Spectrum and normal modes of non-Hermitian quadratic boson operators

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We analyze the spectrum and normal-mode representation of general quadratic bosonic forms H not necessarily Hermitian. It is shown that, in the one-dimensional case, such forms exhibit either a harmonic regime where both H and $H^\dagger$ have a discrete spectrum with biorthogonal eigenstates, and a coherent-like regime where either H or $H^\dagger$ have a continuous complex twofold degenerate spectrum, while its adjoint has no convergent eigenstates. These regimes reflect the nature of the pertinent normal boson operators. Nondiagonalizable cases as well critical boundary sectors separating these regimes are also analyzed. The extension to N-dimensional quadratic systems is also discussed.

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I. INTRODUCTION

The introduction of parity-time-symmetric ($\mathcal{PT}$-symmetric) quantum mechanics [1,2] has significantly enhanced the interest in non-Hermitian Hamiltonians. When possessing $\mathcal{PT}$ symmetry, such Hamiltonians can still exhibit a real spectrum if the symmetry is unbroken in all eigenstates, undergoing a transition to a regime with complex eigenvalues when the symmetry becomes broken [1,2]. A generalization based on the concept of pseudo-Hermiticity was then developed [3–5], which provides a complete characterization of diagonalizable Hamiltonians with real discrete spectrum and is equivalent to the presence of an antilinear symmetry. A similar approach had been already put forward in Ref. [6] in connection with the non-Hermitian bosonization of angular-momentum and fermion operators introduced by Dyson [7,8]. An equivalent formulation of the general formalism based on biorthogonal states can also be made [4,9,10].

Non-Hermitian Hamiltonians were first introduced as effective Hamiltonians for describing open quantum systems [11]. Non-Hermitian Hamiltonians with $\mathcal{PT}$ symmetry have recently provided successful effective descriptions of diverse systems and processes, especially in open regimes with balanced gain and loss. Examples are laser absorbers [12], ultralow threshold phonon lasers [13], defect states and special beam dynamics in optical lattices [14], and other related optical systems [15,16]. $\mathcal{PT}$-symmetric properties have been also observed and investigated in simulations of quantum circuits based on nuclear magnetic resonance [17], superconductivity experiments [18,19], microwave cavities [20], Bose–Einstein condensates [21], spin systems [22], and vacuum fluctuations [23]. Evolution under time-dependent non-Hermitian Hamiltonians has also been discussed in Refs. [24,25].

Of particular interest are non-Hermitian Hamiltonians which are quadratic in coordinates and momenta or, equivalently, boson creation and annihilation operators. They include the so-called Swanson models [26,27] based on one-dimensional $\mathcal{PT}$-symmetric Hamiltonians with real spectra, which have been examined and extended in different ways [28–31]. Effective quadratic non-Hermitian Hamiltonians have also arisen in the description of LRC circuits with balanced gain and loss [32], coupled optical resonators [33], optical trimers [34], and the interpretation of the electromagnetic self-force [35].

The aim of this article is to examine the normal modes, spectrum, and eigenstates of general, not necessarily Hermitian, quadratic bosonic forms in greater detail, extending the methodology of Refs. [36,37] to the present general situation. Such quadratic forms can represent basic systems like a harmonic oscillator with a discrete spectrum, a free particle Hamiltonian with a continuous real spectrum, the square of an annihilation operator, in which case it has a continuous complex spectrum with coherent states [38] as eigenvectors, and the square of a creation operator, in which case it has no convergent eigenstates. We will here show that a general quadratic one-dimensional form belongs essentially to one of these previous categories, as determined by the nature of the normal boson operators, i.e., as whether one, both, or none of them possesses a convergent vacuum. Explicit expressions for eigenstates are provided, together with an analysis of border and "nondiagonalizable" regimes. The extension to N-dimensional quadratic systems is then also discussed.

II. THE ONE-DIMENSIONAL CASE

A. Normal mode representation

We consider a general quadratic form in standard boson creation and annihilation operators $a,a^\dagger$ ($[a,a^\dagger]=1$),

$$H = A\left(a^\dagger a + \frac{1}{2}\right) + \frac{1}{2}(B_+ a^{\dagger 2} + B_- a^2) \tag{1}$$

$$= \frac{1}{2}(a^\dagger \quad a)\mathcal{H}\begin{pmatrix} a \\ a^\dagger \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} A & B_+ \\ B_- & A \end{pmatrix}, \tag{2}$$

where A and $B_\pm$ are in principle arbitrary complex numbers. By extracting a global phase we can always assume, nonetheless, A real non-negative ($A \geqslant 0$), while by a phase transformation $a \to e^{i\phi}a$, $a^\dagger \to e^{-i\phi}a^\dagger$, we can set equal phases on $B_\pm$, such that $B_\pm = |B_\pm|e^{i\theta}$. The Hermitian case corresponds to $\mathcal{H}$ Hermitian and the original Swanson Hamiltonian to $B_\pm$ real [26].

Our first aim is to write H in the normal form

$$H = \lambda\left(\bar{b}^\dagger b + \tfrac{1}{2}\right), \tag{3}$$

where b, $\bar{b}^\dagger$ are related to a and $a^\dagger$ through a generalized Bogoliubov transformation

$$b = ua + va^\dagger, \quad \bar{b}^\dagger = \bar{v}^* a + \bar{u}^* a^\dagger. \tag{4}$$

Here $\bar{b}^\dagger$ may differ from $b^\dagger$ although they still satisfy the bosonic commutation relation

$$[b,\bar{b}^\dagger]=1, \tag{5}$$

which implies

$$u\bar{u}^*-v\bar{v}^*=1. \tag{6}$$

If $\mathcal{H}$ is Hermitian and positive definite ($|B_\pm|<A$), such that H represents a stable bosonic mode, we can always choose u, v, $\bar{u}$, and $\bar{v}$ such that $\bar{b}^\dagger=b^\dagger$. This choice is no longer feasible in the general case.

The transformation (4) can be written as

$$\begin{pmatrix} b\\ \bar{b}^\dagger\end{pmatrix}=\mathcal{W}\begin{pmatrix} a\\ a^\dagger\end{pmatrix},\quad \mathcal{W}=\begin{pmatrix} u & v\\ \bar{v}^* & \bar{u}^*\end{pmatrix}, \tag{7}$$

with $\mathcal{W}$ satisfying $\mathrm{Det}\,\mathcal{W}=1$. We can then rewrite H as

$$H=\frac{1}{2}(\bar{b}^\dagger\quad b)\mathcal{H}'\begin{pmatrix} b\\ \bar{b}^\dagger\end{pmatrix}, \tag{8}$$

$$\mathcal{H}'=\mathcal{M}\mathcal{W}\mathcal{M}\mathcal{H}\mathcal{W}^{-1}=\begin{pmatrix} A' & B'_+\\ B'_- & A'\end{pmatrix}, \tag{9}$$

where $A'=A(u\bar{u}^*+v\bar{v}^*)-B_+u\bar{v}^*-B_-\bar{u}^*v$, $B'_+=B_+u^2+B_-v^2-2Auv$, $B'_-=B_-\bar{u}^{*2}+B_+\bar{v}^{*2}-2A\bar{u}^*\bar{v}^*$, and

$$\mathcal{M}=\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}. \tag{10}$$

It is then seen from Eq. (9) that a *diagonal* $\mathcal{H}'$ ($B'_\pm=0$, $A=\lambda$) and hence a diagonal representation (4) can be obtained *if and only if* (i) the matrix

$$\mathcal{M}\mathcal{H}=\begin{pmatrix} A & B_+\\ -B_- & -A\end{pmatrix}, \tag{11}$$

whose eigenvalues are $\pm\lambda$ with

$$\lambda=\sqrt{A^2-B_+B_-}, \tag{12}$$

is *diagonalizable*, i.e., $\lambda\neq 0$ if $\mathrm{rank}(\mathcal{H})>0$, and (ii) $\mathcal{W}^{-1}$ is a matrix with unit determinant diagonalizing $\mathcal{M}\mathcal{H}$, such that $\mathcal{W}\mathcal{M}\mathcal{H}\mathcal{W}^{-1}=\lambda\mathcal{M}$ and $\mathcal{H}'=\lambda\mathbb{1}$. For instance, assuming $\lambda\neq 0$, we can set

$$u=\bar{u}^*=\sqrt{\frac{A+\lambda}{2\lambda}},$$
$$v=\sqrt{\frac{A-\lambda}{2\lambda}}\sqrt{\frac{B_+}{B_-}},\quad \bar{v}^*=\sqrt{\frac{A-\lambda}{2\lambda}}\sqrt{\frac{B_-}{B_+}}, \tag{13}$$

where signs of v, $\bar{v}^*$ are such that $2\lambda u\bar{v}^*=B_-$, $2\lambda\bar{u}^*v=B_+$. Any further rescaling $b\to\alpha b$, $\bar{b}^\dagger\to\alpha^{-1}\bar{b}^\dagger$, $\alpha\neq 0$, remains feasible, since it will not affect their commutator nor Eq. (3), although the choice (13) directly leads to $\bar{b}^\dagger=b^\dagger$ when $\mathcal{H}$ is Hermitian and positive definite (in which case $0<\lambda\leqslant A$). Equations (13) remain also valid for $B_+\to 0$ or $B_-\to 0$, in which case $\lambda\to A$, $u=\bar{u}^*\to 1$, and $(v,\bar{v}^*)\to(0,\frac{B_-}{2A})$ or $(\frac{B_+}{2A},0)$.

If no further conditions are imposed on $b,\bar{b}^\dagger$, the sign chosen for λ is irrelevant, since (3) can be rewritten as $-\lambda(\bar{b}'^\dagger b'+\frac{1}{2})$ for $\bar{b}'^\dagger=-b$, $b'=\bar{b}^\dagger$ (also satisfying $[b',\bar{b}'^\dagger]=1$). The sign can be fixed by imposing the condition that b (rather than $\bar{b}^\dagger$) has a proper vacuum, as discussed in the next section, in which case the right choice for $A\geqslant 0$ is $\mathrm{Re}(\lambda)\geqslant 0$.

The matrix $\mathcal{M}\mathcal{H}$ determines the commutators of H with a and $a^\dagger$, $[H,a]=-Aa-B_+a^\dagger$, $[H,a^\dagger]=Aa^\dagger+B_-a$:

$$\left[H,\begin{pmatrix} a\\ a^\dagger\end{pmatrix}\right]=-\mathcal{M}\mathcal{H}\begin{pmatrix} a\\ a^\dagger\end{pmatrix}. \tag{14}$$

The normal boson operators $b,\bar{b}^\dagger$ satisfying (3) are then those diagonalizing this semi-algebra:

$$[H,b]=-\lambda b,\quad [H,\bar{b}^\dagger]=\lambda\bar{b}^\dagger. \tag{15}$$

Therefore, if $|\alpha\rangle$ is an eigenvector of H with energy E_α,

$$H|\alpha\rangle=E_\alpha|\alpha\rangle, \tag{16}$$

then $\bar{b}^\dagger|\alpha\rangle$ and $b|\alpha\rangle$ are, respectively, eigenvectors with eigenvalues $E_\alpha\pm\lambda$, *provided* $\bar{b}^\dagger|\alpha\rangle$ and $b|\alpha\rangle$ are *nonzero*:

$$H\bar{b}^\dagger|\alpha\rangle=(\bar{b}^\dagger H+\lambda\bar{b}^\dagger)|\alpha\rangle=(E_\alpha+\lambda)\bar{b}^\dagger|\alpha\rangle, \tag{17}$$

$$Hb|\alpha\rangle=(bH-\lambda b)|\alpha\rangle=(E_\alpha-\lambda)b|\alpha\rangle. \tag{18}$$

As in the standard case, these operators then allow one to move along the spectrum, *even if it is continuous*, as discussed in Sec. II D.

The case where $\mathcal{M}\mathcal{H}$ is *nondiagonalizable* corresponds here to $\mathcal{H}$ of rank 1, and hence to an operator H which is just the square of a linear combination of a and $a^\dagger$:

$$H_{nd}=(\sqrt{B_-}a\pm\sqrt{B_+}a^\dagger)^2/2. \tag{19}$$

Such H leads to $A=\pm\sqrt{B_+B_-}$ and $\lambda=0$. This case, which includes the free-particle case $H\propto P^2$, will be discussed in Sec. II F.

B. The harmonic case

Let $|0_a\rangle$ be the vacuum of a, $a|0_a\rangle=0$, and let us assume that a vacuum $|0_b\rangle$ *exists* such that $b|0_b\rangle=0$. Then, $|0_b\rangle$ is necessarily a Gaussian state of the form [39]

$$|0_b\rangle\propto\exp\left(-\frac{v}{2u}a^{\dagger 2}\right)|0_a\rangle=\sum_{n=0}^{\infty}\left(-\frac{v}{2u}\right)^n\frac{\sqrt{(2n)!}}{n!}|2n_a\rangle. \tag{20}$$

Recalling that $\sum_{n=0}^{\infty}(\frac{z}{4})^n\frac{2n!}{(n!)^2}$ converges to $\frac{1}{\sqrt{1-z}}$ if and only if $|z|\leqslant 1$ and $z\neq 1$ [40], we see that $|0_b\rangle$ has a finite standard norm $\langle 0_b|0_b\rangle$ only if $|v|<|u|$, implying

$$\frac{|B_+|}{|B_-|}<\left|\frac{A+\lambda}{A-\lambda}\right|. \tag{21}$$

Equation (21) imposes an upper bound on $|B_+/B_-|$ for given values of A and B_+B_-. Similarly, assuming that a vacuum $|0_{\bar{b}}\rangle$ exists such that $\bar{b}|0_{\bar{b}}\rangle=0$, then

$$|0_{\bar{b}}\rangle\propto\exp\left(-\frac{\bar{v}}{2\bar{u}}a^{\dagger^2}\right)|0_a\rangle=\sum_{n=0}^{\infty}\left(-\frac{\bar{v}}{2\bar{u}}\right)^n\frac{\sqrt{(2n)!}}{n!}|2n_a\rangle, \tag{22}$$

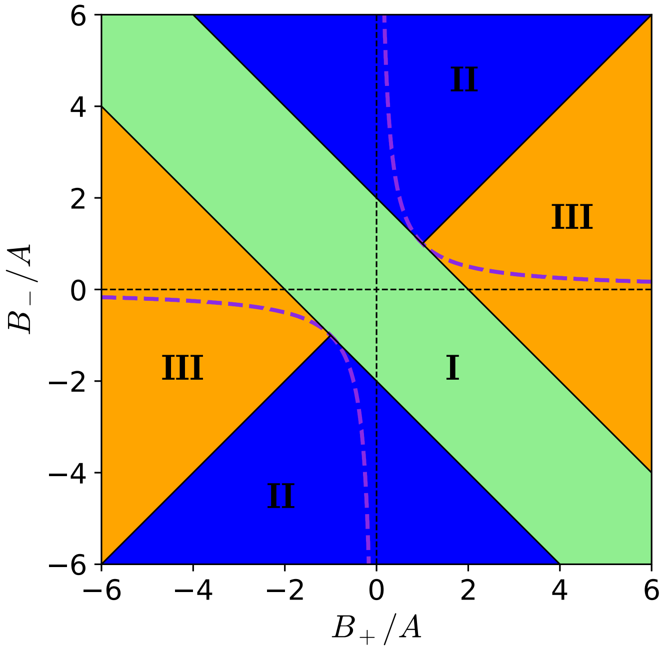

FIG. 1. Regions of distinct spectrum for the operator (1) in the case of $B_\pm$ real (and $A > 0$). **I** denotes the region with discrete positive spectrum [Eq. (32)], **II** that with continuous complex twofold degenerate spectrum [Eq. (46)], and **III** that with no convergent eigenfunctions [Eq. (60)]. The dashed curves depict the set of points where $\mathcal{MH}$ is nondiagonalizable. The Hermitian case corresponds to the line $B_- = B_+$.

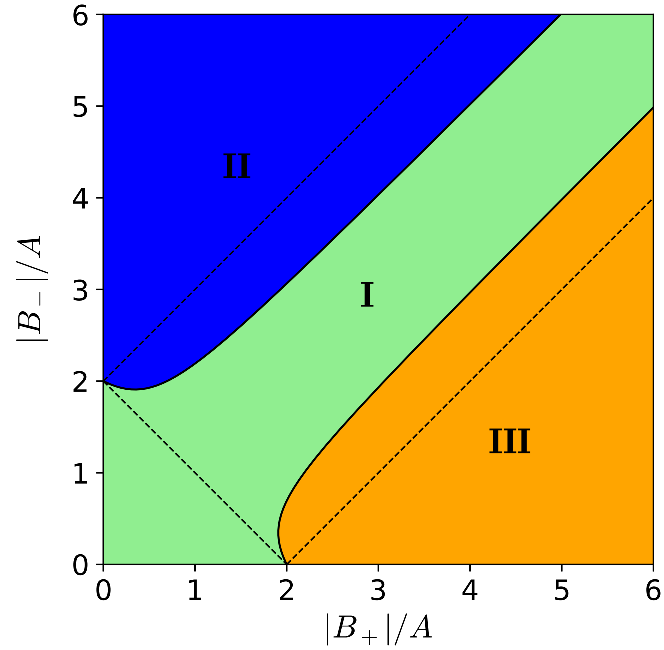

FIG. 2. Regions of distinct spectrum for the operator (1) with complex $B_\pm = |B_\pm|e^{i\theta}$ and $\theta = \pi/6$. Same details as Fig. 1: In **I**, H has a discrete complex spectrum, while in **II** it has a continuous complex spectrum and in **III** no convergent eigenfunctions. The dotted segment $|B_+| + |B_-| = 2A$ indicates the upper limit of region **I** for $\theta = 0$ ($B_\pm$ real and positive) whereas dotted lines $|B_+| - |B_-| = \pm 2A$ indicate the border of **I** for $\theta = \pi/2$ ($B_\pm$ imaginary). For general $\theta \in (0,\pi/2]$ and $|B_\pm| \gg A$, **I** is limited by lines $|B_+| - |B_-| = \pm 2A\sin\theta$.

with $\langle \bar{0}_b|\bar{0}_b\rangle$ convergent only if $|\bar{v}| < |\bar{u}|$, i.e.,

$$\frac{|B_-|}{|B_+|} < \left|\frac{A+\lambda}{A-\lambda}\right|. \tag{23}$$

Equations (21)–(23) determine a common convergence window

$$\frac{|A-\lambda|}{|A+\lambda|} < \frac{|B_+|}{|B_-|} < \frac{|A+\lambda|}{|A-\lambda|}, \tag{24}$$

equivalent to $|A-\lambda| < |B_\pm| < |A+\lambda|$, within which both $|0_b\rangle$ and $|0_{\bar{b}}\rangle$ are well defined. For $A \geqslant 0$, such window can exist only if $A > 0$ and $\mathrm{Re}(\lambda) > 0$, which justifies our previous sign choice of λ. This window corresponds to region **I** in Figs. 1 and 2.

On the other hand, their overlap $\langle 0_{\bar{b}}|0_b\rangle$ converges if and only if

$$\left|\frac{v\bar{v}^*}{u\bar{u}^*}\right| = \left|\frac{A-\lambda}{A+\lambda}\right| \leqslant 1, \tag{25}$$

and $v\bar{v}^* \neq u\bar{u}^*$, but these conditions are always satisfied due to Eq. (6) and the choice $\mathrm{Re}(\lambda) \geqslant 0$ (for $A \geqslant 0$). In particular, if Eq. (24) holds, Eq. (25) is always fulfilled.

It is now natural to define, for $m,n \in \mathbb{N}$, the states

$$|n_b\rangle = \frac{(\bar{b}^\dagger)^n}{\sqrt{n!}}|0_b\rangle, \quad |m_{\bar{b}}\rangle = \frac{(b^\dagger)^m}{\sqrt{m!}}|0_{\bar{b}}\rangle, \tag{26}$$

which, since $[\bar{b}^\dagger b, \bar{b}^\dagger] = \bar{b}^\dagger$ and $[b^\dagger \bar{b}, b^\dagger] = b^\dagger$, satisfy

$$\bar{b}^\dagger b|n_b\rangle = n|n_b\rangle, \quad b^\dagger \bar{b}|m_{\bar{b}}\rangle = m|m_{\bar{b}}\rangle, \tag{27}$$

with

$$\langle m_{\bar{b}}\,|\,n_b\rangle = \delta_{mn}\langle 0_{\bar{b}}|0_b\rangle, \tag{28}$$

implying that $\{|n_b\rangle\}$ and $\{|m_{\bar{b}}\rangle\}$ form a *biorthogonal set* [9]. Adding "normalization" factors $u^{-1/2}$ and $\bar{u}^{-1/2}$ in Eqs. (20)–(22) directly leads to $\langle 0_b|0_{\bar{b}}\rangle = 1$. Note, however, that the $|n_b\rangle$ are not orthogonal among themselves, nor are the $|m_{\bar{b}}\rangle$. Since $\bar{b}^\dagger = C^{-1}[b^\dagger + (u\bar{v} - v\bar{u})^* b]$, with $C = |u|^2 - |v|^2 = [b, b^\dagger]$, the $|n_b\rangle$ are linear combinations of standard Fock states $\propto (b^\dagger)^k|0_b\rangle$ with $k = n, n-2, \ldots$. Similar considerations hold for the $|m_{\bar{b}}\rangle$.

We can then write, in agreement with Eqs. (17) and (18),

$$H|n_b\rangle = \lambda\big(n + \tfrac{1}{2}\big)|n_b\rangle, \tag{29}$$

and also,

$$H^\dagger|m_{\bar{b}}\rangle = \lambda^*\big(m + \tfrac{1}{2}\big)|m_{\bar{b}}\rangle, \tag{30}$$

where $H^\dagger = \lambda^*(b^\dagger \bar{b} + \frac{1}{2})$. Hence, in the interval (24) there is a lower-bounded *discrete spectrum* of both H and $H^\dagger$, as corroborated in Sec. II D.

This discrete spectrum will be proportional to λ. Assuming A real, λ is real and nonzero if and only if B_+B_- is real and satisfies

$$B_+B_- < A^2. \tag{31}$$

For equal phases of $B_\pm$, it then comprises two cases:

(i) $B_\pm$ real ($\theta = 0,\pi$) satisfying Eq. (31), in which case $\lambda = (A^2 - |B_+B_-|)^{1/2} < A$ and u, v, $\bar{v}$ in Eq. (13) are real. Here H is invariant under time reversal, since $\mathcal{T}a\mathcal{T} = a$ and $\mathcal{T}a^\dagger\mathcal{T} = a^\dagger$. This is the Swanson case [26].

(ii) $B_\pm$ imaginary ($\theta = \pm\pi/2$), in which case $\lambda = (A^2 + |B_+B_-|)^{1/2} > A$, with u real and v, $\bar{v}^*$ imaginary. Here H has the antiunitary (or generalized $\mathcal{PT}$) symmetry [41–43] $U\mathcal{T}$, with U being the phase transformation $(a,a^\dagger) \to (-ia,ia^\dagger)$.

For λ real, Eq. (24) implies $|B_+ + B_-| < 2A$ in case (i) and $|B_+ - B_-| < 2A$ in case (ii), which can be summarized, for any case with real λ, as

$$|B_+ + B_-^*| < 2A. \tag{32}$$

Equation (32) is equivalent to $\mathcal{H} + \mathcal{H}^\dagger$ *positive definite*, i.e.,

$$\mathcal{H} + \mathcal{H}^\dagger > 0, \tag{33}$$

such that $\mathrm{Re}[Z^\dagger \mathcal{H} Z] > 0\,\forall\, Z = (z_1,z_2)^T \neq 0$. Therefore, *both H and $H^\dagger$ will exhibit a discrete real positive spectrum if and only if Eq. (33) holds.* Equation (32) then leads to region **I** in Fig. 1, i.e., the stripe $|B_+ + B_-| \leqslant 2A$ when $B_\pm$ are real.

On the other hand, when λ is complex the spectrum of H can be made real just by multiplying H by a phase $\lambda^*/|\lambda|$, as seen from Eq. (29). The ensuing operator H' has the antiunitary symmetry $U\mathcal{T}$, with U being the Bogoliubov transformation $\binom{a}{a^\dagger} \to U\binom{a}{a^\dagger}U^{-1} = (\mathcal{W}^*)^{-1}\mathcal{W}\binom{a}{a^\dagger}$. For complex λ, the stable sector adopts the form depicted in Fig. 2 (sector **I**). For a common phase $\theta = 0$ ($B_\pm$ real and positive) it is just the triangle $|B_+| + |B_-| < 2A$, while for $\theta = \pi/2$ ($B_\pm$ imaginary, equivalent through a phase transformation to $B_\pm$ real with opposite signs) it corresponds to $||B_+| - |B_-|| < 2A$ (sectors delimited by dotted lines). The union of these two sectors leads to the stripe of Fig. 1 for $B_\pm$ arbitrary real numbers. For intermediate phases the stable region is essentially the union of the previous triangle with a narrower stripe, asymptotically delimited by the lines $||B_+| - |B_-|| = 2A\sin\theta$ for $|B_\pm| \gg A$. A similar type of diagram for a nonquadratic system was discussed in Ref. [6].

C. The coordinate representation

We now turn to the representation of H and its eigenstates in terms of coordinate and momentum operators

$$Q = \frac{a + a^\dagger}{\sqrt{2}}, \quad P = \frac{a - a^\dagger}{i\sqrt{2}}, \tag{34}$$

satisfying $[Q,P] = i$. The Hamiltonian (2) becomes

$$H = \frac{1}{2}[\tilde{A}_-P^2 + \tilde{A}_+Q^2 + \tilde{B}(QP + PQ)] \tag{35}$$

$$= (Q \quad P)\tilde{\mathcal{H}}\begin{pmatrix} Q \\ P \end{pmatrix}, \quad \tilde{\mathcal{H}} = \mathcal{S}^\dagger \mathcal{H} \mathcal{S} = \begin{pmatrix} \tilde{A}_+ & \tilde{B} \\ \tilde{B} & \tilde{A}_- \end{pmatrix}, \tag{36}$$

where $\mathcal{S} = \binom{1\ \ i}{1\ -i}/\sqrt{2}$ and

$$\tilde{A}_\pm = A \pm \frac{B_+ + B_-}{2}, \quad \tilde{B} = \frac{B_+ - B_-}{2i}. \tag{37}$$

The Hermitian case corresponds to $\tilde{A}_\pm$ and $\tilde{B}$ real, while the generalized discrete positive spectrum case (33) to $\tilde{\mathcal{H}} + \tilde{\mathcal{H}}^\dagger > 0$. Thus, for $B_\pm$ real the border $|B_+ + B_-^*| = 2A$ corresponds to $\tilde{A}_- = 0$ or $\tilde{A}_+ = 0$, i.e., infinite mass or no quadratic potential, while for $B_\pm$ imaginary to $|\tilde{B}| = A$.

The diagonal form (3) can then be rewritten as

$$H = \frac{\lambda}{2}(P'^2 + Q'^2), \tag{38}$$

where $Q' = \frac{b + \bar{b}^\dagger}{\sqrt{2}}$ and $P' = \frac{b - \bar{b}^\dagger}{i\sqrt{2}}$ satisfy $[Q',P'] = i$ but are in general no longer Hermitian. They are related to Q,P through a general canonical transformation

$$\begin{pmatrix} Q' \\ P' \end{pmatrix} = \tilde{\mathcal{W}}\begin{pmatrix} Q \\ P \end{pmatrix}, \quad \tilde{\mathcal{W}} = \mathcal{S}^\dagger \mathcal{W} \mathcal{S} = \begin{pmatrix} \frac{\alpha + \bar{\alpha}^*}{2} & -\frac{\beta - \bar{\beta}^*}{2i} \\ \frac{\alpha - \bar{\alpha}^*}{2i} & \frac{\beta + \bar{\beta}^*}{2} \end{pmatrix}, \tag{39}$$

where $\binom{\alpha}{\beta} = u \pm v$, $\binom{\bar{\alpha}}{\bar{\beta}} = \bar{u} \pm \bar{v}$, and $\mathrm{Det}(\tilde{\mathcal{W}}) = 1$. Here λ can be expressed as

$$\lambda = \sqrt{\tilde{A}_+\tilde{A}_- - \tilde{B}^2}, \tag{40}$$

with $\pm\lambda$ the eigenvalues of $\tilde{\mathcal{M}}\tilde{\mathcal{H}} = \mathcal{S}^\dagger \mathcal{M}\mathcal{H}\mathcal{S}$.

Setting $Q|x\rangle = x|x\rangle$, the coordinate representations $\psi_0^b(x) \equiv \langle x|0_b\rangle$, $\psi_0^{\bar{b}}(x) \equiv \langle x|0_{\bar{b}}\rangle$ of the vacua can be found from Eqs. (20) and (22). They can also be derived by solving the corresponding differential equations $\langle x|b|0_b\rangle = 0$, $\langle x|\bar{b}|0_{\bar{b}}\rangle = 0$, i.e.,

$$[\alpha x + \beta\partial_x]\psi_0^b(x) = 0, \quad [\bar{\alpha}x + \bar{\beta}\partial_x]\psi_0^{\bar{b}}(x) = 0, \tag{41}$$

and read

$$\psi_0^b(x) \propto \exp\left[-\frac{\alpha}{2\beta}x^2\right], \quad \psi_0^{\bar{b}}(x) \propto \exp\left[-\frac{\bar{\alpha}}{2\bar{\beta}}x^2\right]. \tag{42}$$

Since $\mathrm{Re}[\frac{z_1+z_2}{z_1-z_2}] = \frac{|z_1|^2 - |z_2|^2}{|z_1 - z_2|^2}\ \forall\ z_1 \neq z_2 \in \mathbb{C}$, it is verified that they have finite standard norms if and only if $|v| < |u|$ and $|\bar{v}| < |\bar{u}|$. The wave functions of the excited states $|n_b\rangle$ and $|m_{\bar{b}}\rangle$ can be similarly obtained by applying $\bar{b}^\dagger$ and $b^\dagger$ to the functions (42), according to Eq. (26):

$$\psi_n^b(x) = \frac{1}{\sqrt{n!}}\left[\sqrt{\frac{\bar{\beta}^*}{2\beta}}\right]^n H_n\left(\frac{x}{\gamma}\right)\psi_0^b(x), \tag{43}$$

$$\psi_m^{\bar{b}}(x) = \frac{1}{\sqrt{m!}}\left[\sqrt{\frac{\beta^*}{2\bar{\beta}}}\right]^m H_m\left(\frac{x}{\gamma^*}\right)\psi_0^{\bar{b}}(x), \tag{44}$$

where $\gamma = (\beta\bar{\beta}^*)^{1/2}$ and $H_n(x)$ is a Hermite polynomial of degree n. These functions satisfy the biorthogonality relation (28), i.e., $\int_{-\infty}^{\infty}\psi_m^{\bar{b}*}(x)\psi_n^b(x)dx = \delta_{mn}\langle 0_{\bar{b}}\,|\,0_b\rangle$, with $\langle 0_{\bar{b}}\,|\,0_b\rangle = 1$ if normalization factors $(\sqrt{\pi}\beta)^{-1/2}$ and $(\sqrt{\pi}\bar{\beta})^{-1/2}$ are added in Eq. (42). They are verified to be the finite norm solutions to the Schrödinger equations associated with H and $H^\dagger$, respectively. In the case of $\psi_n^b(x)$, the latter reads

$$-\frac{1}{2}\tilde{A}_-\psi'' - i\tilde{B}\left[x\psi' + \frac{\psi}{2}\right] + \frac{1}{2}\tilde{A}_+x^2\psi = E\psi, \tag{45}$$

with $E = \lambda(n + 1/2)$, while in the case of $\psi_m^{\bar{b}}(x)$, $\tilde{A}_\pm$ and $\tilde{B}$ are to be replaced by $\tilde{A}_\pm^*$ and $\tilde{B}^*$, with $E = \lambda^*(m + 1/2)$.

D. The case of continuous spectrum

If $|v/u| < 1$ but $|\bar{v}/\bar{u}| > 1$, the vacuum $|0_{\bar{b}}\rangle$ of $\bar{b}$ is no longer well defined, since the coefficients of its expansion in the states $|n_a\rangle$, Eq. (22), become increasingly large for large n, and the associated eigenfunction $\psi_0^{\bar{b}}(x)$, Eq. (42), becomes divergent. This situation occurs whenever

$$\frac{|B_+|}{|B_-|} < \frac{|A-\lambda|}{|A+\lambda|}, \tag{46}$$

i.e., below the window (24), and corresponds to regions **II** in Figs. 1 and 2. The same occurs with the excited states $|n_{\bar{b}}\rangle$ defined in Eq. (26).

Instead, it is now the operator $\bar{b}^\dagger$ which has a convergent vacuum; namely,

$$|0_{\bar{b}^\dagger}\rangle \propto \sum_{n=0}^{\infty}\left(-\frac{\bar{u}^*}{2\bar{v}^*}\right)^n \frac{\sqrt{2n!}}{n!}|2n_a\rangle, \tag{47}$$

satisfying $\bar{b}^\dagger|0_{\bar{b}^\dagger}\rangle = 0$. Since we can write H [Eq. (3)] as

$$H = -\lambda[(-b)\bar{b}^\dagger + 1/2], \tag{48}$$

it becomes clear that $H|0_{\bar{b}^\dagger}\rangle = -\lambda/2|0_{\bar{b}^\dagger}\rangle$. Moreover, due to the commutation relation $[\bar{b}^\dagger, -b] = 1$, we may as well formally consider $-b$ as a creation operator and $\bar{b}^\dagger$ as an annihilation operator, and define the states

$$|n_{\bar{b}^\dagger}\rangle = \frac{(-b)^n|0_{\bar{b}^\dagger}\rangle}{\sqrt{n!}}, \tag{49}$$

which then satisfy $-b\bar{b}^\dagger|n_{\bar{b}^\dagger}\rangle = n|n_{\bar{b}^\dagger}\rangle$, and hence

$$H|n_{\bar{b}^\dagger}\rangle = -\lambda\left(n+\tfrac{1}{2}\right)|n_{\bar{b}^\dagger}\rangle. \tag{50}$$

Since the previous states $|0_b\rangle$ and $|n_b\rangle$ remain convergent, and Eq. (29) still holds, it is seen that H possesses in this case two sets of discrete eigenstates constructed from the vacua of b and $\bar{b}^\dagger$, with *opposite energies*. The wave functions of the "negative" band are given by

$$\psi_0^{\bar{b}^\dagger}(x) \propto \exp\left[\frac{\bar{\alpha}^*}{2\bar{\beta}^*}x^2\right],$$
$$\psi_n^{\bar{b}^\dagger}(x) \propto \frac{1}{\sqrt{n!}}\left[\sqrt{\frac{\beta}{2\bar{\beta}^*}}\right]^n H_n\left(\frac{ix}{\gamma}\right)\psi_0^{\bar{b}^\dagger}(x), \tag{51}$$

which are convergent since now $\text{Re}(\bar{\alpha}^*/\bar{\beta}^*) < 0$.

However, these eigenvalues do not exhaust, remarkably, the entire spectrum. The Schrödinger equation (45) has in the present case *two* linearly independent bounded eigenstates $|\nu_b\rangle$ and $|\nu_{\bar{b}^\dagger}\rangle$, for *any complex energy*

$$E_\nu = \lambda\left(\nu + \tfrac{1}{2}\right), \tag{52}$$

with $\nu \in \mathbb{C}$. As demonstrated in the appendix, the associated eigenfunctions $\psi_\nu^b(x) = \langle x\,|\,\nu_b\rangle$ and $\psi_\nu^{\bar{b}^\dagger}(x) = \langle x\,|\,\nu_{\bar{b}^\dagger}\rangle$ are given explicitly by

$$\psi_\nu^b(x) = \Xi(\nu)\left(\sqrt{\frac{\bar{\beta}^*}{2\beta}}\right)^n \exp\left(-\frac{i\tilde{B}+\lambda}{2\tilde{A}_-}x^2\right)$$
$$\times\left[H_\nu\left(\frac{x}{\gamma}\right) + (-1)^n H_\nu\left(-\frac{x}{\gamma}\right)\right], \tag{53}$$

$$\psi_\nu^{\bar{b}^\dagger}(x) = \Xi(\nu)\left(\sqrt{\frac{\beta}{2\bar{\beta}^*}}\right)^n \exp\left(-\frac{i\tilde{B}-\lambda}{2\tilde{A}_-}x^2\right)$$
$$\times\left[H_\nu\left(\frac{ix}{\gamma}\right) + (-1)^n H_\nu\left(-\frac{ix}{\gamma}\right)\right], \tag{54}$$

where $n = \lfloor \text{Re}(\nu)\rfloor$, with $\lfloor x\rfloor$ being the greatest integer lower than x (floor function), and

$$\Xi(\nu) = \begin{cases}\sqrt{(|\nu|-1)!}, & \nu = -1,-2,\ldots\\ \frac{1}{\sqrt{\Gamma(\nu+1)}} & \text{otherwise.}\end{cases} \tag{55}$$

For integer $\nu \geqslant 0$, these functions are proportional to the previous expressions (43) and (51). For general $\nu \in \mathbb{C}$, they satisfy

$$H|\nu_b\rangle = \lambda\left(\nu+\tfrac{1}{2}\right)|\nu_b\rangle, \tag{56}$$

$$H\,|\nu_{\bar{b}^\dagger}\rangle = -\lambda\left(\nu+\tfrac{1}{2}\right)|\nu_{\bar{b}^\dagger}\rangle, \tag{57}$$

with

$$b|\nu_b\rangle \propto \sqrt{\nu}|\nu-1_b\rangle,$$
$$\bar{b}^\dagger|\nu_b\rangle \propto \begin{cases}\sqrt{\nu+1}|\nu+1_b\rangle & (\nu\neq -1)\\ |0_b\rangle & (\nu=-1).\end{cases} \tag{58}$$

$$\bar{b}^\dagger|\nu_{\bar{b}^\dagger}\rangle \propto \sqrt{\nu}|\nu-1_{\bar{b}^\dagger}\rangle,$$
$$(-b)|\nu_{\bar{b}^\dagger}\rangle \propto \begin{cases}\sqrt{\nu+1}|\nu+1_{\bar{b}^\dagger}\rangle & (\nu\neq -1)\\ |0_{\bar{b}^\dagger}\rangle & (\nu=-1),\end{cases} \tag{59}$$

where the proportionality constant is a phase factor. Expressions (56)–(59) are in agreement with Eqs. (17) and (18). They are valid in this region for both real or complex λ.

Note that, if $\bar{b}^\dagger|-1_b\rangle$ would vanish, then $|-1_b\rangle$ would be proportional to $|0_{\bar{b}^\dagger}\rangle$, which is not the case. A similar argument holds for $b|-1_{\bar{b}^\dagger}\rangle$. It is also verified that in the case of discrete spectrum (region **I**), states such as $|-1_b\rangle$ do not exist, i.e., the solution of the first-order differential equation $\langle x|\bar{b}^\dagger|-1_b\rangle = \langle x\,|\,0_b\rangle$ is divergent. In addition, we remark that Eqs. (53) and (54) are always linearly independent solutions of the Schrödinger equation (45), but in region **I** the function (54) is always divergent whereas Eq. (53) is divergent except for $\nu = n = 0,1,2,\ldots$.

E. The case of no convergent eigenstates

If now $|\bar{v}/\bar{u}| < 1$ but $|v/u| > 1$, i.e.,

$$\frac{|B_+|}{|B_-|} > \frac{|A+\lambda|}{|A-\lambda|}, \tag{60}$$

neither b nor $\bar{b}$ have a convergent vacuum, so that the eigenstates $|n_b\rangle$ and $|n_{\bar{b}}\rangle$ of Sec. II B are not well defined. In fact, Eqs. (53) and (54) become *divergent for* any ν, so that H has no convergent eigenfunctions for *any* value of E. This case corresponds to regions **III** in Figs. 1 and 2.

On the other hand, it is the operator $b^\dagger$ which now has a well-defined vacuum $|0_{b^\dagger}\rangle$, in addition to $\bar{b}$, which preserves its vacuum $|0_{\bar{b}}\rangle$. Therefore, one can define the states $|n_{b^\dagger}\rangle$ and $|n_{\bar{b}}\rangle$ in the same way as the treatment of the previous section, and also $|\nu_{b^\dagger}\rangle$ and $|\nu_{\bar{b}}\rangle$ for any $\nu \in \mathbb{C}$, which will be eigenstates of $H^\dagger$. Hence, in this case $H^\dagger$, rather than H, has two linearly independent bounded eigenfunctions for every complex value of E. In contrast, in **II** $H^\dagger$ has no bounded eigenstate.

F. Nondiagonalizable case

The matrix $\mathcal{MH}$ becomes nondiagonalizable when $\lambda = 0$, i.e., rank$(\mathcal{H}) = 1$. This case occurs whenever $B_+B_- = A^2$ and corresponds to the dashed curve in Fig. 1, which lies in regions **II** and **III**. The operator H takes here the single square form (19).

We first analyze the sector lying in region **II**. In the limit $B_+ \to 0$, with $B_- = A^2/B_+ \to \infty$, H becomes proportional to a^2. Its eigenstates then become the well-known *coherent states*

$$|\alpha_a\rangle \propto \exp[\alpha a^\dagger]|0_a\rangle, \tag{61}$$

satisfying $a|\alpha_a\rangle = \alpha|\alpha_a\rangle$, $\alpha \in \mathbb{C}$, with $\frac{2B_+}{A^2}H|\pm\alpha_a\rangle \to \alpha^2|\pm\alpha_a\rangle$. This implies a *continuous twofold degenerate spectrum*, as in the rest of region **II**. The spectrum of H in **II** is then similar to that of a^2, reflecting the fact that here both b and $\bar{b}^\dagger$ have a convergent vacuum and are then annihilation operators.

In fact, for $\lambda \to 0$ and $A > 0$, the operators b and $\bar{b}^\dagger$ of Eq. (4) become *proportional*, i.e., $\bar{b}^\dagger \to \sqrt{B_-/B_+}b$, such that $H \propto b^2$ at leading order. At the curve $\lambda = 0$ and within region **II**, H takes the exact form

$$H = \frac{|B_-| - |B_+|}{2}\tilde{b}^2, \quad \tilde{b} = \frac{\sqrt{B_-}a + \sqrt{B_+}a^\dagger}{\sqrt{|B_-| - |B_+|}}, \tag{62}$$

where $\tilde{b}$ fulfills $[\tilde{b},\tilde{b}^\dagger] = 1$ and has a *convergent* vacuum $|0_{\tilde{b}}\rangle$ since here $|B_+| < |B_-|$. It then represents a proper *annihilation* operator. The eigenstates of H become its coherent states $|\alpha_{\tilde{b}}\rangle \propto \exp[\alpha\tilde{b}^\dagger]|0_{\tilde{b}}\rangle$ satisfying $\tilde{b}|\alpha_{\tilde{b}}\rangle = \alpha|\alpha_{\tilde{b}}\rangle$, such that

$$H|\pm\alpha_{\tilde{b}}\rangle = \frac{|B_-| - |B_+|}{2}\alpha^2|\pm\alpha_{\tilde{b}}\rangle, \tag{63}$$

with $\alpha \in \mathbb{C}$. The spectrum is then complex, continuous and twofold degenerate, as in the rest of sector **II**. The eigenfunctions become

$$\psi_\alpha(x) = \langle x\,|\,\alpha_{\tilde{b}}\rangle \propto \exp\left[-\frac{1}{2}\frac{\sqrt{B_-}+\sqrt{B_+}}{\sqrt{B_-}-\sqrt{B_+}} \times \left(x - \sqrt{2}\alpha\frac{\sqrt{|B_-|-|B_+|}}{\sqrt{B_-}+\sqrt{B_+}}\right)^2\right]. \tag{64}$$

On the other hand, in region **III**, $|B_+| > |B_-|$ and along the curve $\lambda = 0$ we have instead

$$H = \frac{|B_+| - |B_-|}{2}\tilde{b}^{\dagger 2}, \quad \tilde{b}^\dagger = \frac{\sqrt{B_-}a + \sqrt{B_+}a^\dagger}{\sqrt{|B_+| - |B_-|}}, \tag{65}$$

with $\tilde{b}^\dagger$ a proper *creation* operator satisfying $[\tilde{b},\tilde{b}^\dagger] = 1$ and having no bounded vacuum. Hence, here H has no bounded eigenstates while $H^\dagger$ has a continuous complex spectrum.

Finally, in the Hermitian limit $|B_+| = |B_-| = A$, i.e., when the curve $\lambda = 0$ crosses the border between **II** and **III**, $H \to \frac{A}{2}(e^{-i\phi}a + e^{i\phi}a^\dagger)^2$, becoming proportional to Q^2 for $\phi = 0$ (or equivalently, to P^2 if $\phi = \pi/2$). It then possesses a continuous twofold degenerate nonnegative *real* spectrum, although with non-normalizable eigenstates ($|x\rangle$ or $|p\rangle$). This case corresponds in Fig. 1 to the two "critical" points where *all three regions* **I, II, III** *merge*, i.e., $|v/u| = |\bar{v}/\bar{u}| = 1$. Thus, at the nondiagonalizable curve $\lambda = 0$, H is proportional to the square of: an annihilation operator inside region **II**, a creation operator inside region **III**, and a coordinate or momentum operator at the crossing with the Hermitian case.

G. Intermediate regions

We finally discuss the border between regions **I** and **II** or **III**. These intermediate lines have either $|v/u| = 1$ or $|\bar{v}/\bar{u}| = 1$. When crossing from **I** to **II** (**III**), $\bar{b}$ (b) undergoes an *annihilation* $\to$ *creation* transition, loosing its bounded vacuum and becoming at the crossing a coordinate or momentum.

As can be verified from Eqs. (53) and (54) when $\tilde{A}_- \neq 0$, at the border between **I** and **II** H has still a discrete spectrum and satisfies Eq. (29), since Eq. (53) remains convergent just for $\nu = n$. On the other hand, (54) has no longer a finite norm since $(i\tilde{B} - \lambda)/(2\tilde{A}_-)$ is an imaginary number. However, the dual states $|0_{\bar{b}}\rangle$ and $|n_{\bar{b}}\rangle$, while also lacking a finite norm $\langle n_{\bar{b}}\,|\,n_{\bar{b}}\rangle$, still have *finite* biorthogonal norms $\langle m_{\bar{b}}\,|\,n_b\rangle$, fulfilling Eq. (28). In contrast, at the border **I–III** H ceases to have convergent eigenfunctions for any value of ν, since $|n_b\rangle$ stops being convergent, while dual states $|n_{\bar{b}}\rangle$ remain convergent.

When $\tilde{A}_- = 0$, which corresponds to the case $B_\pm$ real and $B_+ + B_- = 2A$ (the border between **I** and regions **II-III** in Fig. 1), we have $\bar{v} = \bar{u}$. In this case, and for $A \neq B_-$, Eq. (45) becomes of first order and has a unique solution given by

$$\psi_\nu^b(x) \propto e^{-\frac{Ax^2}{2(B_- - A)}}x^\nu, \tag{66}$$

where we have set $E = \lambda(\nu + 1/2)$, with $\lambda = B_- - A$, along this line. Hence, at the border with region **III** ($B_- < A$) Eq. (66) is always divergent for $|x| \to \infty$, while at the border with **II** it is always convergent for $|x| \to \infty$ yet regular at $x = 0$ *just for* $\nu = n = 0,1,2,\ldots$, as in the previous case. For these values, Eq. (66) becomes proportional to Eq. (43).

Regarding the dual states, at this line $\bar{b} = \bar{b}^\dagger = \sqrt{2}\bar{u}Q$ (since $\bar{u}$ is real), and as such $|0_{\bar{b}}\rangle$ is the state with $Q = 0$, i.e., $\langle x\,|\,0_{\bar{b}}\rangle \propto \delta(x)$. In fact, for $\bar{v} \to \bar{u}$ the coordinate representation of the state $|0_{\bar{b}}\rangle$ in Eq. (22) becomes a δ function, as also seen from Eq. (42):

$$\langle x\,|\,0_{\bar{b}}\rangle \to \frac{e^{-x^2/2}}{\pi^{1/4}}\sum_{n=0}^{\infty}\frac{H_{2n}(x)H_{2n}(0)}{2^{2n}(2n)!} = \pi^{1/4}\delta(x), \tag{67}$$

where we have used $\delta(x) = \langle x\,|\,0_{\bar{b}}\rangle = \sum_{n=0}^{\infty}\langle x|n_a\rangle\langle n_a|0_{\bar{b}}\rangle$. It is then still verified that $\langle 0_{\bar{b}}\,|\,0_b\rangle$ is a finite number. The same holds for the remaining states $|n_{\bar{b}}\rangle$, with $\langle x\,|\,n_{\bar{b}}\rangle$ involving derivatives of the δ function, such that Eq. (28) still holds.

III. THE GENERAL N-DIMENSIONAL CASE

We now discuss the main features of the N-dimensional case. We consider a general quadratic form in boson operators

$a_i, a_j^\dagger$ satisfying $[a_i,a_j^\dagger]=\delta_{ij}$, $[a_i,a_j]=0$, $i,j=1,\dots,N$:

$$H=\sum_{i,j}A_{ij}\left(a_i^\dagger a_j+\frac{1}{2}\delta_{ij}\right)+\frac{1}{2}(B_{ij}^+a_i^\dagger a_j^\dagger+B_{ij}^-a_ia_j) \quad (68)$$

$$=\frac{1}{2}(a^\dagger \quad a)\mathcal{H}\begin{pmatrix}a\\ a^\dagger\end{pmatrix},\quad \mathcal{H}=\begin{pmatrix}A & B_+\\ B_- & A^T\end{pmatrix}. \quad (69)$$

Here $a=(a_1,\dots,a_N)$ and $B_\pm$ are *symmetric* $N\times N$ matrices of elements $B_{ij}^\pm$, such that $\mathcal{H}$ satisfies

$$\mathcal{H}^T=\mathcal{R}\mathcal{H}\mathcal{R},\quad \mathcal{R}=\begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}. \quad (70)$$

Following the treatment of Ref. [36] for the general Hermitian case, we define new operators b_i, $\bar{b}_i^\dagger$ through a generalized Bogoliubov transformation

$$\begin{pmatrix}b\\ \bar{b}^\dagger\end{pmatrix}=\mathcal{W}\begin{pmatrix}a\\ a^\dagger\end{pmatrix},\quad \mathcal{W}=\begin{pmatrix}U & V\\ \bar{V}^* & \bar{U}^*\end{pmatrix}, \quad (71)$$

where again $\bar{b}_i^\dagger$ may not coincide with $b_i^\dagger$ although the bosonic commutation relations are preserved:

$$[b_i,\bar{b}_j^\dagger]=\delta_{ij},\quad [b_i,b_j]=[\bar{b}_i^\dagger,\bar{b}_j^\dagger]=0. \quad (72)$$

These conditions imply [36,37]

$$\mathcal{W}\mathcal{M}\mathcal{R}\mathcal{W}^T\mathcal{R}=\mathcal{M} \quad (73)$$

[$\mathcal{M}$ is the matrix (10) extended to $2N\times 2N$], i.e.,

$$U(\bar{U}^*)^T-V(\bar{V}^*)^T=\mathbb{1}, \quad (74)$$

$$VU^T-UV^T=0,\quad \bar{V}\bar{U}^T-\bar{U}\bar{V}^T=0. \quad (75)$$

We can then rewrite H exactly as in Eqs. (8) and (9):

$$H=\frac{1}{2}(\bar{b}^\dagger \quad b)\mathcal{H}'\begin{pmatrix}b\\ \bar{b}^\dagger\end{pmatrix},\quad \mathcal{H}'=\mathcal{M}\mathcal{W}\mathcal{M}\mathcal{H}\mathcal{W}^{-1}, \quad (76)$$

where $\mathcal{H}'$ has again the form (69) and satisfies (70) due to Eq. (73). The problem of obtaining a normal-mode representation

$$H=\sum_i\lambda_i\left(\bar{b}_i^\dagger b_i+\frac{1}{2}\right), \quad (77)$$

leads then to the diagonalization of the matrix

$$\mathcal{M}\mathcal{H}=\begin{pmatrix}A & B_+\\ -B_- & -A^T\end{pmatrix}, \quad (78)$$

which is that representing the commutation relations of Eq. (14) in the present general case: $[H,\binom{a}{a^\dagger}]=\mathcal{M}\mathcal{H}\binom{a}{a^\dagger}$.

A basic result is that the eigenvalues of (78) *always come in pairs of opposite sign*, as in the Hermitian case [36] (see also Ref. [44]): Noting that $\mathcal{R}\mathcal{M}=-\mathcal{M}\mathcal{R}$ and $\mathcal{M}^2=\mathcal{R}^2=\mathbb{1}$, Eq. (70) implies

$$(\mathcal{M}\mathcal{H}-\lambda\mathbb{1})^T=\mathcal{R}\mathcal{H}\mathcal{R}\mathcal{M}-\lambda\mathbb{1}=\mathcal{R}\mathcal{M}(\mathcal{M}\mathcal{H}+\lambda\mathbb{1})\mathcal{R}\mathcal{M},$$

and hence $\mathrm{Det}[\mathcal{M}\mathcal{H}-\lambda\mathbb{1}]=\mathrm{Det}[\mathcal{M}\mathcal{H}+\lambda\mathbb{1}]$, entailing that if λ is an eigenvalue of $\mathcal{M}\mathcal{H}$, so is $-\lambda$.

From Eq. (70) we also see that if Z_i are eigenvectors of $\mathcal{M}\mathcal{H}$ satisfying $\mathcal{M}\mathcal{H}Z_i=\lambda_iZ_i$, then $Z_i^T\mathcal{R}\mathcal{M}Z_j(\lambda_i+\lambda_j)=0$, implying the orthogonality relations

$$Z_i^T\mathcal{R}\mathcal{M}Z_j=0\;(\lambda_i\neq-\lambda_j). \quad (79)$$

The pairs $(b_i,\bar{b}_i^\dagger)$ emerge then from the eigenvectors $Z_i,Z_{\bar{i}}$ associated with *opposite* eigenvalues $\pm\lambda_i$, which are to be scaled such that

$$Z_i^T\mathcal{R}\mathcal{M}Z_{\bar{i}}=1. \quad (80)$$

Writing $Z_i=(\bar{U}^* \; -\bar{V}^*)_i^T$ and $Z_{\bar{i}}=(-V \;\; U)_i^T$, we can form with them the eigenvector matrix $\mathcal{W}^{-1}$, with Eqs. (79) and (80) ensuring that $\mathcal{W}$ will satisfy Eq. (73).

Therefore, if $\mathcal{M}\mathcal{H}$ is *diagonalizable*, a diagonalizing matrix $\mathcal{W}$ satisfying Eq. (73) will exist such that H can be written in the diagonal form (77). The N-dimensional H can then be reduced to a sum of N commuting one-dimensional systems (*complex normal modes*) described by operators $H_i=\lambda_i(\bar{b}_i^\dagger b_i+\frac{1}{2})$. The normal operators b_i, $\bar{b}_i^\dagger$, satisfy

$$[H,b_i]=-\lambda_ib_i,\quad [H,\bar{b}_i^\dagger]=\lambda_i\bar{b}_i^\dagger, \quad (81)$$

diagonalizing the commutator algebra with H and satisfying then Eqs. (17) and (18) $\forall\, b=b_i$.

Now, if a common vacuum $|0_b\rangle$ exists such that

$$b_i|0_b\rangle=0, \quad (82)$$

for $i=1,\dots,N$, it must necessarily be of the form [39]

$$|0_b\rangle\propto\exp\left[-\frac{1}{2}\sum_{i,j}(U^{-1}V)_{ij}a_i^\dagger a_j^\dagger\right]|0_a\rangle, \quad (83)$$

where $U^{-1}V$ is a *symmetric* matrix due to Eq. (75). Equation (83) can be directly checked by application of b_i. Similarly, assuming a common vacuum $|0_{\bar{b}}\rangle$ exists such that

$$\bar{b}_i|0_{\bar{b}}\rangle=0, \quad (84)$$

for $i=1,\dots,N$, it must be of the form

$$|0_{\bar{b}}\rangle\propto\exp\left[-\frac{1}{2}\sum_{i,j}(\bar{U}^{-1}\bar{V})_{ij}a_i^\dagger a_j^\dagger\right]|0_a\rangle. \quad (85)$$

Assuming these series are convergent, which implies that $U^{-1}V$ and $\bar{U}^{-1}\bar{V}$ have both all singular values $\sigma_i<1$, $\bar{\sigma}_i<1$, we can define the states

$$|n_1,\dots,n_{Nb}\rangle=\left(\prod_i\frac{(\bar{b}_i^\dagger)^{n_i}}{\sqrt{n_i!}}\right)|0_b\rangle, \quad (86)$$

$$|m_1,\dots,m_{N\bar{b}}\rangle=\left(\prod_i\frac{(b_i^\dagger)^{m_i}}{\sqrt{m_i!}}\right)|0_{\bar{b}}\rangle. \quad (87)$$

Due to the commutation relations (72), these states form again a biorthogonal set,

$$\langle m_1,\dots,m_{N\bar{b}}|n_1,\dots,n_{Nb}\rangle=\delta_{m_1n_1}\dots\delta_{m_Nn_N}\langle 0_{\bar{b}}|0_b\rangle, \quad (88)$$

and satisfy

$$H|n_1,\ldots,n_{Nb}\rangle = \left[\sum_i \lambda_i\left(n_i+\frac{1}{2}\right)\right]|n_1,\ldots,n_{Nb}\rangle, \quad (89)$$

$$H^\dagger|m_1,\ldots,m_{N\bar{b}}\rangle = \left[\sum_i \lambda_i^*\left(m_i+\frac{1}{2}\right)\right]|m_1,\ldots,m_{N\bar{b}}\rangle. \quad (90)$$

Thus, both H and $H^\dagger$ possess in this case a *discrete* spectrum. Such a spectrum can be real if H has some antilinear (generalized $\mathcal{PT}$) symmetry (for instance, $\mathcal{H}$ real).

In a general situation, a common vacuum may exist just for a certain subset of operators b_i and $\bar{b}_i$, leading to terms H_i with properties similar to those encountered in the previous section. An important difference is to be found in the nondiagonalizable cases: The corresponding modes may not necessarily be of the form (19), and are not necessarily associated with vanishing eigenvalues $\lambda_i = 0$, since Jordan forms of higher dimension can arise, as was already shown in two-dimensional systems [37,45] in the context of Hermitian yet unstable Hamiltonians. Besides, $\mathcal{MH}$ may remain diagonalizable in the presence of vanishing eigenvalues [37,46].

IV. CONCLUSIONS

We have first analyzed the spectrum and normal modes of a general one-dimensional quadratic bosonic form, showing that it can exhibit three distinct regimes:

(i) A harmonic phase characterized by a discrete spectrum of both H and $H^\dagger$, with bounded eigenstates constructed from Gaussian vacua, which form a biorthogonal set. Such a phase, which comprises the cases considered in Refs. [26,27], arises when the deviation from the stable Hermitian case is not "too large" [Eq. (24), equivalent to Eqs. (32) and (33) for $\lambda > 0$], in which case the generalized normal boson operators $\bar{b}^\dagger$, b can be considered as creation and annihilation operators, respectively. According to the phase of λ, the discrete spectrum can be real or complex, but in the latter it can be made real by applying a trivial phase factor (as opposed to discrete regimes in nonquadratic Hamiltonians [47]).

(ii) A coherent-like phase where H exhibits a complex twofold degenerate continuous spectrum while $H^\dagger$ has no bounded eigenstates. It corresponds to large deviations from the Hermitian harmonic case. The normal operators $\bar{b}^\dagger$, b can be considered as a pair of annihilation operators, each with a convergent vacuum yet still satisfying a bosonic commutator. The spectrum is then similar to that of a square of a bosonic annihilation operator.

(iii) An adjoint coherent phase where $H^\dagger$ has a continuous complex spectrum while H has no bounded eigenstates. Here the normal modes are a pair of creation operators. While **(ii)** and **(iii)** might be considered as having no proper biorthogonal eigenstates, the convergent eigenstates (of H or $H^\dagger$) constitute a generalization of the standard coherent states, which arise here in the particular case of a nondiagonalizable matrix $\mathcal{MH}$. These regimes may be considered to correspond to a broken generalized $\mathcal{PT}$ symmetry, since there are complex eigenvalues. Nonetheless, the latter do not emerge from the coalescence of two or more real eigenvalues [2] but from the onset of convergence of eigenstates with complex quantum number ν.

We have also analyzed the transition curves between these previous regimes, where one of the operators changes from creation to annihilation (or vice versa). At these curves such operator is actually a coordinate (or momentum), and even though there is just a discrete spectrum (with bounded eigenstates) of either H or $H^\dagger$, the biorthogonality relations are still preserved. Explicit expressions for eigenfunctions were provided in all regimes.

The normal-mode decomposition of the N-dimensional non-Hermitian case has also been discussed, together with the corresponding harmonic regime. It opens the way to investigate in detail along these lines the spectrum of more complex specific non-Hermitian quadratic systems.

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APPENDIX: SOLUTIONS OF THE SCHRÖDINGER EQUATION IN THE CASE OF CONTINUOUS SPECTRUM

The solutions to the Schrödinger equation (45) can be obtained by making the substitution

$$\psi(x) = \exp\left[-\frac{i\tilde{B}+\lambda}{2\tilde{A}_-}x^2\right]\phi\left(\frac{x}{\gamma}\right). \quad \text{(A1)}$$

We obtain the Hermite equation [48]:

$$\phi''(z) - 2z\phi'(z) + 2\nu\phi(z) = 0, \quad \text{(A2)}$$

with $z = x/\gamma$ and $\nu = (2E-\lambda)/(2\lambda)$. For complex ν, four solutions are

$$\phi_\nu^{(1)}(z) = H_\nu(z), \quad \phi_\nu^{(2)}(z) = H_\nu(-z),$$
$$\phi_\nu^{(3)}(z) = e^{z^2}H_{-\nu-1}(iz), \quad \phi_\nu^{(4)}(z) = e^{z^2}H_{-\nu-1}(-iz), \quad \text{(A3)}$$

where H_ν are the Hermite functions [48]. Since the Hermite equation is of second order, any of these solutions can be written as a linear combination of two others. For instance, for real $A, B_+, B_- > 0$:

$$H_\nu(z) = \frac{2^\nu\Gamma(\nu+1)}{\sqrt{\pi}}e^{z^2}[e^{\nu\pi i/2}H_{-\nu-1}(iz) + e^{-\nu\pi i/2}H_{-\nu-1}(-iz)]. \quad \text{(A4)}$$

Additionally, note that, for integer $\nu \geqslant 0$, $\phi_1 = (-1)^\nu\phi_2$ whereas for integer $\nu < 0$, $\phi_3 = (-1)^{\nu+1}\phi_4$.

The asymptotic behavior of the Hermite functions for $|\arg z| < 3/4$ goes as follows:

$$H_\nu(z) \sim (2z)^\nu + O(|z|^{\nu-2}), \quad \text{(A5)}$$

and for $\pi/4 + \delta \leqslant \arg z \leqslant 5\pi/4 - \delta$ (which includes z on the real negative axis):

$$H_\nu(z) \sim (2z)^\nu[1 + O(|z|^{-2})] - \frac{\sqrt{\pi}e^{\nu\pi i}}{\Gamma(-\nu)}e^{z^2}z^{-\nu-1}[1+O(|z|^{-2})]. \quad \text{(A6)}$$

Note that

$$e^{z^2}\exp\left[-\frac{i\tilde{B}+\lambda}{2\tilde{A}_-}x^2\right]=\exp\left[-\frac{i\tilde{B}-\lambda}{2\tilde{A}_-}x^2\right]. \quad (A7)$$

For Hermitian H, $\tilde{B}$ is either a real number or zero, and λ determines whether the eigenfunctions are bounded or not (i.e., if λ is real and positive then there are *some* bounded eigenfunctions, whereas for λ negative or imaginary every eigenfunction is divergent). In such a case, for positive integer ν only $\phi_\nu^{(1)}$ (and $\phi_\nu^{(2)}$, since they are linearly dependent) may be bounded [see Eq. (A6)], and for other values of ν there are no bounded eigenfunctions. On the other hand, for non-Hermitian H, the convergence of both linearly independent eigenstates may be assured provided that $\text{Re}[(i\tilde{B}-\lambda)/\tilde{A}_-]>0$, which is fulfilled in region **II**, i.e., when both b and $\bar{b}^\dagger$ have convergent vacua. Moreover, both linearly independent eigenstates may be convergent even if λ is an imaginary number or zero, which implies for real $A, B_\pm$, that region **II** extends into the imaginary part of the spectrum in Fig. 1.

The eigenfunctions of H must then be constructed from (A3) in such a way that they behave as the eigenstates $|n_b\rangle$ and $|n_{\bar{b}^\dagger}\rangle$, i.e., they satisfy Eqs. (26) and (27), and they must be even or odd with respect to coordinate inversion $x \to -x$ (since the Hamiltonian is parity invariant). These considerations lead to the eigenfunctions (53) and (54).

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