Quantifying entanglement in two-mode Gaussian states

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Entangled two-mode Gaussian states are a key resource for quantum information technologies such as teleportation, quantum cryptography, and quantum computation, so quantification of Gaussian entanglement is an important problem. Entanglement of formation is unanimously considered a proper measure of quantum correlations, but for arbitrary two-mode Gaussian states no analytical form is currently known. In contrast, logarithmic negativity is a measure that is straightforward to calculate and so has been adopted by most researchers, even though it is a less faithful quantifier. In this work, we derive an analytical lower bound for entanglement of formation of generic two-mode Gaussian states, which becomes tight for symmetric states and for states with balanced correlations. We define simple expressions for entanglement of formation in physically relevant situations and use these to illustrate the problematic behavior of logarithmic negativity, which can lead to spurious conclusions.

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I. INTRODUCTION

Entanglement is a nonclassical physical property, emerging from the quantum mechanical superposition principle. Theoretically, it can be described as the inability to separate a global quantum state of a composite system into a product of individual subsystems. Experimentally, it is manifested as the correlations of the observables of different subsystems, which cannot be classically reproduced.

In order to quantify entanglement of bipartite systems, we employ the axiomatic theory of entanglement measures [1,2], where an entanglement measure, $\mathcal{E}$, should satisfy the following postulates: (i) $\mathcal{E}$ vanishes on separable states, and (ii) $\mathcal{E}$ does not increase on average under local operations and classical communication (strong monotonicity). Besides the above postulates, there are several other mathematical properties that it is desirable for $\mathcal{E}$ to satisfy, such as additivity, strong superadditivity, convexity, and asymptotic continuity.

For pure states, entropy of entanglement is the *bona fide* measure of quantum correlations, defined as $\mathcal{E}(|\psi\rangle) := \mathcal{S}(\mathrm{tr}_B|\psi\rangle\langle\psi|)$, where $\mathcal{S}(\rho) := -\mathrm{tr}(\rho\log_2\rho)$ is the von Neumann entropy, and tr_B denotes the partial trace over subsystem B [3]. For mixed states, entanglement can be measured via different quantifiers, which, in general, do not coincide with each other. One of them is entanglement of formation, defined as the convex-roof extension of the von Neumann entropy, $\mathcal{E}_F(\rho) := \inf\{\sum_i p_i \mathcal{S}(\mathrm{tr}_B|\psi_i\rangle\langle\psi_i|)\}$, where the infimum is taken over all ensembles $\{p_i,\psi_i\}$ of $\rho := \sum_i p_i|\psi_i\rangle\langle\psi_i|$ [4]. Specifically, for two-mode Gaussian states, where $\mathcal{E}_F$ has been proven to be additive [5] (and thus strongly superadditive as well [6]), it coincides with the entanglement cost, $\mathcal{E}_C(\rho) := \lim_{n\to\infty}\mathcal{E}_F(\rho^{\otimes n})/n$ [7]. For a given state ρ, entanglement cost has a clear operational meaning, since it quantifies the minimum entanglement needed (cost of quantum resources) to produce ρ [7], which is of great importance in quantum technologies. In discrete-variable bipartite systems, an explicit form of entanglement of formation has been found for generic states (qubits) [8], while in the continuous-variable regime, and specifically for two-mode Gaussian systems, there are only two families for which the entanglement of formation can be analytically calculated: (a) for symmetric states [9] and (b) for nonsymmetric extremal (maximally and minimally) entangled states for fixed global and local purities [10–13]. An explicit form of the measure for arbitrary two-mode Gaussian states is yet considered an open problem.

The inability to define entanglement of formation through an explicit closed form for arbitrary states led researchers to use other, more easily computable measures. Specifically, in two-mode Gaussian systems the most widely used quantifier is the logarithmic negativity, $\mathcal{E}_N(\rho) := \log_2\|\tilde{\rho}\|$, where $\tilde{\rho}$ denotes the partially transposed density matrix ρ, and $\|x\| := \mathrm{tr}\sqrt{x^\dagger x}$ is the trace norm [14–16]. However, unlike $\mathcal{E}_F$, $\mathcal{E}_N$ does not satisfy convexity, asymptotic continuity, and strong superadditivity [1,2,17]. Asymptotic continuity and strong superadditivity are requirements for an entanglement measure to satisfy the widely accepted extremality of Gaussian states; i.e., for a given covariance matrix the entanglement is minimized by Gaussian states [17,18]. Not only does logarithmic negativity fail to satisfy these requirements, but also counterexamples have been found, showing that $\mathcal{E}_N$ can actually defy the extremality of Gaussian states, leading to an overestimation of entanglement [17]. Furthermore, since logarithmic negativity is not asymptotically continuous, it does not reduce to the entropy of entanglement in all pure states [1], which is why it is usually referred to as a monotone, instead of a measure.

In this work we provide a clear physical interpretation of the entanglement of formation and we derive an analytical lower bound of it for arbitrary two-mode Gaussian states, which saturates for symmetric states and for states with balanced correlations. For the rest of the states, the bound provides a measure of necessary correlations needed to construct the state, closely approximating the exact value (computed numerically) of the entanglement of formation. Our approach leads to simple exact expressions for the $\mathcal{E}_F$ of two-mode squeezed states after passage through typical communication channels, which we use to illustrate significant qualitative differences between $\mathcal{E}_N$ and $\mathcal{E}_F$.

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II. GAUSSIAN STATES

We begin by briefly reviewing two-mode Gaussian states [19,20]. Any two-mode state can be fully described by a covariance matrix (assuming for simplicity that its mean value is 0), which in standard form [21,22] is written as

$$\boldsymbol{\sigma}^{\text{sf}} = \begin{bmatrix} A & C \\ C & B \end{bmatrix}, \tag{1}$$

which is a real and positive definite matrix, with $A = \text{diag}(a,a)$, $B = \text{diag}(b,b)$, and $C = \text{diag}(c_1,c_2)$. Its elements are proportional to the second-order moments of the quadrature field operators, $\hat{x}_j := \hat{a}_j + \hat{a}_j^\dagger$ and $\hat{p}_j := i(\hat{a}_j^\dagger - \hat{a}_j)$, where $\hat{a}_j$ and $\hat{a}_j^\dagger$ are the annihilation and creation operators, respectively, with $[\hat{a}_i,\hat{a}_j^\dagger] = \delta_{ij}$. In continuous-variable optical systems entanglement is manifested by the correlations of the field operators $\hat{x}$ and $\hat{p}$, and it is typically created by pumping a nonlinear crystal in a nondegenerate optical parametric amplifier. This process is described by a symplectic map $\boldsymbol{\sigma} \to S_2(r)\boldsymbol{\sigma} S_2^T(r)$, known as the two-mode squeezing operation defined by

$$S_2(r) := \begin{bmatrix} \cosh r\, \mathbb{1} & \sinh r\, Z \\ \sinh r\, Z & \cosh r\, \mathbb{1} \end{bmatrix},$$

where $r \in \mathbb{R}$ is the squeezing parameter and $Z := \text{diag}(1,-1)$. By applying $S_2(r)$ to a couple of vacuums, we obtain a pure state called the two-mode squeezed vacuum, with $a = b = \frac{1+\chi^2}{1-\chi^2}$ and $c_1 = -c_2 = \frac{2\chi}{1-\chi^2}$, where $\chi = \tanh r \in [0,1)$.

For any covariance matrix $\boldsymbol{\sigma}$, there exists a symplectic transformation S, such that $\boldsymbol{\sigma} = S\boldsymbol{\nu}S^T$, with $\boldsymbol{\nu} = \nu_- \mathbb{1} \oplus \nu_+ \mathbb{1}$, where $1 \leqslant \nu_- \leqslant \nu_+$. The quantities ν_i are called symplectic eigenvalues [15]. The necessary and sufficient separability criterion for a two-mode Gaussian state $\boldsymbol{\sigma}$ has been shown to be the positivity of the partial transposed state $\tilde{\boldsymbol{\sigma}}$ [21–23]. This is equivalent to checking the condition $\tilde{\nu}_- \geqslant 1$ [10], where $\tilde{\nu}_-$ is the lowest symplectic eigenvalue of $\tilde{\boldsymbol{\sigma}}$.

III. ENTANGLEMENT OF FORMATION: LOWER BOUND

Any state $\boldsymbol{\sigma}^{\text{sf}}$ can be decomposed (proof can be found in Appendix A) as

$$\boldsymbol{\sigma}^{\text{sf}} = L(r_1,r_2)S_2(r)\boldsymbol{\sigma}_{\text{c}} S_2^T(r)L^T(r_1,r_2), \tag{2}$$

with $L(r_1,r_2) = S(r_1) \oplus S(r_2)$, where $S(r_i) := \text{diag}(e^{-r_i},e^{r_i})$ is the local squeezing symplectic map for each mode, and $\boldsymbol{\sigma}_{\text{c}} \geqslant \mathbb{1}$ is a classical state [see Fig. 1(a)]. We call *optimum* the decomposition with the least two-mode squeezing, r_o, i.e., $\boldsymbol{\sigma}^{\text{sf}} = L(r_{1_o},r_{2_o})S_2(r_o)\boldsymbol{\sigma}_{\text{c}_o} S_2^T(r_o)L^T(r_{1_o},r_{2_o})$. Gaussian entanglement of formation [24], which has been proven to be equal to the general entanglement of formation in two-mode Gaussian systems [12], is equal to the entanglement of formation of the pure state with a covariance matrix given by $\boldsymbol{\sigma}_p(r_o) = L(r_{1_o},r_{2_o})S_2(r_o)\mathbb{1} S_2^T(r_o)L^T(r_{1_o},r_{2_o})$ (with the corresponding symplectic eigenvalue, $\tilde{\nu}_{o_-} = e^{-2r_o}$), and thus $\mathcal{E}_F(\boldsymbol{\sigma}) = \mathcal{E}_F[\boldsymbol{\sigma}_p(r_o)]$, so we have [25]

$$\mathcal{E}_F(\boldsymbol{\sigma}) = \cosh^2 r_o \log_2(\cosh^2 r_o) - \sinh^2 r_o \log_2(\sinh^2 r_o). \tag{3}$$

Thus, *entanglement of formation quantifies the minimum amount of two-mode squeezing needed to prepare an entangled state starting from a classical one*. The optimum decomposition, and consequently r_o, cannot in general be found analytically [5,12,24,26]

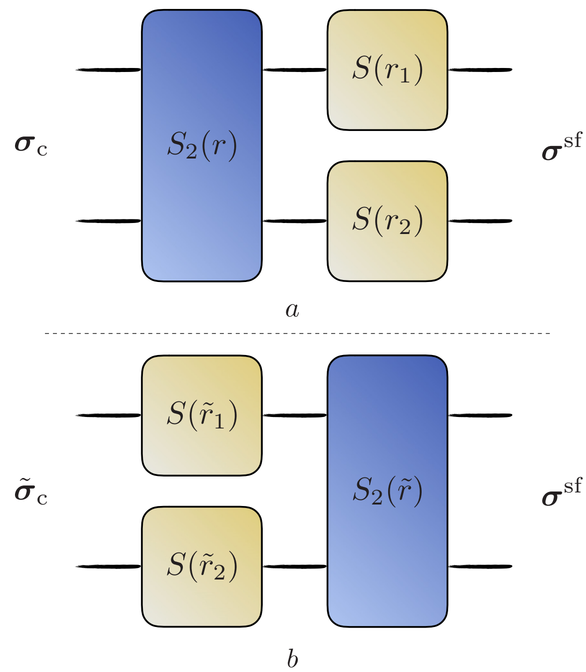

FIG. 1. State decompositions. Any state, $\boldsymbol{\sigma}^{\text{sf}}$, can be constructed by applying a sequence of (a) two-mode squeezing $S_2(r)$ followed by local squeezing $S(r_i)$ to a classical state $\boldsymbol{\sigma}_{\text{c}}$ or, reversely, (b) local squeezing $S(\tilde{r}_i)$ followed by two-mode squeezing $S_2(\tilde{r})$ to a classical state $\tilde{\boldsymbol{\sigma}}_{\text{c}}$.

Another way to decompose a state is as

$$\boldsymbol{\sigma}^{\text{sf}} = S_2(\tilde{r})L(\tilde{r}_1,\tilde{r}_2)\tilde{\boldsymbol{\sigma}}_{\text{c}} L^T(\tilde{r}_1,\tilde{r}_2)S_2^T(\tilde{r}), \tag{4}$$

since we can always disentangle a state by antisqueezing it and then apply the corresponding local squeezing to make the separable state classical, i.e., $\tilde{\boldsymbol{\sigma}}_{\text{c}} \geqslant \mathbb{1}$ [see Fig. 1(b)]. In order to make a state separable we have to solve the inequality $\tilde{\nu}_-[S_2(-\tilde{r})\boldsymbol{\sigma}^{\text{sf}} S_2^T(-\tilde{r})] \geqslant 1$, which is satisfied for the range of $\tilde{r}_- \leqslant \tilde{r} \leqslant \tilde{r}_+$, with

$$\tilde{r}_\pm = \frac{1}{2}\ln\sqrt{\frac{\kappa \pm \sqrt{\kappa^2 - \lambda_+\lambda_-}}{\lambda_-}}, \tag{5}$$

where we have set $\kappa = 2(\det\boldsymbol{\sigma} + 1) - (a-b)^2$ and $\lambda_\pm = \det A + \det B - 2\det C + 2[(ab - c_1c_2) \pm (c_1 - c_2)(a+b)]$. The physical meaning of $\tilde{r}_-$ is that it quantifies the minimum amount of two-mode squeezing needed to disentangle a state (in its standard form). For symmetric states, i.e., $a = b$, and for states with balanced correlations, i.e., $c_1 = -c_2$, we have $\tilde{r}_- = r_o$, but in general $\tilde{r}_- \leqslant r_o$ (the proof of that statement can be found in Appendix B), and thus we have a lower bound of the entanglement of formation

$$\tilde{\mathcal{E}}_F(\boldsymbol{\sigma}) = \mathcal{E}_F[\boldsymbol{\sigma}_p(\tilde{r}_-)] \leqslant \mathcal{E}_F[\boldsymbol{\sigma}_p(r_o)] = \mathcal{E}_F(\boldsymbol{\sigma}). \tag{6}$$

We note that for fixed global and local purities, the more imbalanced the correlations, the larger the deviation of the lower bound $\tilde{\mathcal{E}}_F(\boldsymbol{\sigma})$ from the real value $\mathcal{E}_F(\boldsymbol{\sigma})$.

In Fig. 2 we compare the entanglement of formation (calculated numerically using the approach developed at [24])

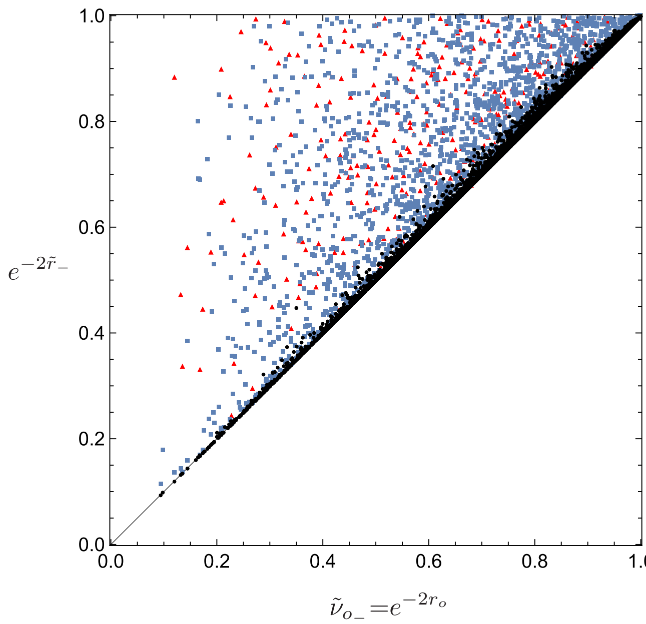

FIG. 2. Lower bound for entanglement of formation. We plot with black dots the optimum symplectic eigenvalue $\tilde{\nu}_{o_-} = e^{-2r_o}$ (calculated numerically [24]) versus the corresponding value based on $\tilde{r}_-$, i.e., $e^{-2\tilde{r}_-}$, for randomly generated states. The symplectic eigenvalue is a bounded value $\in (0,1]$, which shows (a) that $\tilde{\mathcal{E}}_F(\boldsymbol{\sigma}) \leqslant \mathcal{E}_F(\boldsymbol{\sigma})$ and (b) that the bound is also tight for separable and infinite entangled states. We also depict, with blue squares [27] and red triangles [28], the corresponding values we get from the previously known lower bounds. The closer the dots are to the diagonal, the smaller the deviation from the real value of entanglement. It is clear that our bound is, on average, tighter than previous bounds. All quantities plotted are dimensionless.

and its lower bound for randomly generated states. The significant progress over the previously known lower bounds of the measure derived in Refs. [27] and [28] is also depicted. As we see, the former lower bounds deviate significantly from the real value and, sometimes, even imply separability for an entangled state.

For many quantum communication protocols, Gaussian channels describe the decoherence introduced by the environment to a quantum state and represent the basic models of communication lines such as optical fibers [19]. Let us assume that a single mode of a two-mode squeezed vacuum state, i.e., $a = b = \frac{1+\chi^2}{1-\chi^2}$ and $c_1 = -c_2 = \frac{2\chi}{1-\chi^2}$, with $\chi = \tanh r \in [0,1)$, is sent through a Gaussian channel. One-mode Gaussian channels can be defined as the transformation of the covariance matrix of the mode $\boldsymbol{\gamma}$, i.e., $\boldsymbol{\gamma} \to \mathcal{U}\boldsymbol{\gamma}\,\mathcal{U}^T + \mathcal{V}$ [19]. Typically, these channels are phase invariant and so produce states with balanced correlations that saturate the lower bound, i.e., $\tilde{r}_- = r_o$. The value of r_o, derived from $\tilde{r}_-$ in Eq. (5), for three fundamental Gaussian channels is presented here:

(a) The lossy channel, $\mathcal{L}(\tau)$, is defined as $\mathcal{U} = \sqrt{\tau}\mathbb{1}$ and $\mathcal{V} = (1-\tau)\mathbb{1}$, with transmissivity $0 \leqslant \tau \leqslant 1$. Thus we have

$$r_o = \frac{1}{2}\ln\frac{1+\chi\sqrt{\tau}}{1-\chi\sqrt{\tau}}.$$

(b) The amplifier channel, $\mathcal{A}(\tau)$, is defined as $\mathcal{U} = \sqrt{\tau}\mathbb{1}$ and $\mathcal{V} = (\tau-1)\mathbb{1}$, with transmissivity $\tau \geqslant 1$. Equation (5) takes the form

$$r_o = \frac{1}{2}\ln\frac{\sqrt{\tau}+\chi}{\sqrt{\tau}-\chi}.$$

(c) The classical noise channel, $\mathcal{C}(v)$, is defined as $\mathcal{U} = \mathbb{1}$ and $\mathcal{V} = v\mathbb{1}$, with $v \geqslant 0$. The optimum squeezing parameter for $0 \leqslant v \leqslant 2$ is

$$r_o = \frac{1}{2}\ln\frac{2+v+\chi(2-v)}{2+v+\chi(v-2)},$$

while for $u > 2$, r_o vanishes, i.e., entanglement-breaking channel.

The deterministic upper bound of entanglement for a channel, i.e., the amount of entanglement assuming that an infinitely squeezed state is sent through the same channel [29], is reached for $\chi \to 1$. This bound allows us to investigate physical limits, like the calculation of the maximum possible amount of quantum correlations that can possibly exist after a specific decohering channel.

IV. COMPARISON WITH LOGARITHMIC NEGATIVITY

As mentioned before, besides entanglement of formation, other quantifiers have also been used to compute entanglement for these kinds of states, so it would be interesting to give a direct comparison with the most popular of these (due to its computability), i.e., the logarithmic negativity, which is defined, for two-mode Gaussian states, as $\mathcal{E}_N(\boldsymbol{\sigma}) := \max[0, -\log_2\tilde{\nu}_-]$ [10,14–16]. In order to have a clear operational meaning of this monotone, we can define the generalized *EPR* correlations $\hat{u} = \frac{\hat{x}_1 - g_x\hat{x}_2}{\sqrt{1+g_x{}^2}}$ and $\hat{v} = \frac{\hat{p}_1 + g_p\hat{p}_2}{\sqrt{1+g_p{}^2}}$, where $g_x, g_p \in \mathbb{R}$ are experimentally variable gains. For these operators the separability criterion [23] takes the form $\beta = \frac{V_x V_p}{(1+g_x g_p)^2} \geqslant 1$, with $V_x = \langle(\hat{x}_1 - g_x\hat{x}_2)^2\rangle$ and $V_p = \langle(\hat{p}_1 + g_p\hat{p}_2)^2\rangle$ being the conditional variances. For $\beta < 1$ we have an entangled state, and its minimum value β_- is equal to $\tilde{\nu}_-^2$ [30]. We should note that this equality, i.e., $\beta_- = \tilde{\nu}_-^2$, holds for any two-mode Gaussian state. So, *logarithmic negativity quantifies the maximum possible violation of the separability criterion*.

The logarithmic negativity is, in general, not directly related to the squeezing of the state, which is a major drawback, since squeezing is considered the resource of the quantum correlations in the system and is, experimentally, the primary figure of merit. Furthermore, in Fig. 3 it is apparent that $\mathcal{E}_N$ fails, in general, to satisfy the extremality of the entanglement cost (which coincides with entanglement of formation in these systems), i.e., $\mathcal{E}_i \leqslant \mathcal{E}_C$ [31], which was expected since logarithmic negativity is not asymptotically continuous. This results in an inconsistent behavior of $\mathcal{E}_N$, which, for finite squeezing, can be either an upper or a lower bound of $\mathcal{E}_F$, depending on the channel that the state is sent through. A specific example of how $\mathcal{E}_N$ can lead to a qualitatively different evaluation of the entanglement sent through a physically relevant channel compared to $\mathcal{E}_F$ is shown in Fig. 3. To sum up, logarithmic negativity is a quantifier widely used in the literature, since it has the merit of being analytically computable in various quantum systems

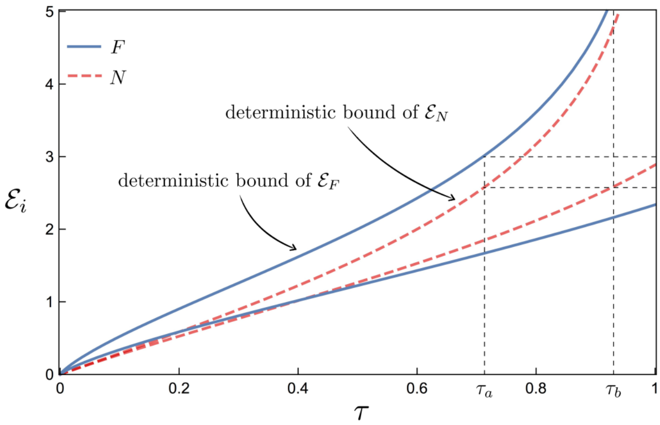

FIG. 3. Comparison between entanglement of formation (solid blue line) and logarithmic negativity (dashed red line). Assuming that a two-mode squeezed vacuum state with $r = 1$ is sent through a lossy channel of transmissivity $0 \leqslant \tau \leqslant 1$, we compare the two measures. The deterministic upper bounds (upper lines), i.e., the amount of entanglement assuming that an infinitely squeezed state is sent through the same channel, are also depicted, since they provide further insight regarding the qualitative differences between entanglement of formation and logarithmic negativity. The deterministic bound for logarithmic negativity can be found in Ref. [29]. Specifically, for logarithmic negativity, the deterministic bound of a state with transmissivity value τ_a can also be reached by sending the squeezed state ($r = 1$) through a channel of transmissivity τ_b, with $\tau_b > \tau_a$. However, in contrast, entanglement of formation predicts that we cannot reach the deterministic bound with a squeezed state ($r = 1$) regardless of how much we raise the transmissivity. This is a critical difference, since the two quantifiers disagree on whether or not a physical upper bound has been reached. All quantities plotted are dimensionless.

but, from an information-theoretic point of view, is inferior to entanglement of formation.

V. CONCLUSION

In conclusion, we have found a lower bound of entanglement of formation which is tight for symmetric states and for states with balanced correlations, while it deviates from the real value for states with asymmetric correlations. The deviation, though, is relatively small, which practically makes this lower bound an analytical approximation of the entanglement of formation for experimental purposes. We also have shown via physical examples that this measure should be favored over logarithmic negativity. We also introduced an alternative interpretation of the measure in Gaussian systems, proving that entanglement of formation is intrinsically related to the amount of antisqueezing needed to disentangle a state up to the point where the state becomes classical, which might also be helpful for the quantification of entanglement of several other families of states, e.g., multipartite Gaussian and non-Gaussian states.

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APPENDIX A: STATE DECOMPOSITION

Any two-mode Gaussian state can be written in the standard form as [5,12,26]

$$\boldsymbol{\sigma}^{\rm sf} = L(r_1,r_2)[\boldsymbol{\sigma}_p^{\rm sf}(r) + \boldsymbol{\varphi}]L^T(r_1,r_2), \tag{A1}$$

with $L(r_1,r_2){=}S(r_1){\oplus}S(r_2)$, where $S(r_i) := \text{diag}(e^{-r_i},e^{r_i})$ is the local squeezing symplectic map for each mode, and $\boldsymbol{\varphi}$ is a positive semidefinite matrix. So, we have

$$\boldsymbol{\sigma}^{\rm sf} = L(r_1,r_2)\left[\underbrace{S_2(r)\mathbb{1}S_2^T(r)}_{\boldsymbol{\sigma}_p^{\rm sf}(r)} + S_2(r)\underbrace{S_2(-r)\boldsymbol{\varphi}S_2^T(-r)}_{\boldsymbol{\theta}}S_2^T(r)\right] \times L^T(r_1,r_2). \tag{A2}$$

where

$$S_2(r) := \begin{bmatrix} \cosh r\,\mathbb{1} & \sinh r\,Z \\ \sinh r\,Z & \cosh r\,\mathbb{1}\end{bmatrix}$$

is the two-mode squeezing symplectic map, with $Z := \text{diag}(1,-1)$. Since $\boldsymbol{\varphi}$ has a structure identical to a covariance matrix, but not necessarily in the standard form, i.e.,

$$\boldsymbol{\varphi} = \begin{bmatrix} n_1 & 0 & d_1 & 0 \\ 0 & n_2 & 0 & d_2 \\ d_1 & 0 & m_1 & 0 \\ 0 & d_2 & 0 & m_2 \end{bmatrix}, \tag{A3}$$

then $\boldsymbol{\theta} = S_2(-r)\boldsymbol{\varphi}S_2^T(-r)$ is also in the same form as $\boldsymbol{\varphi}$ and, thus, a Hermitian matrix, so, based on Wigner's theorem [32], we know that $\boldsymbol{\theta} \geqslant 0$. So, we can write

$$\boldsymbol{\sigma}^{\rm sf} = L(r_1,r_2)\left[S_2(r)\underbrace{\{\mathbb{1}+\boldsymbol{\theta}\}}_{\boldsymbol{\sigma}_{\rm c}}S_2^T(r)\right]L^T(r_1,r_2), \tag{A4}$$

but $\mathbb{1}+\boldsymbol{\theta}$ can always represent a classical state, $\boldsymbol{\sigma}_{\rm c}$, where $\boldsymbol{\theta}$ is interpreted as the random correlated displacements applied to a couple of vacuums, and thus we have

$$\boldsymbol{\sigma}^{\rm sf} = L(r_1,r_2)S_2(r)\boldsymbol{\sigma}_{\rm c}S_2^T(r)L^T(r_1,r_2). \tag{A5}$$

APPENDIX B: LOWER BOUND

Any state can be decomposed as

$$\boldsymbol{\sigma}^{\rm sf} = S_2(\tilde{r})L(\tilde{r}_1,\tilde{r}_2)\tilde{\boldsymbol{\sigma}}_{\rm c}L^T(\tilde{r}_1,\tilde{r}_2)S_2^T(\tilde{r}), \tag{B1}$$

since we can always disentangle a state by antisqueezing it and then apply the corresponding local squeezing to make the separable state classical, i.e., $\tilde{\boldsymbol{\sigma}}_{\rm c} \geqslant \mathbb{1}$. In order to make a state separable we have to solve the inequality $\tilde{\nu}_-[S_2(-\tilde{r})\boldsymbol{\sigma}^{\rm sf}S_2^T(-\tilde{r})] \geqslant 1$, which is satisfied for the range of $\tilde{r}_- \leqslant \tilde{r} \leqslant \tilde{r}_+$, with

$$\tilde{r}_\pm = \frac{1}{2}\ln\sqrt{\frac{\kappa \pm \sqrt{\kappa^2 - \lambda_+\lambda_-}}{\lambda_-}}, \tag{B2}$$

where we have set $\kappa = 2(\det\boldsymbol{\sigma} + 1) - (a-b)^2$ and $\lambda_\pm = \det A + \det B - 2\det C + 2[(ab - c_1c_2) \pm (c_1 - c_2)(a+b)]$. Given an $\tilde{r}$ which disentangles $\boldsymbol{\sigma}^{\mathrm{sf}}$, we can always analytically calculate the local squeezing parameters $\tilde{r}_1$ and $\tilde{r}_2$ needed to remove any nonclassicality. The entanglement needed to construct a state for an arbitrary decomposition of this form is equivalent to the entanglement of the corresponding pure state

$$\boldsymbol{\sigma}_p = S_2(\tilde{r})L(\tilde{r}_1,\tilde{r}_2)\mathbb{1}L^T(\tilde{r}_1,\tilde{r}_2)S_2^T(\tilde{r}), \tag{B3}$$

but the covariance matrix of this pure state is always identical to the covariance matrix constructed in the following way:

$$\boldsymbol{\sigma}_p = L(r_1',r_2')S_2(r')\mathbb{1}S_2^T(r')L^T(r_1',r_2'), \tag{B4}$$

where

$$r'(\tilde{r},\tilde{r}_1,\tilde{r}_2) = \cosh^{-1}\left(\frac{1}{2}\sqrt{e^{-\tilde{r}_1-\tilde{r}_2}\sqrt{\cosh(2\tilde{r})(e^{2\tilde{r}_1}+e^{2\tilde{r}_2})+e^{2\tilde{r}_1}-e^{2\tilde{r}_2}}\sqrt{\cosh(2\tilde{r})(e^{2\tilde{r}_1}+e^{2\tilde{r}_2})-e^{2\tilde{r}_1}+e^{2\tilde{r}_2}}+2}\right), \tag{B5}$$

$$r_1'(\tilde{r},\tilde{r}_1,\tilde{r}_2) = \log\left(\frac{e^{\frac{\tilde{r}_1+\tilde{r}_2}{2}}\sqrt[4]{e^{2\tilde{r}_1}\cosh^2(\tilde{r})+e^{2\tilde{r}_2}\sinh^2(\tilde{r})}}{\sqrt[4]{e^{2\tilde{r}_1}\sinh^2(\tilde{r})+e^{2\tilde{r}_2}\cosh^2(\tilde{r})}}\right), \tag{B6}$$

and

$$\begin{aligned} r_2'(\tilde{r},\tilde{r}_1,\tilde{r}_2) = {}& \frac{1}{(e^{2\tilde{r}_1}+e^{2\tilde{r}_2})\sqrt[4]{e^{2\tilde{r}_1}\cosh^2(\tilde{r})+e^{2\tilde{r}_2}\sinh^2(\tilde{r})}}\log\Bigg(\operatorname{csch}(\tilde{r})\operatorname{sech}(\tilde{r})e^{\frac{3(\tilde{r}_1+\tilde{r}_2)}{2}}\sqrt[4]{e^{2\tilde{r}_1}\sinh^2(\tilde{r})+e^{2\tilde{r}_2}\cosh^2(\tilde{r})} \\ &\times\sqrt{e^{-\tilde{r}_1-\tilde{r}_2}\sqrt{e^{2\tilde{r}_1}\sinh^2(\tilde{r})+e^{2\tilde{r}_2}\cosh^2(\tilde{r})}\sqrt{e^{2\tilde{r}_1}\cosh^2(\tilde{r})+e^{2\tilde{r}_2}\sinh^2(\tilde{r})}-1} \\ &\times\sqrt{e^{-\tilde{r}_1-\tilde{r}_2}\sqrt{e^{2\tilde{r}_1}\sinh^2(\tilde{r})+e^{2\tilde{r}_2}\cosh^2(\tilde{r})}\sqrt{e^{2\tilde{r}_1}\cosh^2(\tilde{r})+e^{2\tilde{r}_2}\sinh^2(\tilde{r})}+1}\Bigg). \end{aligned} \tag{B7}$$

Let us assume that we have the optimum decomposition for the entanglement of formation, i.e., $\boldsymbol{\sigma}^{\mathrm{sf}} = L(r_{1_o},r_{2_o})S_2(r_o)\boldsymbol{\sigma}_{\mathrm{c}_o}S_2^T(r_o)L^T(r_{1_o},r_{2_o})$, which corresponds to $\boldsymbol{\sigma}^{\mathrm{sf}} = S_2(\tilde{r}_o)L(\tilde{r}_{1_o},\tilde{r}_{2_o})\tilde{\boldsymbol{\sigma}}_{\mathrm{c}_o}L^T(\tilde{r}_{1_o},\tilde{r}_{2_o})S_2^T(\tilde{r}_o)$ with $\tilde{r}_- \leqslant \tilde{r}_o \leqslant \tilde{r}_+$. We know that r_o must be a function of r', i.e., $r_o = r'(\tilde{r}_o,\tilde{r}_{1_o},\tilde{r}_{2_o})$. It is straightforward to prove that $r'(\tilde{r},\tilde{r}_1,\tilde{r}_2) \geqslant r'(\tilde{r},\tilde{r}_1 = \tilde{r}_2)$, since $\frac{\partial r'}{\partial \tilde{r}_1} = \frac{\partial r'}{\partial \tilde{r}_2} = 0 \Rightarrow r_1 = r_2$ and $\frac{\partial^2 r'}{\partial \tilde{r}_1^2} \geqslant 0$, $\frac{\partial^2 r'}{\partial \tilde{r}_2^2} \geqslant 0$ for any $\tilde{r} > 0$. So, for the case of $\tilde{r}_{1_o} = \tilde{r}_{2_o} = 0$, $r'(\tilde{r}_o,\tilde{r}_{1_o},\tilde{r}_{2_o}) \geqslant r'(\tilde{r}_o,\tilde{r}_{1_o} = \tilde{r}_{2_o} = 0)$ should hold as well. It is apparent that $r'(\tilde{r}_o,\tilde{r}_{1_o} = \tilde{r}_{2_o} = 0) = \tilde{r}_o$, and thus

$$\tilde{r}_- \leqslant r_o \Rightarrow \tilde{\mathcal{E}}_F(\boldsymbol{\sigma}) = \mathcal{E}_F[\boldsymbol{\sigma}_p(\tilde{r}_-)] \leqslant \mathcal{E}_F[\boldsymbol{\sigma}_p(r_o)] = \mathcal{E}_F(\boldsymbol{\sigma}), \tag{B8}$$

where $\tilde{\mathcal{E}}_F(\boldsymbol{\sigma})$ is the lower bound of the entanglement of formation. The reason this lower bound is tight for balanced states is that for these states the local squeezing parameters of the optimum decomposition are found to be $r_1 = r_2 = 0$ [5,12,26], and thus the two decompositions, i.e., Eq. (A5) and Eq. (B1), coincide. For symmetric states, where the local squeezing parameters of the optimal decomposition are $r_1 = r_2 = \sqrt{\frac{a+c_2}{a-c_1}}$ [5,12,24,26], the bound is tight since the operation $L(r_1,r_2)S_2(r)S_2^T(r)L^T(r_1,r_2)$ is identical to $S_2(r)L(r_1,r_2)L^T(r_1,r_2)S_2^T(r)$ for $r = r_1 = r_2$, so again, the two decompositions [Eq. (A5) and Eq. (B1)] coincide.

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