m\omega}} \frac{1}{1 + a_s^2 m\omega}, \quad (49)$$

where  $\mathcal{C}$  is the contact density [32,33]. The contact density is directly related to the trace anomaly near unitarity

$$\Delta_{\text{Tr}} P = \frac{\mathcal{C}}{12\pi m a_s}. \quad (50)$$

In the high-temperature limit  $\mathcal{C}$  can be computed using the virial expansion [34]. Near unitarity we find

$$\mathcal{C} = 4\pi n^2 \lambda^2 \left\{ 1 + \frac{1}{\sqrt{2}} \left( \frac{\lambda}{a_s} \right) + \dots \right\}, \quad (51)$$

which implies

$$\zeta(\omega) \sim \lambda^{-3} \left( \frac{z\lambda}{a_s} \right)^2 \left( \frac{T}{\omega} \right)^{3/2}, \quad (52)$$

where  $\lambda = [(2\pi)/(mT)]^{1/2}$  is the thermal de Broglie wavelength and  $z = \frac{1}{2} n \lambda^3$  is the fugacity of the gas. Finally, there is a sum rule for the bulk viscosity spectral function [35–37],

$$\frac{1}{\pi} \int d\omega \zeta(\omega) = \frac{1}{72\pi m a^2} \left. \frac{\partial \mathcal{C}}{\partial a_s^{-1}} \right|_{s/n}. \quad (53)$$

In the next section we will combine these constraints with the fluctuation bound to provide a simple model of the viscosity spectral function.

### C. Fluctuation bound

The cutoff dependent term in the bulk viscosity,

$$\zeta_\Lambda = \frac{1}{18\pi^2} \left( \frac{A_T \Lambda}{2D_T} + \frac{A_\Gamma \Lambda}{\Gamma} \right), \quad (54)$$

has to combine with the bare bulk viscosity to determine the physical bulk viscosity of the fluid. We can view this result as arising from a renormalization-group procedure, where fluid dynamics is matched to a microscopic theory at the scale  $\Lambda$ , and then the evolution of  $G_R(\omega)$  below the scale  $\Lambda$  is computed using stochastic fluid dynamics. For this procedure to be consistent the bare viscosity at the cutoff scale must be positive, and the physical viscosity must be larger than  $\zeta_\Lambda$ . This bound increases with the cutoff scale  $\Lambda$ . The largest possible  $\Lambda$  is determined by the breakdown scale of fluid dynamics, because above that scale stochastic fluid dynamics is not reliable. Of course, the viscosity at the cutoff scale must depend on  $\Lambda$ , so that the physical viscosity  $\zeta(0)$  is cutoff independent. The

same conclusion also follows from the spectral density given in Eq. (48). We observe that the nonanalytic  $\sqrt{\omega}$  term is negative. If this term is the dominant correction to the spectral density below the breakdown scale of fluid dynamics,  $\omega \lesssim \omega_{br}$ , then spectral positivity implies that  $\zeta(0)$  cannot be arbitrarily small.

In order to determine the maximum momentum where fluid dynamics can be trusted we can study the dispersion relation of diffusive heat modes and propagating sound waves, and determine the maximum momentum for which higher-order corrections are small compared to leading-order terms.

(1) Diffusive modes: heat modes are characterized by  $\omega \sim D_T k^2$ . Corrections arise from higher-order terms in the derivative expansion. For nonzero frequency the leading correction is due to the relaxation time. We get  $\omega \sim D_T k^2 \ll \tau_\kappa^{-1}$ . For this relation to be maintained for all  $k < \Lambda$  we have to require that  $\Lambda \lesssim \Lambda_T$  with  $\Lambda_T = (\tau_\kappa D_T)^{-1/2}$ . In kinetic theory  $\tau_\kappa = (m\kappa)/(c_P T)$  and

$$\Lambda_T \simeq \frac{1}{D_T} \left( \frac{T}{m} \right)^{1/2}. \quad (55)$$

Equation (55) implies that the expansion parameter of the low-frequency expansion,  $\epsilon = \omega/(D_T \Lambda^2)$ , is of order  $\epsilon \sim (m D_T)(\omega/T)$ . For a nearly perfect fluid  $D_T \sim m^{-1}$  [38] and the low-frequency expansion is valid all the way up to  $\omega \sim T$ . In the case of a poor fluid  $D_T \gg m^{-1}$  and the range of validity of the low-frequency expansion is smaller. We also note that Eq. (55) ensures that the expansion parameter  $(D_T \Lambda/c_s)$  is indeed small.

(2) Sound channel: in the sound channel we have  $\omega \sim c_s k \ll \Gamma k^2$ . This implies  $k \lesssim \Lambda_\Gamma$  with

$$\Lambda_\Gamma \simeq \frac{1}{\Gamma} \left( \frac{\partial P}{\partial \rho} \right)_{s/n}^{1/2}. \quad (56)$$

For a weakly interacting gas we get  $(\partial P)/(\partial \rho)_{s/n} \simeq (5T)/(3m)$ . We can either use the two estimates Eqs. (55) and (56) in the respective channels, or use the smaller of the two values. In the weak-coupling limit, where  $\text{Pr} \sim 1$ , these two estimates are numerically very similar. Using the first method, we obtain the bound

$$\zeta_{\min} = \left( \frac{A_T}{2D_T} + \frac{\sqrt{5}A_\Gamma}{\sqrt{3}\Gamma^2} \right) \sqrt{\frac{T}{m}}. \quad (57)$$

We observe that there is a minimum value of  $\zeta$  that is solely controlled by  $(\Delta_{\text{Tr}} P/D_T)^2$  and  $(\Delta_{\text{Tr}} P/\Gamma)^2$ . This implies that if there is scale breaking in the equation of state, and if the shear viscosity and thermal conductivity are finite, then the bulk viscosity cannot be zero. Fluctuation bounds on the shear viscosity were studied in [11,12]. We observe that the bound on  $\zeta$  has the same structure as the bound on  $\eta$ , but is suppressed by a factor  $(\Delta_{\text{Tr}} P/P)^2$ .

Finally, we provide some numerical estimates. For this purpose we assume that the bare bulk viscosity is zero and that the shear viscosity and thermal conductivity are described by kinetic theory,  $\eta = \eta_0(mT)^{3/2}$  and  $\kappa = \kappa_0 m^{1/2} T^{3/2}$  with  $\eta_0 = 15/(32\sqrt{\pi})$  and  $\kappa_0 = 225/(128\sqrt{\pi})$  [30,39]. In the case of the shear viscosity this is known to be a good approximation even close to the critical temperature [40]. We also use the results for  $c_s^2$ ,  $c_P$ , and  $\Delta_{C_P}$  in the dilute limit; see the Appendix.

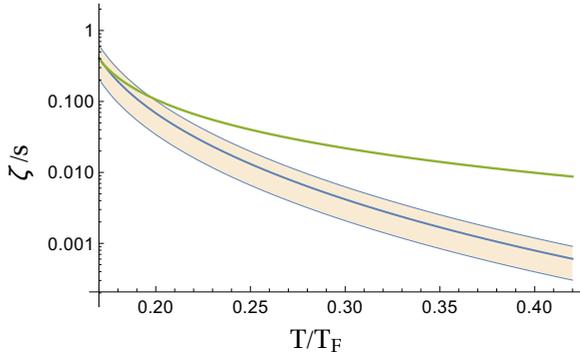


FIG. 2. Fluctuation bound (blue line) on  $\zeta/s$  for the dilute Fermi gas as a function of  $T/T_F$ . We show the regime  $T > T_c$  with  $T_c/T_F \simeq 0.17$ . As explained in the text we estimate the equation of state and transport properties using results in the high-temperature limit. We have also chosen  $a_s/\lambda = 1$ . The error band corresponds to a 50% error in  $\Lambda_T$  and  $\Lambda_T$ . For comparison we show the kinetic theory result for  $\zeta/s$  as the green line.

The bound on  $\zeta/s$  as a function of  $T/T_F$  is shown in Fig. 2. The width of the band reflects a 50% error related to the choice of  $\Lambda$ . For comparison we also show the kinetic theory result  $\zeta/n = z^2/(24\sqrt{2}\pi\lambda^3)(\lambda/a_s)^2$  [21]. At high temperature the fluctuation bound is very small, but near  $T_c \simeq 0.17T_F$  [41] the bound is comparable to the kinetic theory result, indicating that the bulk viscosity must be at least as big as predicted by kinetic theory. Note that we have extrapolated the bound on  $\zeta/s$  all the way to  $T_c$ , despite the fact that several estimates involve approximations that are only reliable for  $T \gg T_c$ . Similar to the kinetic theory estimates discussed above, it is known that in the case of  $\eta/s$  this procedure provides a numerically accurate estimate of the bound near  $T_c$ .

We note that  $\zeta/s$  is given in units of  $\hbar/k_B$ . Both the hydrodynamic and the kinetic theory calculation are completely classical. Planck's constant enters the hydrodynamic calculation via the equation of state, and it appears in the kinetic theory calculation in terms of both the equation of state and the quasiparticle dispersion relation.

In Fig. 3 we summarize the available information on the spectral function  $\zeta(\omega)$ . We plot  $\zeta(\omega)/s$  as a function of  $\omega/T$ . For small  $\omega$  we show the hydrodynamic prediction in Eq. (48) where  $\zeta(0)$  is assumed to be the fluctuation bound. For large  $\omega$  we show the tail predicted by the operator product expansion; see Eq. (49). We have chosen  $T/T_F = 0.2$  and  $\lambda/a_s = 1$ . We conclude that a smooth extrapolation of the large frequency tail to  $\omega = 0$  is consistent with a bulk viscosity  $\zeta(0)$  which is somewhat larger than the fluctuation bound. As an example we show the green dotted line which corresponds to  $\zeta = \zeta_{\min} + \zeta_{\text{micro}} - c\sqrt{\omega}$  with  $\zeta_{\text{micro}}/s = 0.04$  and  $c$  given by Eq. (49). This function smoothly matches the high-frequency tail. Integrating the low-frequency model and the high-frequency tail over the entire range  $\omega \in [0, \infty]$  saturates 65% of the sum rule in Eq. (53). We conclude that a reasonable model of the bulk viscosity spectral function can be obtained by matching the high-frequency tail to the hydrodynamic spectral function combined with a small microscopic viscosity.

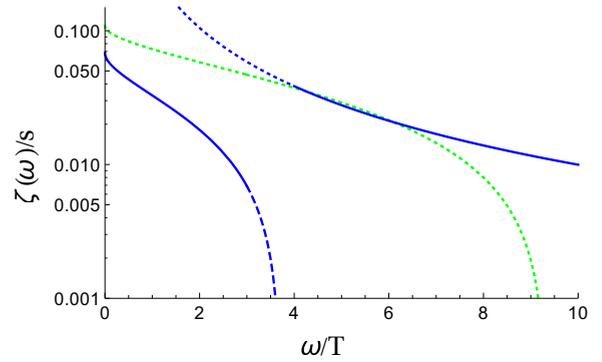


FIG. 3. Schematic form of the bulk viscosity spectral function. This figure shows  $\zeta/s$  as a function of the frequency  $\omega$  in units of  $T$ . We have chosen  $a_s/\lambda = 1$  and  $T/T_F = 0.2$ . The low-frequency part shows the function  $\zeta(\omega) = \zeta_{\min} - c\sqrt{\omega}$ , where  $\zeta_{\min}$  is the bound in Eq. (57) and  $c$  is the universal coefficient given in Eq. (48). The high-frequency part is the function given in Eq. (49). The green dotted line shows a model for the low-frequency spectral function where we have added a microscopic contribution  $\zeta_{\text{micro}}/s = 0.04$  to the hydrodynamic result. The microscopic contribution was chosen to smoothly match the high-frequency tail.

## VII. OUTLOOK

In this work we have studied the role of hydrodynamic fluctuations in the bulk stress correlation function. We have shown that fluctuations provide a lower bound on the bulk viscosity that only depends on the thermal conductivity and shear viscosity as well as scale breaking in the equation of state. The physical mechanism for the bound can be understood in terms of the rate of equilibration of thermal fluctuations. Consider a fluid in equilibrium at density  $\rho$  and temperature  $T$ . Fluctuations in this fluid are controlled by the entropy functional in Eq. (13). If the fluid is compressed then the equilibrium density and temperature change, and as a result the mean-square fluctuations in  $\rho, T, \mathbf{v}$  have to change as well. However, the mechanism for fluctuations to adjust involves diffusion of heat and momentum, and does not take place instantaneously. As a consequence the fluid is slightly out of equilibrium, entropy increases, and the effective bulk viscosity is not zero. This mechanism is particularly relevant in fluids which have no significant microscopic sources of bulk viscosity.

An example of a very good fluid that does not have a simple microscopic mechanism for generating bulk viscosity is the dilute Fermi gas near unitarity. Our estimates indicate that the ratio of bulk viscosity to entropy density near the phase transition and for  $|\lambda/a_s| \gtrsim 1$  is  $\zeta/s \gtrsim 0.1$ . This is within reach of experiments involving hydrodynamic expansion [42]. The effects might be even more significant in two-dimensional gases. In these systems bulk viscosity has been studied using the damping of monopole oscillations [43,44]. It may also be possible to observe the nonanalyticity of the spectral function or the long-time tail in the Kubo integrand using numerical simulations [45].

Our work can be extended in several directions. One interesting question is the role of critical fluctuations in the vicinity of a second-order phase transition [7,8]. In that case loop diagrams similar to the graphs studied in this work lead

to an enhancement of the bulk viscosity near the critical point. Another important problem is the study of fluctuations in relativistic fluids; see [9,10,46,47]. In that case it has been conjectured that the quark gluon plasma phase transition has a critical end point which is in the universality class of model H [26,48], and that critical fluctuations can be observed in the relativistic heavy-ion collisions [49].

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#### APPENDIX: THERMODYNAMIC QUANTITIES

We assume that the equation of state is given in the form  $P = P(\mu, T)$ . A specific example is the virial expansion which provides the equation of state in the form

$$P = \frac{\nu T}{\lambda^3} (z + b_2(T)z^2 + \dots), \quad (\text{A1})$$

where  $\nu$  is the number of degrees of freedom ( $\nu = 2$  in the unitary Fermi gas),  $\lambda = [(2\pi)/(mT)]^{1/2}$  is the thermal wavelength, and  $z = \exp(\mu/T)$  is the fugacity. Note that we have set  $\hbar = k_B = 1$ . Near unitarity  $b_2 = b_2^0 + \delta b_2$ , where  $b_2^0 = -1/(4\sqrt{2})$  is due to quantum statistics and [21]

$$\delta b_2 = \frac{1}{\sqrt{2}} \left( 1 + \frac{2}{\sqrt{\pi m T a_s}} + \dots \right). \quad (\text{A2})$$

Derivatives of the pressure with respect to  $\mu$  and  $T$  determine the entropy density and pressure

$$s = \left. \frac{\partial P}{\partial T} \right|_{\mu}, \quad n = \left. \frac{\partial P}{\partial \mu} \right|_T. \quad (\text{A3})$$

The energy density is determined by the relation

$$\mathcal{E} = \mu n + sT - P \quad (\text{A4})$$

and the enthalpy per particle is  $h = (\mathcal{E} + P)/n$ . In order to compute the specific heat at constant volume we use  $V = N/n$  and write

$$\begin{aligned} c_V &= \left. \frac{T}{V} \frac{\partial S}{\partial T} \right|_V = \frac{\partial(s, V)}{\partial(T, V)} = \frac{\partial(s, V)/\partial(T, \mu)}{\partial(T, V)/\partial(T, \mu)} \\ &= T \left[ \left. \frac{\partial s}{\partial T} \right|_{\mu} - \frac{[(\partial n/\partial T)|_{\mu}]^2}{(\partial n/\partial \mu)|_T} \right], \end{aligned} \quad (\text{A5})$$

where we have defined the Jacobian

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}. \quad (\text{A6})$$

In order to compute  $c_P$  we make use of the relation between  $c_P - c_V$  and the thermal expansion coefficient  $\alpha = (1/V)(\partial V/\partial T)|_P$ . This relation is given by

$$c_P - c_V = -\frac{T}{V} \frac{[(\partial V/\partial T)|_P]^2}{(\partial V/\partial P)|_T}. \quad (\text{A7})$$

The partial derivatives are

$$\begin{aligned} \left. \frac{1}{V} \frac{\partial V}{\partial T} \right|_P &= \frac{1}{n} \left[ \left. \frac{s}{n} \frac{\partial n}{\partial \mu} \right|_T - \left. \frac{\partial n}{\partial T} \right|_{\mu} \right], \\ \left. \frac{1}{V} \frac{\partial V}{\partial P} \right|_T &= -\left. \frac{1}{n^2} \frac{\partial n}{\partial \mu} \right|_T. \end{aligned} \quad (\text{A8})$$

The second of these relations defines the bulk modulus  $\kappa_T^{-1} = -V^{-1}(\partial V)/(\partial P)|_T$ . We get

$$c_P = c_V + T \frac{[\frac{s}{n}(\partial n/\partial \mu)|_T - (\partial n/\partial T)|_{\mu}]^2}{(\partial n/\partial \mu)|_T}. \quad (\text{A9})$$

The isothermal and the adiabatic speed of sound are defined by

$$c_T^2 = \left. \frac{\partial P}{\partial \rho} \right|_T, \quad c_s^2 = \left. \frac{\partial P}{\partial \rho} \right|_{s/n}. \quad (\text{A10})$$

We have

$$c_T^2 = \frac{n}{m} \left[ \left. \frac{\partial n}{\partial \mu} \right|_T \right]^{-1}, \quad c_s^2 = \frac{c_P}{c_V} c_T^2, \quad (\text{A11})$$

and the thermal expansion coefficient can be written as

$$\alpha = \frac{1}{T} \left[ \frac{1}{c_T^2} \frac{T}{m} \frac{c_P - c_V}{n} \right]^{1/2}. \quad (\text{A12})$$

Finally, we can determine the first-order derivatives that appear in the expansion in Eq. (14). We get

$$\begin{aligned} \left. \frac{\partial P}{\partial \rho} \right|_T &= c_T^2, \quad \left. \frac{\partial \mathcal{E}}{\partial \rho} \right|_T = \frac{h}{m} - \frac{\alpha \kappa_T T}{\rho^2}, \quad \left. \frac{\partial P}{\partial T} \right|_{\rho} = \alpha \kappa_T, \\ \left. \frac{\partial \mathcal{E}}{\partial T} \right|_{\rho} &= c_V, \end{aligned} \quad (\text{A13})$$

where  $h = (\mathcal{E} + P)/n$  is the enthalpy per particle. Partial derivatives of these results with respect to  $T$  and  $\rho$  determine the second-order coefficients in Eqs. (15)–(17).

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