Non-negativity of conditional von Neumann entropy and global unitary operations

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Conditional von Neumann entropy is an intriguing concept in quantum information theory. In the present work, we examine the effect of global unitary operations on the conditional entropy of the system. We start with a set containing states with a non-negative conditional entropy and find that some states preserve the non-negativity under unitary operations on the composite system. We call this class of states the absolute conditional von Neumann entropy non-negative (ACVENN) class. We characterize such states for $2 \otimes 2$–dimensional systems. From a different perspective the characterization accentuates the detection of states whose conditional entropy becomes negative after the global unitary action. Interestingly, we show that this ACVENN class of states forms a set which is convex and compact. This feature enables the existence of Hermitian witness operators. With these we can distinguish the unknown states which will have a negative conditional entropy after the global unitary operation. We also show that this has immediate application to superdense coding and state merging, as the negativity of the conditional entropy plays a key role in both these information processing tasks. Some illustrations followed by analysis are also provided to probe the connection of such states with absolutely separable states and absolutely local states.

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I. INTRODUCTION

Entanglement [1], which lies at the heart of quantum mechanics, is not only of deep philosophical interest [2] but also established as the most pivotal resource in various information processing tasks: teleportation [3], superdense coding [4], key generation [5,6], secret sharing [7], remote entanglement distribution [8], and many more [9–11]. However, not all entangled states can be directly used for an information processing task; pertinent mentions in this regard are the bound entangled states [12]. However, these entangled states are available when we go beyond $2 \otimes 2$ and $2 \otimes 3$ systems, where we do not have necessary sufficient conditions like the Peres-Horodecki criterion [13] for detection of entanglement. Some entangled states have to be processed by local filtering [14] before they can be used in a task. As a consequence, teleportation witnesses and thermodynamical witnesses [13,16] have been devised which can identify useful entangled states for various tasks. In multiqubit systems concepts like "task-oriented entangled" states [17] have been introduced.

The ubiquitous role of entanglement in information processing tasks has motivated recent research in the generation of entangled states from separable states. Global unitary operations can play a significant role in this scenario, as local unitaries cannot generate entanglement. However, there are some separable states, termed absolutely separable (AS) [18], from which no entanglement can be produced even with any arbitrary global unitary operation. Characterization of such states has been an active line of research recently [19]. This notion of "absoluteness" was extended to define absolutely Bell-CHSH local states and absolute unsteerability [20,21]. The notion of absoluteness indicates that the state preserves a certain characteristic trait under global unitary transformations. For absolutely separable states it is separability; for absolutely Bell-CHSH local states it is their nature of being Bell-CHSH local.

The conditional von Neumann entropy is another such characteristic trait of quantum states. Unlike its classical counterpart this quantity can be negative [22], providing yet again a departure from classical information theory. An operational interpretation of the quantum conditional entropy was provided in [23], in terms of state merging. The negativity of the conditional entropy also indicates the signature of entanglement, although the converse of the statement is not true, as there are entangled states with non-negative conditional entropy. Conditional entropy also plays a key role in dense coding [24], as a bipartite quantum state is useful for dense coding in the sense that it will have a quantum advantage if and only if it has a negative conditional entropy.

Negativity of the conditional entropy being such an important yardstick, our present work probes whether it is always possible to start with a state having a non-negative conditional entropy and arrive at a state having a negative conditional entropy via global unitaries. We find that there is a class of states which preserve the non-negativity of the conditional entropy under global unitary transformations. The characterization also enables one to identify useful states whose conditional entropy becomes negative with a global unitary. It is interesting to find that this class of state, the absolute conditional von Neumann entropy non-negative (ACVENN) class, which preserves the non-negativity of the conditional entropy, is a convex and compact set. This in

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principle guarantees that we can create a witness operator to detect these states which can arrive at a negative conditional entropy in spite of starting with a non-negative conditional entropy using global unitary operations. Since separability and nonlocality are also important distinctive features of quantum mechanics, we also discuss the connections of these states preserving the non-negativity of the conditional entropy under a global unitary with the AS states and the recently introduced absolutely Bell-CHSH local (AL) states [20].

Our work has immediate applications in information processing tasks like superdense coding [4] and state merging [23]. In superdense coding states with a negative conditional entropy give us quantum advantages, while in state merging the same states are useful as a potential future resource. One starts with some seemingly useless states having a non-negative conditional entropy; then, using global unitaries as a resource, one can turn these states into states having a negative conditional entropy. Since the ACVENN class is convex and compact, it is in principle possible to create a witness operator to detect these transformed states.

In Sec. II we give an introduction to all the related concepts that are relevant to this article. In Sec. III we report a general necessary and sufficient condition to characterize the ACVENN class of states in the state space of two-qubit systems. In Sec. IV, we show that this ACVENN class of states is convex and compact, which in principle allows us to construct a witness operator for identifying those states which do not belong to this class. In Sec. V we connect this ACVENN class of states with the absolutely separable and absolutely local states. In Sec. VI we show the potential application of characterizing such states in various information processing tasks like superdense coding and state merging. Finally, we conclude in Sec. VII.

II. USEFUL DEFINITIONS AND RELATED CONCEPTS

In this section, we briefly introduce various concepts which will be useful and are related to the main theme of our paper. We present these concepts in different subsections.

A. General two-qubit states

In this work we consider Bloch representations of generalized two-qubit states. A general two-qubit state is represented in canonical form as

$$\rho = \frac{1}{4}\left[\mathbb{I}_2 \otimes \mathbb{I}_2 + \sum_{i=1}^{3} r_i \sigma_i \otimes \mathbb{I}_2 + \sum_{i=1}^{3} s_i \mathbb{I}_2 \otimes \sigma_i + \sum_{i,j=1}^{3} t_{ij} \sigma_i \otimes \sigma_j\right], \tag{1}$$

where $r_i = \mathrm{Tr}[\rho(\sigma_i \otimes \mathbb{I}_2)]$ and $s_i = \mathrm{Tr}[\rho(\mathbb{I}_2 \otimes \sigma_i)]$ are local Bloch vectors. The correlation matrix is given by $T = [t_{ij}]$, where $t_{ij} = \mathrm{Tr}[\rho(\sigma_i \otimes \sigma_j)]$ with $[\sigma_i;\ i = \{1,2,3\}]$ are $2 \otimes 2$ Pauli matrices and $\mathbb{I}_2$ denotes identity.

In this paper, we use the notation Q to denote the set of all two-qubit states.

B. Separable and absolutely separable classes of states

When we go beyond the one-qubit system to the two-qubit system, we come across the notion of entanglement, states which cannot be written as a convex combination of the tensor products of one-qubit systems. The exact complement of this are states for which the composite system can be written as a convex combination of the tensor products of subsystems. However, the definition is not as straightforward when we go beyond two-qubit pure states. For a mixed quantum system consisting of two subsystems the general definition of being separable is when the density matrix can be written as $\sigma_{\mathrm{sep}} = \sum \lambda_i \sigma^A \otimes \sigma^B$ ($\sum \lambda_i = 1,\ \lambda_i \geqslant 0$), where σ^A and σ^B are the density matrices for the two subsystems A and B [13]. Set S denotes the class of separable states. Lately people have identified absolutely separable states [18,19], which are states that remain separable under all global unitary operations, i.e., $\mathrm{AS} = \{\sigma_{\mathrm{as}} : U\sigma_{\mathrm{as}}U^\dagger \text{ is separable } \forall\ U\}$.

C. Local and absolutely local classes of states

We denote the set of all states which do not violate the Bell-CHSH inequality L [15]. Recall that any density matrix for two qubits can be written in the canonical form, where T denotes the correlation matrix corresponding to ρ. The function $M(\rho)$ is defined as the sum of the maximum two eigenvalues of $T^t T$. Any state with $M(\rho) \leqslant 1$ is considered local with respect to the Bell-CHSH inequality [25]. The set of states that do not violate Bell-CHSH inequality is denoted $L = \{\sigma_L : M(\sigma_L) \leqslant 1\}$. Recently researchers were able to characterize the states which do not violate Bell-CHSH inequality under any global unitary. The set containing these states is denoted AL [20] and is defined as $\mathrm{AL} = \{\sigma_{\mathrm{al}} : M(U\sigma_{\mathrm{al}}U^\dagger) \leqslant 1,\ \forall\ U\}$.

D. Witness operator and geometric form of the Hahn-Banach theorem

A geometric form of the Hahn-Banach theorem states that given a set that is convex and compact, there exists a hyperplane that can separate any point lying outside the set from the given set [26]. A witness operator W pertaining to a convex and compact set S will be a Hermitian operator that satisfies the following conditions: (a) $\mathrm{Tr}(W\sigma) \geqslant 0$, for all states $\sigma \in S$; and (b) $\mathrm{Tr}(W\chi) < 0$, for any state $\chi \notin S$ [13,16].

E. Dense coding capacity

Quantum superdense coding involves the sending of classical information from one sender to the receiver when they share a quantum resource in the form of an entangled state. More specifically, superdense coding is a technique used in quantum information theory to transmit classical information by sending quantum systems. It is quite well known that if we have a maximally entangled state in $H_d \otimes H_d$ as our resource, then we can send $2 \log d$ bits of classical information. In the asymptotic case, we know one can send $\log d + S(\rho)$ number of bits. It has been shown that the number of classical bits one can transmit using a non–maximally entangled state in $H_d \otimes H_d$ as a resource is $(1 + p_0 \frac{d}{d-1}) \log d$, where p_0 is the smallest Schmidt coefficient. However, when the state is maximally entangled in its subspace then one can send up to $2 \log(d-1)$ bits [4,24].

F. State merging

Another important information processing task is state merging. In the classical setting, the idea of state merging is essentially the following: Consider two parties, Alice and Bob, where Bob has some prior information B and Alice has some missing information A (where A and B are random variables). At this point one important question is, If Bob wants to learn about A, how much additional information does Alice need to send him? It has been shown that only $H(A|B)$ bits suffices. In the quantum setting, Alice and Bob each possess a system in some unknown quantum state with joint density operator ρ_{AB}. Assuming that Bob is correlated with Alice, one asks how much additional quantum information Alice needs to send him, so that he has knowledge about the entire state. The amount of partial quantum information [23] that Alice needs to send Bob is given by the quantum conditional entropy, $S(A|B) = S(\rho_{AB}) - S(\rho_A)$. Ideally this conditional entropy can be positive $[S(A|B) > 0]$, negative $[S(A|B) < 0]$, or $0\,[S(A|B) = 0]$. If it is positive, the sender needs to communicate that number of quantum bits to the receiver; if it is 0, there is no need for such communication. However, if it is negative, the sender and receiver gain the same amount of potential for future quantum communication.

III. CHARACTERIZATION OF THE ABSOLUTE CONDITIONAL VON NEUMANN ENTROPY NON-NEGATIVE CLASS

In this section we introduce the class of states for which the conditional von Neumann entropy remains non-negative even after application of the global unitary operator. The characterization of these states enables us to identify states which can be made useful for some information processing task. The von Neumann entropy of a system ρ_{AB} with two subsystems, A and B, is denoted $S(\rho_{AB})$. The conditional von Neumann entropy for ρ_{AB} entropy is defined as $S(\rho_{AB}) - S(\rho_A)$, where $S(\rho_A)$ denotes the von Neumann entropy of subsystem A. We note the class of states for which the conditional von Neumann entropy is non-negative. We denote this class CVENN, defined as CVENN $= \{\sigma_{cv} : S(\sigma_{cv}) - S((\sigma_{cv})_A) \geqslant 0\}$.

A. ACVENN

The set of states whose conditional von Neumann entropy remains non-negative under any global unitary operations is denoted ACVENN $= \{\sigma_{ac} : S(U\sigma_{ac}U^\dagger) - S[(U\sigma_{ac}U^\dagger)_A] \geqslant 0, \forall\, U\}$. The von Neumann entropy remains invariant under global unitary transformations, however, the conditional entropy can change. We are interested in characterizing the set of states that preserves the non-negativity of the conditional entropy under unitary action on the composite system.

Theorem 1. A state $\sigma_{ac} \in$ ACVENN iff $S(\sigma_{ac}) \geqslant 1$.

Proof. Let $\sigma_{ac} \in$ ACVENN. Then $S(U\sigma_{ac}U^\dagger) - S[(U\sigma_{ac}U^\dagger)_A] \geqslant 0, \quad \forall\, U$. This implies $(\sigma_{ac}) - S[(U\sigma_{ac}U^\dagger)_A] \geqslant 0, \;\forall\, U$, as the von Neumann entropy is invariant under changes in the basis of σ_{ac}, i.e., $S(\sigma_{ac}) = S(U\sigma_{ac}U^\dagger)$, with U being any unitary transformation. Hence, we have $S(\sigma_{ac}) \geqslant S[(U\sigma_{ac}U^\dagger)_A], \forall\, U$. The maximum value of $[S(U\sigma_{ac}U^\dagger)_A]$ is obtained at $(U\sigma_{ac}U^\dagger)_A = \frac{\mathbb{I}}{2}$ and the maximum value is 1. There always exists a unitary that converts the σ_{ac} to a Bell diagonal σ_{bell} for a given spectrum. And we know that for a Bell diagonal state the reduced subsystem $(\sigma_{bell})_A$ is $\frac{\mathbb{I}}{2}$. Therefore, $S(\sigma_{ac}) \geqslant S[(U\sigma_{ac}U^\dagger)_A]$, $\forall\, U \Rightarrow \mathrm{S}(\sigma_{ac}) \geqslant S[(\sigma_{bell})_A] = S(\frac{\mathbb{I}}{2}) = 1$.

Conversely, let $S(\sigma_{ac}) \geqslant 1$; one can note that the maximum achievable von Neumann entropy of a subsystem is 1 in the case of a two-qubit system, as under a unitary transformation, the entropy of the subsystem alone changes. Hence, for any state σ_{ac} whose von Neumann entropy is greater than or equal to 1, we know that this state cannot have a negative conditional entropy under any global unitary operations. Therefore, any state σ_{ac} whose $S(\sigma_{ac}) \geqslant 1$ will $\in$ ACVENN. ∎

One may quickly note the following observations.

(a) Any pure separable state has a non-negative conditional entropy and can be brought by some unitary to a maximally entangled state which now possesses a negative conditional entropy, and thus pure separable states can never belong to our desired class. Pure entangled states themselves have a negative conditional entropy. Therefore, pure states are not eligible members of ACVENN.

(b) The fact that some mixed states will be members of ACVENN is exemplified by the maximally mixed state, which remains invariant under any global unitary operation and thus preserves the non-negativity of the conditional entropy. However, the maximally mixed state only constitutes a trivial example and we find that the class contains some very nontrivial states.

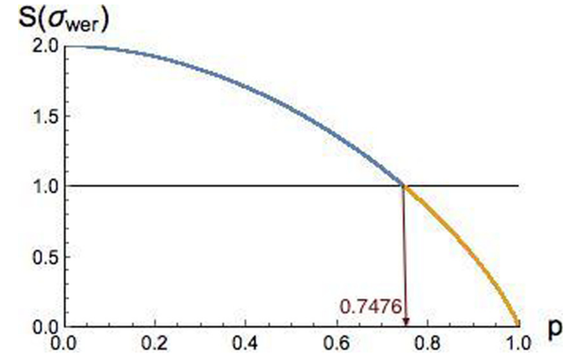

FIG. 1. von Neumann entropy of the Werner state σ_{wer} versus the classical mixing parameter p.

B. Example A: The Werner state

As an example, we first consider the Werner state. The density matrix representation of a Werner state is given by

$$\sigma_{wer} = (1-p)(\mathbb{I}/4) + p|\psi\rangle\langle\psi|, \tag{2}$$

where $|\psi\rangle = 1/\sqrt{2}(|00\rangle + |11\rangle)$ is the Bell state, p is the classical mixing parameter, and $\mathbb{I}$ denotes identity.

In the Fig. 1 we have plotted the von Neumann entropy of the Werner state with respect to the mixing parameter p. Interestingly, we find that for all values of $p \in [0, \approx 0.7476]$, we have $S(\sigma_{wer}) \geqslant 1$. This clearly indicates that the Werner state for values of $p \in [0, \approx 0.7476]$ falls within the ACVENN class.

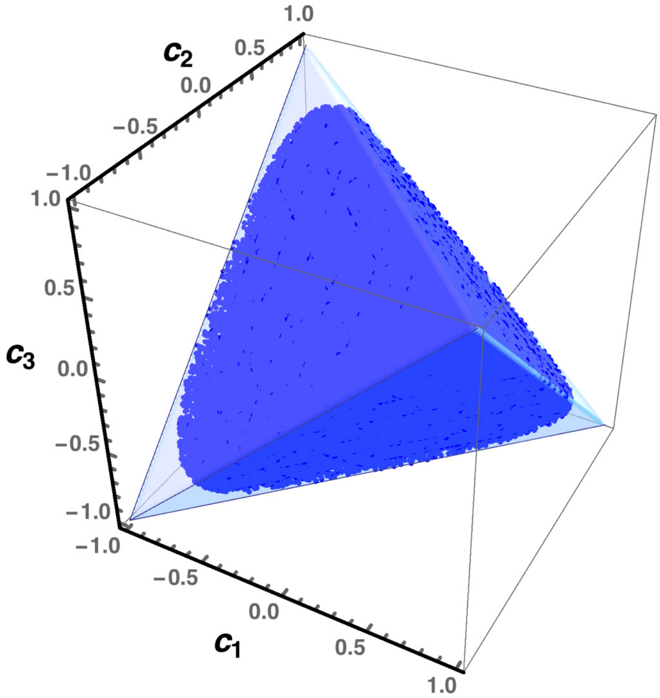

FIG. 2. von Neumann entropy of the Bell diagonal state σ_{bell} versus the parameter c_i.

C. Example B: Bell diagonal states

Bell diagonal states can be expressed as $\sigma_{\text{bell}} = \{\vec{0},\vec{0},T^b\}$, where $\vec{0}$ is the Bloch vector, which is a null vector, and the correlation matrix is $T^b = (c_1,c_2,c_3)$ with $-1 \leqslant c_i \leqslant 1$. The eigenvalues λ_1, λ_2, λ_3, and λ_4 of Bell diagonal states are expressed as $\lambda_1 = \frac{1}{4}(\chi - 2c_1)$, $\lambda_2 = \frac{1}{4}(\chi - 2c_2)$, $\lambda_3 = \frac{1}{4}(\chi - 2c_3)$, and $\lambda_4 = \frac{1}{4}(2-\chi)$, where $\chi = 1 + c_1 + c_2 + c_3$. Therefore, the necessary and sufficient condition for a Bell diagonal state to lie in ACVENN is given by $S(\sigma_{\text{bell}}) \geqslant 1$, which, in terms of c_1, c_2, c_3, and χ, becomes

$$\begin{aligned}&\log((\chi-2c_2)(\chi-2c_3)(2-\chi)(\chi-2c_1))\\&\quad+c_1\log\left(\frac{(\chi-2c_2)(\chi-2c_3)}{(2-\chi)(\chi-2c_1)}\right)\\&\quad+c_2\log\left(\frac{(\chi-2c_3)(\chi-2c_1)}{(\chi-2c_2)(2-\chi)}\right)\\&\quad+c_3\log\left(\frac{(\chi-2c_2)(\chi-2c_1)}{(\chi-2c_3)(2-\chi)}\right)\leqslant 4. \qquad (3)\end{aligned}$$

In Fig. 2, we consider an exhaustive ensemble of 10^5 states within which the dark-blue area at the center of the octahedron determines the class of states for which $S(\sigma_{\text{bell}}) \geqslant 1$ and falls into our ACVENN class. The light-blue areas at the corners are areas whose conditional entropy can be made negative after the application of some global unitary transformation. It is evident from Fig. 2 that the non-negativity of the conditional entropy for most of the Bell diagonal states remains invariant after the application of a global unitary transformation.

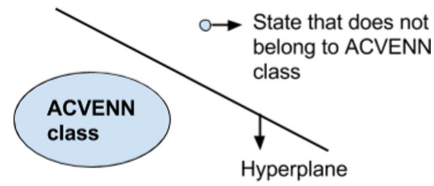

FIG. 3. The set ACVENN is convex and compact, and using the Hahn-Banach theorem [26] it follows that any state not belonging to ACVENN can be separated from the states that belong to ACVENN by a hyperplane, thus providing for the existence of a witness.

IV. CONVEXITY AND COMPACTNESS OF THE ACVENN CLASS: EXISTENCE OF A WITNESS

In this section we show that the ACVENN class, which is a subset of class Q, is a convex and compact set. This helps us identify states whose conditional entropy remains negative even after the application of a global unitary. We now present the proof that the set ACVENN is convex and compact.

A. Existence of a witness

The theorems below support the existence of witness operators to distinguish ACVENN states from states which are not in ACVENN. See Fig. 3 for reference.

Theorem 2. ACVENN is convex.

Proof. Consider $\sigma_1,\sigma_2 \in$ ACVENN. Therefore, $S(\sigma_i) \geqslant 1$, $i = 1,2$. Now by the concavity of the von Neumann entropy $S(\lambda\sigma_1 + (1-\lambda)\sigma_2) \geqslant 1$, where $\lambda \in [0,1]$. Hence, $\lambda\sigma_1 + (1-\lambda)\sigma_2 \in$ ACVENN, implying that ACVENN is convex. ■

Theorem 3. ACVENN is a compact subset of Q.

Proof. Let us define a function $f : Q \to \mathbb{R}$ as

$$f(\rho) = S(\rho); \qquad (4)$$

as ACVENN $= \{\sigma_{\text{ac}} : S(\sigma_{\text{ac}}) \geqslant 1\}$, and f will have a maximum value of 2, we can say that ACVENN $= f^{-1}[1,2]$. f is a continuous function, as S is a continuous function [27]. Therefore, ACVENN $= f^{-1}[1,2]$ is a closed set in Q defined under the trace norm. The set ACVENN is bounded, as every density matrix has a bounded spectrum, i.e., their eigenvalues lie between 0 and 1. This proves that the ACVENN class is compact. ■

The theorem now guarantees the existence of Hermitian operators to successfully identify states that do not belong to ACVENN.

Next we estimate the size of the ACVENN class by taking the maximum and minimum distance from the identity $(\frac{\mathbb{I}}{2} \otimes \frac{\mathbb{I}}{2})$. The distance measure we have used in this context is the Frobenius norm, which is given by $\|X\| = \sqrt{\text{Tr}(X^\dagger X)}$. Having already proved that ACVENN is a convex set, we try to determine the maximum and minimum distances from $\frac{\mathbb{I}}{2} \otimes \frac{\mathbb{I}}{2}$.

For any general $\widetilde{\varrho}$, the distance from $\frac{\mathbb{I}}{2} \otimes \frac{\mathbb{I}}{2}$ is given by $\|\widetilde{\varrho} - \frac{\mathbb{I}}{4}\| = \sqrt{\text{Tr}((\widetilde{\varrho} - \frac{\mathbb{I}}{4})^\dagger(\widetilde{\varrho} - \frac{\mathbb{I}}{4}))}$, which, upon solving, further results in $\sqrt{\text{Tr}(\widetilde{\varrho}^2) - \frac{1}{4}}$. To calculate the maximum distance we needed to maximize $|\sigma - \frac{\mathbb{I}}{4}\|$, over all $\sigma \in$ ACVENN. Here we solve this problem numerically. After going through 2×10^5 ACVENN states the maximum distance we have is 0.645 966,

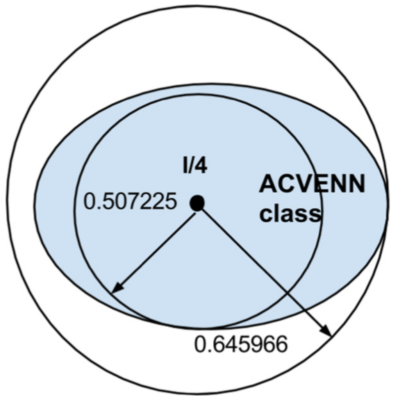

FIG. 4. Approximate size of the ACVENN class.

for the state whose eigenvalues were $\lambda_1 = 0.809\,161$, $\lambda_2 = 0.052\,114\,1$, $\lambda_3 = 0.059\,544\,8$, and $\lambda_4 = 0.079\,180\,5$.

To calculate the minimum distance we needed to minimize $|\rho - \frac{\mathbb{I}}{4}\|$, over all $\rho \notin$ ACVENN. Going through 1×10^5 non-ACVENN states numerically, we attained the minimum distance as 0.507 225. This is given by a state whose eigenvalues were $\lambda_1 = 0.000\,143\,47$, $\lambda_2 = 0.000\,551\,157$, $\lambda_3 = 0.436\,523$, and $\lambda_4 = 0.562\,783$.

In Fig. 4 we show a rough estimation of the size of the ACVENN class.

V. RELATION BETWEEN THE AS, ACVENN, AND AL CLASSES

In this section we present a comparative picture of three classes of states. These classes remain invariant from the context of separability (AS), nonviolation of Bell's inequlity (AL), and non-negative conditional entropy (ACVENN) under global unitary transformation.

A. AS vs ACVENN

Figure 5 shows the relation between AS, ACVENN, and separable states.

Lemma 4. AS $\subseteq$ ACVENN.

Proof. Absolutely separable states preserve separability under any global unitary action. The non-negativity of conditional entropies is a necessary condition for separability [28]. All separable states have a non-negative conditional von Neumann entropy. AS states remain separable under any unitary transformation. AS states will always have a non-negative conditional von Neumann entropy. So they will form a subset of the ACVENN class. ■

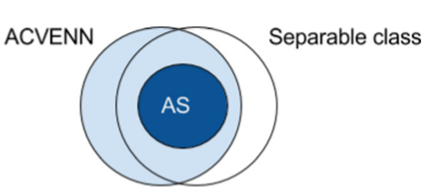

FIG. 5. Relation between ACVENN, AS, and separable classes of states.

1. Illustration A: Absolutely separable Werner states

Let us consider the Werner states $\sigma_{\text{wer}} = p|\psi\rangle\langle\psi| + \frac{1-p}{4}\mathbb{I}$, where $|\psi\rangle$ is the Bell state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. The state σ_{wer} belongs to ACVENN for values of p that satisfy the equation $3(1-p)\log(1-p) + (1+3p)\log(1+3p) \leqslant 4$. Solving the inequality we get $p \in [0,\approx 0.7476]$ as obtained earlier. For the states to be in AS, one must have $a_1 \leqslant a_3 + 2\sqrt{a_2 a_4}$, where $a_1 = \frac{(1+3p)}{4}$, $a_2 = \frac{(1-p)}{4}$, $a_3 = \frac{(1-p)}{4}$, and $a_4 = \frac{(1-p)}{4}$ are all the eigenvalues in descending order. σ_{wer} belongs to AS for values of $p \in [0, \frac{1}{3}]$. This provides an example of an absolutely separable state which is contained in the ACVENN class.

2. Illustration B: States that are incoherent in the computational basis

Next, we give an example of a class of states which are incoherent in the computational basis:

$$\sigma_{\text{comp}} = a_1|00\rangle\langle 00 + a_2|01\rangle\langle 01| + a_3|10\rangle\langle 10| + a_4|11\rangle\langle 11|. \tag{5}$$

Taking the example of a state with eigenvalues $a_1 = \frac{5}{10}$, $a_2 = \frac{3}{10}$, $a_3 = \frac{2}{10}$, and $a_4 = 0$, $S(\sigma_{\text{comp}}) = -\sum a_i \log_2 a_i \approx 1.485 \geqslant 1$. Hence, this state $\in$ ACVENN.

We see that $a_1 \geqslant a_2 \geqslant a_3 \geqslant a_4$. For the states to be in AS, one must have $a_1 \leqslant a_3 + 2\sqrt{a_2 a_4}$. In this specific case $a_1 = 0.5$, $a_3 + 2\sqrt{a_2 a_4} = 0.2$. Clearly, this shows that AS is a subset of ACVENN.

Theorem 5. AS $\subset$ ACVENN.

Proof. In Lemma 4 it has been shown that AS $\subseteq$ ACVENN. In fact, we can say more than that. In view of the example of Werner states in Sec. V A 1, we have seen that there are states that do not belong to AS but belong to ACVENN. This shows that absolutely separable states form a proper subset of ACVENN. ■

After proving that AS $\subset$ ACVENN we want to estimate the minimum and maximum entropies recorded by the states belonging to AS. We solve this problem numerically as well. After going through 1×10^5 AS states the minimum entropy that we obtain is 1.586 62. This is attained for a state with eigenvalues $\lambda_1 = 0.341\,023$, $\lambda_2 = 0.331\,417$, $\lambda_3 = 0.327\,411$, and $\lambda_4 = 0.000\,148\,614$. We already know that the maximum entropy for AS is 2. This gives us a rough estimate of the number of AS states lying within ACVENN in terms of entropy.

B. AL vs ACVENN

The Werner states are absolutely local for the visibility factor $p \leqslant 1/\sqrt{2}$, and they belong to ACVENN for $p \leqslant 0.7476$. Therefore, the *absolutely Bell-CHSH local* Werner states form a subset of the ACVENN class. This is an interesting result, as this would mean that there are states that violate Bell-CHSH inequality and still under any unitary cannot be improved to a state with a negative conditional entropy. However, it is difficult to comment in general on the relation between the AL and the ACVENN classes.

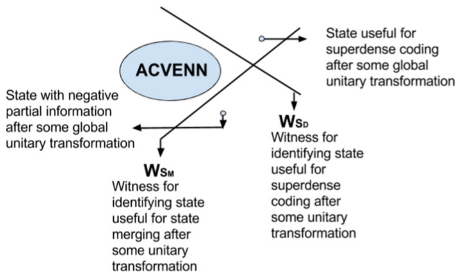

FIG. 6. Role of the witness in detecting states which are useful for superdense coding and state merging after global unitary transformation.

VI. APPLICATIONS: STATE MERGING AND SUPERDENSE CODING

In this section we show how the characterization of this ACVENN class of states helps to identify states which are not useful for information processing tasks like superdense coding and state merging but are made useful with the help of global unitary transformations. In either of these tasks we are able to detect a class of states which can be converted into superdense coding and state merging resources by applying global unitary transformations. In Fig. 6 we give a pictorial description of two types of witness operators that can be created. W_{S_D} and W_{S_M} act as hyperplanes that distinguish states useful for superdense coding and state merging, respectively, from states belonging to the ACVENN class which can never be made useful for these information theoretic tasks by using global unitary operations.

A. Superdense coding

In particular, the superdense coding capacity for a mixed state ρ_{AB} in $D(H_d \otimes H_d)$ is defined as

$$\mathcal{C}_{AB} = \max\{\log_2 d, \log_2 d + S(\rho_B) - S(\rho_{AB})\}, \quad (6)$$

where $\rho_B = \mathrm{Tr}_A[\rho_{AB}]$ [4,24]. $\mathcal{C}_{AB}$ is nothing but the amount of classical information that can be sent from system A to system B. Here we note that the expression $S(\rho_B) - S(\rho_{AB})$ can be either positive or negative. If it is positive, then one can use the shared state to transfer bits greater than the classical limit of $\log_2 d$ bits. This is known as the quantum advantage where we can do more than the classical limit. For pure states, $S(\rho_{AB}) = 0$; then the superdense coding capacity is given by

$$\mathcal{C}_{AB} = \log_2 d + S(\rho_B) = \log_2 d + E(\rho_{AB}), \quad (7)$$

where the entanglement entropy $E(\rho_{AB})$ of a pure state ρ_{AB} is nothing but the von Neumann entropy $S(\rho_B)$ of the reduced subsystem ρ_B. The capacity will be maximum for Bell states, as $S(\rho_B)$ will be equal to 1. In a nutshell, a state ρ_{AB} for which this expression $S(\rho_B) - S(\rho_{AB})$ is positive will give us a quantum advantage for superdense coding. In other words, a state with a negative conditional entropy $S(A|B)$ will be useful. It is obvious that not all states will have a negative conditional entropy. The next important question is, If we apply a global unitary operator, can we make a state which is not useful for superdense coding into a useful resource? In other words, Can we change the conditional entropy of the state from positive to negative? The answer is yes; however, there will be some states for which we cannot do this. These sets of invariant states are nothing but the previously described ACVENN class of states which can never be useful from the perspective of superdense coding, even after the application of global unitary operators. As we have seen previously that this class of state is convex and compact, in principle it will be possible to create a witness operator (W_{S_D} as shown in Fig. 6) to detect states which are initially not useful but are made useful for superdense coding. It is important to mention here that this witness operator is not a witness operator to detect states which are useful for superdense coding as opposed to nonuseful states. The class of states useful for superdense coding is not a convex and compact set; neither is the class of states that are not useful for superdense coding. This witness operator detects those states which need not be useful initially but can be made useful after global unitary transformation. Further, we provide examples to show all these kinds of states.

Illustrations

For our first example let us consider a mixed separable state in $D(H_2 \otimes H_2)$ given by [13,16]

$$\rho = \begin{vmatrix} a & 0 & b & 0 \\ 0 & 0 & 0 & 0 \\ b & 0 & 1-a & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}. \quad (8)$$

The eigenvalues for this state are $\frac{1-q}{2}$, $\frac{1+q}{2}$, 0, and 0 and the eigenvalues of subsystem A are $\frac{1-q}{2}$ and $\frac{1+q}{2}$, where $q = \sqrt{1-4a+4a^2+4b^2}$. The state $\rho \in$ ACVENN iff $S(\rho) \geqslant 1$. In the current scenario this occurs only when $q = 0$. However, for no real values of a and b is $q = 0$. Therefore we know that for no real values of a and b does this state belong to ACVENN. Thus $S(\rho_{A|B}) = 0$ for all real values of a and b, which clearly does not have a negative conditional entropy and, therefore, provides no quantum advantage. But upon application of the unitary operator,

$$U_1 = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ -1 & 0 & 0 & 1 \end{vmatrix} \quad (9)$$

becomes

$$\rho' = \begin{vmatrix} \frac{a}{2} & 0 & \frac{b}{\sqrt{2}} & \frac{-a}{2} \\ 0 & 0 & 0 & 0 \\ \frac{b}{\sqrt{2}} & 0 & 1-a & \frac{-b}{\sqrt{2}} \\ \frac{-a}{2} & 0 & \frac{-b}{\sqrt{2}} & \frac{a}{2} \end{vmatrix}. \quad (10)$$

While the eigenvalues of ρ' remain unchanged, the eigenvalues of subsystem ρ'_A become $\frac{1-q'}{2}$ and $\frac{1+q'}{2}$, where $q' = \sqrt{1-2a+a^2+2b^2}$. For all values of a and b, where $q > q'$ the state can be made useful for superdense coding. One such example is when $a = 0.5$ and $b = 0.4$. Thus, we provide an example of a state which was not useful for superdense coding initially but, after a unitary transformation, was made useful for superdense coding.

B. State merging and partial quantum information

In information processing scenarios, it is important to ask this question: If an unknown quantum state is distributed over two systems, say A and B, how much quantum communication is needed to transfer the full state to one system? This communication measures the *partial information* one system needs conditioned on its prior information. Remarkably, this is given by the conditional entropy $S(A|B)$ (if it is from A to B) of the system. It is interesting to note that in principle this entropy can be positive $[S(A|B) > 0]$, negative $[S(A|B) < 0]$, or $0\,[S(A|B) = 0]$, where each has a different meaning in the context of state merging. If the partial information is positive, its sender needs to communicate this number of quantum bits to the receiver; if it is 0, there is no need for such communication; and if it is negative, the sender and receiver instead gain the corresponding potential for future quantum communication. So given a quantum state ρ_{AB}, shared between A and B, three possible cases arise, and we characterize the state based on these cases; namely, for states with $S(A|B) > 0$ we denote them $\rho_{S(A|B)>0}$, and similarly, other states as $\rho_{S(A|B)<0}$, $\rho_{S(A|B)=0}$. It is always useful from the information-theoretic point of view to look out for states $\rho_{S(A|B)<0}$, as they have the potential for future communication. Of course, not all states will be of this type. So the next question that becomes important in this context of a global unitary is detecting states which are initially of type $\rho_{S(A|B)>0}$ but can be converted to type $\rho_{S(A|B)<0}$ after global unitary operations. These states for which the conditional entropy remains positive even after all possible global unitary operations are nothing but the previously defined ACVENN class. Since we have already proved that the ACVENN class is always convex and compact, this means that we can detect states whose partial information can be made negative after the global unitary operation with the help of a witness operator (W_{S_M} as shown in Fig. 6). As in the case of superdense coding it is important to mention here, also, that we are showing not that the set $\rho_{S(A|B)>0}$ is convex and compact, but that the set for which the partial information remains positive after the global unitary operation [say $U\rho_{S(A|B)}U^\dagger > 0$] is convex and compact. As a result, we are not detecting the state for which the partial information is negative instead of those states whose partial information can be made negative [say $U\rho_{S(A|B)}U^\dagger < 0$] after the application of a global unitary. We give examples to identify all these classes.

Illustrations

Let us take a mixed two-qubit state,

$$\rho_{AB} = \tfrac{3}{4}|00\rangle\langle 00| + \tfrac{1}{4}|11\rangle\langle 11|; \tag{11}$$

we see that $S(\rho_{B|A}) = 0$. After the application of a unitary transformation $U_2 = U_1^{-1}$, where U_1 is defined in Eq. (9). The state ρ_{AB} transforms to

$$\rho'_{AB} = \tfrac{1}{2}|00\rangle\langle 00| + \tfrac{1}{4}|00\rangle\langle 11| + \tfrac{1}{4}|11\rangle\langle 00| + \tfrac{1}{2}|11\rangle\langle 11|. \tag{12}$$

$S(\rho'_{B|A})$ is -0.1887. Thus we give an example of how a state which has a non-negative conditional entropy initially can be made negative with the help of a unitary transformation and also, in principle, one can construct a witness operator to detect this kind of state.

In this subsection we also ask this question: For a given spectrum of density matrix, for which states will the minimum state-merging cost be achieved?

Theorem 6. For a given spectrum of density matrix the minimum state-merging cost will be achieved for the Bell diagonal states.

Proof. We know that the conditional entropy for the quantum state ρ_{AB} is given by the difference, $S(B|A) = S(\rho_{AB}) - S(\rho_A)$. Let us assume that the spectrum ρ_{AB} is fixed with eigenvalues a_i, $i = 1,2,3,4$. Since the spectrum is fixed we have the freedom to apply a global unitary operator. Now the problem is to minimize $S(B|A)$, by using only global unitary operations. Since the global unitary operations will not change $S(\rho_{AB})$, we need to maximize $S(\rho_A)$. Now $S(\rho_A)$ is maximized if $\rho_A = \mathbb{I}/2$. The reduced density matrices for the Bell diagonal state is $\mathbb{I}/2$. Therefore if one reaches the Bell diagonal state by some global unitary, no further maximization of $S(\rho_A)$ is possible. Thus for a given spectrum of density matrices the minimal merging cost is attained for the Bell diagonal state. ■

VII. CONCLUSION

In this work, for a general two-qubit system we are able to characterize a class of states, ACVENN, whose von Neumann entropy will remain positive even after the application of a global unitary operator. More specifically, we are able to show that states with a von Neumann entropy greater than 1 are the same ACVENN class of states.

We also find that this class of states is convex and compact, which guarantees the existence of a witness for detecting states which could have a positive conditional entropy initially but have a negative conditional entropy after the application of a unitary operator. This in turn provides the power to identify states which are not initially useful but can be made useful for information processing tasks like superdense coding and state merging.

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