Circuit configurations which may or may not show superradiant phase transitions

Motoaki Bamba[1,2,*] and Nobuyuki Imoto[1]

[1]*Department of Materials Engineering Science, Osaka University, 1-3 Machikaneyama, Toyonaka, Osaka 560-8531, Japan*
[2]*PRESTO, Japan Science and Technology Agency, 4-1-8 Honcho, Kawaguchi, Saitama 332-0012, Japan*

(Received 9 March 2017; published 27 November 2017)

Several superconducting circuit configurations are examined on the existence of superradiant phase transitions (SRPTs) in thermal equilibrium. For some configurations consisting of artificial atoms, whose circuit diagrams are however not specified, and an LC resonator or a transmission line, we confirm the absence of SRPTs in the thermal equilibrium following the similar analysis as the no-go theorem for atomic systems. We also show some other configurations where the absence of SRPTs cannot be confirmed.

I. INTRODUCTION

A superradiant phase transition (SRPT), i.e., a spontaneous appearance of (static) coherent amplitude of transverse electromagnetic fields in the thermal equilibrium due to the light-matter interaction, was first proposed theoretically around 1970 [1–3]. It is different from the so-called superradiance or superfluorescence [4], i.e., a collective spontaneous emission from many atoms. It is also different from the exciton superradiance (one-photon superradiance) [5], i.e., an emission-rate enhancement by spatial broadening of wave function of an excitation. In contrast to these nonequilibrium phenomena, SRPTs are phase transitions in the thermal equilibrium. Since the first proposals [1–3], its absence (no-go theorem) in atomic systems has been discussed based on the so-called A^2 term [6–9], P^2 term [10,11], gauge invariance [12–14], and minimal-coupling Hamiltonian [14,15]. Influences of the longitudinal dipole-dipole interaction have also been discussed recently [16–20].

SRPTs require an ultrastrong light-matter interaction [21,22], i.e., the interaction strength (vacuum Rabi splitting or absorption or emission rate in single-photon level) must be comparable to or larger than frequencies of electromagnetic waves and of transitions in matters. In recent years, the ultrastrong interactions have been realized experimentally in a variety of systems [23–37]. The presence of the so-called vacuum photons [21] and the Schrödinger-cat-like state [21,38,39] are expected in the ground state under the ultrastrong interaction, and recent experiments are indicating a signature of them [37]. However, the coherent amplitude of the electromagnetic fields (expectation value of annihilation operator of a photon) does not appear even in such a ground state, but it is obtained only after a SRPT. Currently, SRPTs are not yet observed experimentally in the thermal equilibrium, while nonequilibrium analogs were proposed theoretically [40] and observed experimentally in cold atoms driven by laser light [41,42].

Instead of the atomic systems [23–26,30–32,35], which are basically described by the minimal-coupling Hamiltonian [43], the possibility of the thermal-equilibrium SRPTs in superconducting circuits [27–29,36,37] has been discussed [44–48]. The existence of a SRPT was proposed for a superconducting

circuit with capacitive coupling between (two-level) artificial atoms and a resonator by estimating the A^2 term to be relatively small [44]. However, its estimation was doubted through a standard description of superconducting circuit systems [45]. After that, the existence of a SRPT was proposed again for superconducting circuit with three-level artificial atoms as a result of the modification of the sum rule (and then of the A^2 term) [46]. In these three works, their Hamiltonians were guessed for standard circuit configurations but without specifying circuit diagrams in detail, although the derivation of exact Hamiltonians is crucial for discussing the possibility of SRPTs. Recently, the absence of SRPTs was confirmed for a superconducting circuit diagram with capacitive coupling between an LC resonator and charge qubits by deriving its Hamiltonian in the standard quantization procedure [47]. Almost at the same time, for a circuit diagram consisting of an LC resonator coupled with Josephson junctions through inductors, the existence of a SRPT was proposed also in the standard quantization procedure [48]. No doubt is raised until now.

A remarkable feature of SRPTs is a decrease of the zero-point energy in the whole system due to the light-matter interaction [48–51]. Chemical reactions [52] and work functions [53] were reported to be modified by the ultrastrong interaction with the vacuum electromagnetic fields. The free energy, i.e., thermodynamic behaviors at finite temperatures, should also be modified as suggested in Ref. [54], while its experimental and theoretical evaluations are still under debate [55,56]. In the superconducting circuit proposed in Ref. [48], an external magnetic flux bias or π junctions [57] are inevitable for realizing the SRPT in the thermal equilibrium. The external magnetic flux increases the zero-point energy of the circuit. While the zero-point energy is certainly decreased by the increase in the photon-atom interaction strength, it cannot be lower than the zero-point energy in the absence of the external magnetic flux. It is still open to dispute whether there is a lower bound of the zero-point energy in superconducting circuits. If there exists a superconducting circuit showing a SRPT without the external magnetic flux or π junctions, the zero-point energy should be purely decreased by increasing the strength of the interaction with the transverse electromagnetic fields, and the thermodynamic properties, e.g., the superconducting transition temperature, of the circuit might be modified.

In order to find such a circuit structure, in this paper, we show some hopeless circuit configurations where SRPTs are

*bamba@qi.mp.es.osaka-u.ac.jp

absent even in the presence of an external magnetic flux or π junctions. There are a large number of degrees of freedom in designing circuit structures, and there is not a standard Hamiltonian corresponding to the minimal-coupling one for the atomic systems. In order to rule out a wide range of circuit structures, we treat artificial atoms as a black box, i.e., we do not specify their circuit diagrams. We consider some capacitive- and inductive-coupling configurations between the black box and an LC resonator or a transmission line. The absence of SRPTs in those configurations are confirmed following the similar analysis as the no-go theorem for the atomic systems [14,15] by deriving Hamiltonians in the flux- [58] or charge-based [59] standard quantization procedure. In the analyses based on the A^2 term [6–9,44–46], on the P^2 term [10,11,47], or on the softening of transition frequency [44,48–50], we must specify circuit diagrams of whole systems in detail. In contrast, in this paper, the artificial atoms are treated as a black box following the no-go theorem [14,15], but we need to specify only the connection between the black box and a resonator.

We also show some other circuit configurations where the absence of SRPTs in the thermal equilibrium cannot be confirmed. The circuit structure proposed in Ref. [48] is certainly included in these configurations. While our analysis does not depend on whether an external magnetic flux or π junctions are absent or not, it does not rule out the possibility of SRPTs without the external magnetic flux and π junctions.

This paper is organized as follows. We first review the no-go theorem for atomic systems in Sec. II. Following the similar analysis, in Sec. III, we show the absence of SRPTs in three circuit configurations by deriving Hamiltonians without specifying circuit diagrams of artificial atoms. In Sec. IV, we show some other configurations where the absence of SRPTs cannot be confirmed. The discussion is summarized in Sec. V.

II. NO-GO THEOREM FOR ATOMIC SYSTEMS

In this section, we review the no-go theorem of SRPTs in atomic systems described by the minimal-coupling Hamiltonian. It was mainly discussed in Refs. [14,15] based on the c-number substitution [3,15,60,61], which is also used in the semiclassical analysis of Ref. [48].

The minimal-coupling Hamiltonian is expressed as [43]

$$
\hat{\mathcal{H}}_{\min} = \int d\boldsymbol{r} \left\{ \frac{\varepsilon_0 \hat{\boldsymbol{E}}_\perp(\boldsymbol{r})^2}{2} + \frac{\hat{\boldsymbol{B}}(\boldsymbol{r})^2}{2\mu_0} \right\}
$$
$$
+ \sum_{j=1}^{N} \frac{[\hat{\boldsymbol{p}}_j - e_j \hat{\boldsymbol{A}}(\hat{\boldsymbol{r}}_j)]^2}{2m_j} + \hat{V}(\{\hat{\boldsymbol{r}}_j\}). \quad (1)
$$

Here, the second last term is the kinetic energy of charged particles. N is the number of the particles. $\hat{\boldsymbol{r}}_j$ and $\hat{\boldsymbol{p}}_j$ are operators of a position and a momentum, respectively, of the jth particle with a mass m_j and a charge e_j. They satisfy $[\hat{\boldsymbol{r}}_j, \hat{\boldsymbol{p}}_{j'}] = \delta_{j,j'} i\hbar \mathbf{1}$. The last term $\hat{V}$ represents the Coulomb interaction between the charged particles, and it depends only on the particles' positions $\{\hat{\boldsymbol{r}}_j\}$. The first and second terms represent the energies of the transverse electric field $\hat{\boldsymbol{E}}_\perp(\boldsymbol{r}) = -\hat{\boldsymbol{\Pi}}(\boldsymbol{r})/\varepsilon_0$ and the magnetic flux density $\hat{\boldsymbol{B}}(\boldsymbol{r}) = \nabla \times \hat{\boldsymbol{A}}(\boldsymbol{r})$, respectively. Here, $\hat{\boldsymbol{A}}(\boldsymbol{r})$ is the vector potential and $\hat{\boldsymbol{\Pi}}(\boldsymbol{r})$ is its

conjugate momentum satisfying

$$
[\hat{\boldsymbol{A}}(\boldsymbol{r}), \hat{\boldsymbol{\Pi}}(\boldsymbol{r}')] = i\hbar \boldsymbol{\delta}_\perp(\boldsymbol{r} - \boldsymbol{r}'), \quad (2)
$$

where $\boldsymbol{\delta}_\perp(\boldsymbol{r} - \boldsymbol{r}')$ is the transverse delta function [43]. We rewrite these fields by annihilation and creation operators as

$$
\hat{\boldsymbol{A}}(\boldsymbol{r}) = \sum_{k=1}^{M} \boldsymbol{e}_k \sqrt{\frac{\hbar}{2\varepsilon_0 \omega_k}} f_k(\boldsymbol{r})(\hat{a}_k + \hat{a}_k^\dagger), \quad (3a)
$$

$$
\hat{\boldsymbol{\Pi}}(\boldsymbol{r}) = -\sum_{k=1}^{M} \boldsymbol{e}_k i \sqrt{\frac{\hbar \varepsilon_0 \omega_k}{2}} f_k(\boldsymbol{r})(\hat{a}_k - \hat{a}_k^\dagger). \quad (3b)
$$

Here, $\hat{a}_k$ annihilates a photon in the kth mode of the electromagnetic wave with a frequency of ω_k. $f_k(\boldsymbol{r})$ is the wave function of the kth mode, $\boldsymbol{e}_k$ is the unit vector in its polarization direction, and ε_0 is the vacuum permittivity. M is the number of modes. The minimal-coupling Hamiltonian in Eq. (1) is rewritten as

$$
\hat{\mathcal{H}}_{\min} = \sum_{k=1}^{M} \hbar \omega_k \left(\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2} \right)
$$
$$
+ \sum_{j=1}^{N} \frac{[\hat{\boldsymbol{p}}_j - e_j \hat{\boldsymbol{A}}(\hat{\boldsymbol{r}}_j)]^2}{2m_j} + \hat{V}(\{\hat{\boldsymbol{r}}_j\}). \quad (4)
$$

For simplicity, as discussed in Ref. [14], we apply the long-wavelength approximation (electric-dipole approximation), i.e., the vector potential is rewritten as

$$
\hat{\boldsymbol{A}}(\hat{\boldsymbol{r}}_j) \simeq \hat{\boldsymbol{A}}(\boldsymbol{R}_j), \quad (5)
$$

where $\boldsymbol{R}_j$ is the rough position of the jth particle (e.g., position of lattice site). The long-wavelength approximation is justified when the amplitude of the vector potential varies only slightly by the distance $\hat{\boldsymbol{r}}_j - \boldsymbol{R}_j$. In other words, $\hat{\boldsymbol{r}}_j - \boldsymbol{R}_j$ is much shorter than the wavelength of the electromagnetic wave in the frequency range of interest. A more general discussion beyond the long-wavelength approximation is shown in Ref. [15].

Expanding the kinetic energy of the charged particles in Eq. (1) or Eq. (4), we get $-\sum_{j=1}^{N}(e_j/m_j)\hat{\boldsymbol{p}}_j \cdot \hat{\boldsymbol{A}}(\hat{\boldsymbol{r}}_j)$ and $\sum_{j=1}^{N} e^2 \hat{\boldsymbol{A}}(\hat{\boldsymbol{r}}_j)^2/(2m_j)$. The former leads to the light-matter interaction term, and the latter leads to the A^2 term [6–9]. The absence of SRPTs by the presence of the A^2 term can be confirmed when we specify the atomic systems of interest, especially the shape of $\hat{V}(\{\hat{\boldsymbol{r}}_j\})$. In contrast, the following no-go theorem shows the absence of SRPTs generally in the minimal-coupling Hamiltonian, i.e., without specifying the systems in detail.

The thermodynamic properties at a finite temperature T are analyzed by the partition function for $\beta = 1/(k_{\mathrm{B}}T)$ as

$$
Z(T) = \mathrm{Tr}[e^{-\beta \hat{\mathcal{H}}_{\min}}]. \quad (6)
$$

As discussed in Refs. [3,15,60,61], we replace the trace over the photonic variables by the integral over the coherent state as

$$
\bar{Z}(T) = \int \left(\prod_k \frac{d^2 \alpha_k}{\pi} \right) \mathrm{Tr}[e^{-\beta \hat{\mathcal{H}}'_{\min}}], \quad (7)
$$

where the photon operators $\{\hat{a}_k, \hat{a}_k^\dagger\}$ and vector potential $\hat{A}(\boldsymbol{R}_j)$ are replaced by c numbers as

$$
\hat{\mathcal{H}}'_{\min} = \sum_{k=1}^{M} \hbar\omega_k \left(|\alpha_k|^2 + \frac{1}{2} \right) \\
+ \sum_{j=1}^{N} \frac{[\hat{\boldsymbol{p}}_j - e_j \boldsymbol{A}(\boldsymbol{R}_j)]^2}{2m_j} + \hat{V}(\{\hat{\boldsymbol{r}}_j\}). \quad (8)
$$

Here, $\alpha_k \in \mathbb{C}$ is an amplitude of a coherent state $|\alpha_k\rangle_k$ in the kth mode giving $\hat{a}_k |\alpha_k\rangle_k = \alpha_k |\alpha_k\rangle_k$. The c-number vector potential is expressed as

$$
\boldsymbol{A}(\boldsymbol{r}) = \sum_{k=1}^{M} \boldsymbol{e}_k \sqrt{\frac{\hbar}{2\varepsilon_0 \omega_k}} f_k(\boldsymbol{r})(\alpha_k + \alpha_k^*). \quad (9)
$$

The replacement (approximation) performed in Eq. (7) is called the c-number substitution [15,61], and the analysis based on it is called the semiclassical analysis in Ref. [48], since the photonic operators are treated as the c numbers.

For justifying this c-number substitution, we must consider the thermodynamic limit $N \to \infty$. Further, in the early study by Wang and Hioe [3], they note that this substitution is justified on the following two assumptions:

Assumption 1: The limits as $N \to \infty$ of the field operator $\hat{a}/\sqrt{N}$ and $\hat{a}^\dagger/\sqrt{N}$ exist.

Assumption 2: The order of the double limit in the exponential series $\lim_{N\to\infty} \lim_{R\to\infty} \sum_{r=1}^{R} (-\beta\hat{\mathcal{H}})^r/r!$ can be interchanged.

The first assumption implies that $\alpha_k/\sqrt{N}$ should be of a finite value after the SRPTs in the thermodynamic limit $N \to \infty$. On the other hand, it is hard to check the second assumption for arbitrary systems. Instead, we follow the justification discussed in Ref. [60]. The exact partition function $Z(T)$ in Eq. (6) and the approximated one $\bar{Z}(T)$ in Eq. (7) satisfy the following relation [60]:

$$
\bar{Z}(T) \leqslant Z(T) \leqslant \exp\left(\frac{1}{k_{\mathrm{B}}T} \sum_{k=1}^{M} \hbar\omega_k \right) \bar{Z}(T). \quad (10)
$$

From this, the free energy $-(k_{\mathrm{B}}T/N)\ln Z(T)$ per atom satisfies

$$
-\frac{1}{N} \sum_{k=1}^{M} \hbar\omega_k - \frac{k_{\mathrm{B}}T}{N} \ln \bar{Z}(T) \leqslant -\frac{k_{\mathrm{B}}T}{N} \ln Z(T)
$$

$$
\leqslant -\frac{k_{\mathrm{B}}T}{N} \ln \bar{Z}(T). \quad (11)
$$

Therefore, in the thermodynamic limit $N \to \infty$, $Z(T)$ is well approximated by $\bar{Z}(T)$, if systems of interest satisfy

Assumption A: $\displaystyle \lim_{N\to\infty} \sum_{k=1}^{M} \frac{\hbar\omega_k}{N} \ll \left| \frac{k_{\mathrm{B}}T}{N} \ln \bar{Z}(T) \right|.$

This condition can be checked when we specify atomic systems of interest. It is satisfied for ensemble of two-level atoms [60], i.e., in the Dicke Hamiltonian. For superconducting circuits, it was checked numerically for the circuit proposed in Ref. [48]. In this paper, we implicitly consider that the systems of interest satisfy *Assumptions 1 and 2, or A*

in the thermodynamic limit $N \to \infty$, while we do not specify the systems in detail. In other words, we cannot discuss the absence of SRPTs in systems that do not satisfy these assumptions since we cannot rewrite the partition function as Eq. (7) and the following analysis is not justified.

The no-go theorem [14] for atomic systems in the long-wavelength approximation is discussed based on the partition function in Eq. (7) described by the minimal-coupling Hamiltonian in Eq. (8) under the c-number substitution. If there exists a state $|\psi(\{\alpha_k\})\rangle$ that minimizes the energy $\langle\psi(\{\alpha_k\})|\hat{\mathcal{H}}'_{\min}|\psi(\{\alpha_k\})\rangle$ for a nonzero amplitude $\alpha_k \neq 0$, the transverse electromagnetic fields get an amplitude spontaneously in the ground state (and also in the thermal equilibrium for $T > 0$), i.e., the system shows a SRPT. However, the absence of such a superradiant ground state is confirmed as seen in the following.

Here, we introduce a unitary operator

$$
\hat{U}_c \equiv \exp\left[\frac{i}{\hbar} \sum_{j=1}^{N} e_j \hat{\boldsymbol{r}}_j \cdot \boldsymbol{A}(\boldsymbol{R}_j) \right]. \quad (12)
$$

Using this, we get

$$
\hat{U}_c^\dagger \hat{\boldsymbol{p}}_j \hat{U}_c = \hat{\boldsymbol{p}}_j + e_j \boldsymbol{A}(\boldsymbol{R}_j). \quad (13)
$$

Then, since the Coulomb interaction $\hat{V}$ does not depend on the momentum $\{\hat{\boldsymbol{p}}_j\}$ of the charged particles, we get

$$
\hat{\mathcal{H}}''_{\min} \equiv \hat{U}_c^\dagger \hat{\mathcal{H}}'_{\min} \hat{U}_c = \sum_{k=1}^{M} \hbar\omega_k \left(|\alpha_k|^2 + \frac{1}{2} \right) + \hat{\mathcal{H}}_{\mathrm{atom}}, \quad (14)
$$

where $\hat{\mathcal{H}}_{\mathrm{atom}}$ is the Hamiltonian of the charged particles without the interaction with the transverse electromagnetic fields as

$$
\hat{\mathcal{H}}_{\mathrm{atom}} \equiv \sum_{j=1}^{N} \frac{\hat{\boldsymbol{p}}_j^{\,2}}{2m_j} + \hat{V}(\{\hat{\boldsymbol{r}}_j\}). \quad (15)
$$

Since $\hat{U}_c$ is a unitary operator, the partition function in Eq. (7) can be rewritten as

$$
\bar{Z}(T) = \int \left(\prod_k \frac{d^2\alpha_k}{\pi} \right) \mathrm{Tr}[e^{-\beta\hat{\mathcal{H}}''_{\min}}]. \quad (16)
$$

Then, the problem is reduced to the minimization of $\langle\psi(\{\alpha_k\})|\hat{\mathcal{H}}''_{\min}|\psi(\{\alpha_k\})\rangle$ for trial state $|\psi(\{\alpha_k\})\rangle$. Since $\hat{\mathcal{H}}_{\mathrm{atom}}$ in Eq. (14) is simply the Hamiltonian of the charged particles, the minimum energy is obtained for the following state:

$$
|\psi''_{\min,g}\rangle = |\psi_g\rangle_{\mathrm{atom}} \otimes |\{\alpha_k = 0\}\rangle_{\mathrm{em}}, \quad (17)
$$

where $|\psi_g\rangle_{\mathrm{atom}}$ is the ground state of $\hat{\mathcal{H}}_{\mathrm{atom}}$ and $|\{\alpha_k = 0\}\rangle_{\mathrm{em}}$ represents a classical state with zero amplitude for all the photonic modes. In this way, the photonic modes do not spontaneously get an amplitude in the ground state (and also in thermal equilibrium). This is the basic logic of the no-go theorem of SRPTs in atomic systems discussed in Refs. [14,15].

On the other hand, from the minimal-coupling Hamiltonian $\hat{\mathcal{H}}_{\min}$ in Eq. (4) without the c-number substitution, we can

get the Hamiltonian $\hat{\mathcal{H}}_{\text{dip}}$ of the length form [16–20,43], in contrast to $\hat{\mathcal{H}}_{\text{min}}$ called the velocity form. Recovering the vector potential as an operator in the unitary operator as

$$\hat{U} = \exp\left[\frac{i}{\hbar}\sum_{j=1}^{N} e_j \hat{\boldsymbol{r}}_j \cdot \hat{\boldsymbol{A}}(\boldsymbol{R}_j)\right], \tag{18}$$

the Hamiltonian of the length form is obtained in the long-wavelength approximation as

$$\hat{\mathcal{H}}_{\text{dip}} = \hat{U}^{\dagger}\hat{\mathcal{H}}_{\text{min}}\hat{U} \tag{19}$$

$$= \int d\boldsymbol{r}\left\{\frac{[\hat{\boldsymbol{D}}_{\perp}(\boldsymbol{r}) - \hat{\boldsymbol{P}}_{\perp}(\boldsymbol{r})]^2}{2\varepsilon_0} + \frac{\hat{\boldsymbol{B}}(\boldsymbol{r})^2}{2\mu_0}\right\} + \hat{\mathcal{H}}_{\text{atom}} \tag{20}$$

$$= \sum_{k=1}^{M} \hbar\omega_k\left(\hat{a}_k^{\dagger}\hat{a}_k + \frac{1}{2}\right) - \frac{1}{\varepsilon_0}\int d\boldsymbol{r}\,\hat{\boldsymbol{P}}_{\perp}(\boldsymbol{r}) \cdot \hat{\boldsymbol{D}}_{\perp}(\boldsymbol{r})$$

$$+ \frac{1}{2\varepsilon_0}\int d\boldsymbol{r}\,\hat{\boldsymbol{P}}_{\perp}(\boldsymbol{r})^2 + \hat{\mathcal{H}}_{\text{atom}}. \tag{21}$$

Here, $\hat{\boldsymbol{P}}_{\perp}(\boldsymbol{r})$ is the transverse component of the electric polarization $\boldsymbol{P}(\boldsymbol{r}) = \sum_j e_j \hat{\boldsymbol{r}}_j \delta(\boldsymbol{r} - \hat{\boldsymbol{r}}_j)$, while a more general definition is required beyond the long-wavelength approximation (Power-Zienau-Woolley transformation) [17,19,43]. The last term in the first line of Eq. (21) represents the light-matter interaction mediated by $\hat{\boldsymbol{P}}_{\perp}(\boldsymbol{r})$ and the transverse component of the electric displacement field $\hat{\boldsymbol{D}}_{\perp}(\boldsymbol{r})$, which corresponds to the conjugate momentum of the vector potential as $\hat{\boldsymbol{D}}_{\perp}(\boldsymbol{r}) = -\hat{\boldsymbol{\Pi}}(\boldsymbol{r})$ in the length form. The second last term in Eq. (21) is called the P^2 term, by which the absence of SRPTs can also be confirmed [10,11] in the similar manner as the A^2 term.

The ground state $|\psi''_{\text{min},g}\rangle$ of $\hat{\mathcal{H}}''_{\text{min}}$ is not the exact ground state $|\psi_{\text{dip},g}\rangle$ of $\hat{\mathcal{H}}_{\text{dip}}$. However, the absence of SRPTs itself can be confirmed as discussed above if systems of interest satisfy *Assumptions 1 and 2, or A* in the thermodynamic limit. When the transverse electric polarization $\boldsymbol{P}_{\perp}(\boldsymbol{r}) = \langle\psi_g|\hat{\boldsymbol{P}}_{\perp}(\boldsymbol{r})|\psi_g\rangle_{\text{atom}}$ gets an amplitude spontaneously in the ground state $|\psi_g\rangle_{\text{atom}}$ of the charged particles, the electric displacement field can be induced as $\boldsymbol{D}_{\perp}(\boldsymbol{r}) = \boldsymbol{P}_{\perp}(\boldsymbol{r})$, while the electric field is basically zero $\boldsymbol{E}_{\perp} = (\boldsymbol{D}_{\perp} - \boldsymbol{P}_{\perp})/\varepsilon_0 = 0$, by simply considering the minimization of the first term in Eq. (20) as the classical analysis in Ref. [48]. Even though the photonic amplitude can get an amplitude as $\langle\psi_{\text{dip},g}|\hat{\boldsymbol{\Pi}}(\boldsymbol{r})|\psi_{\text{dip},g}\rangle \approx -\boldsymbol{D}_{\perp}(\boldsymbol{r})$ in the ground state of $\hat{\mathcal{H}}_{\text{dip}}$, we do not call it a SRPT in this paper because the appearance of the photonic amplitude originates from the system of charged particles $\hat{\mathcal{H}}_{\text{atom}}$, not from the light-matter interaction.

While the possibility of SRPTs in atomic systems is still under debate especially beyond the long-wavelength approximation [15,19,20], the above logic is basically valid if the c-number substitution performed in Eq. (7) is justified, i.e., if systems of interest satisfy *Assumptions 1 and 2, or A* in the thermodynamic limit $N \to \infty$. Following this semiclassical analysis, we examine the possibility of SRPTs in some superconducting circuit configurations in the following sections.
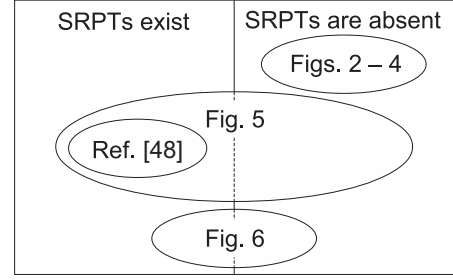

FIG. 1. Map of circuit configurations discussed in this paper and the circuit proposed in Ref. [48]. SRPTs are absent in the configurations depicted in Figs. 2–4, if systems of interest satisfy *Assumptions 1 and 2, or A*. The absence of SRPTs cannot be confirmed in the configurations depicted in Figs. 5 and 6. The circuit proposed in Ref. [48] [depicted in Fig. 5(c)] shows a SRPT and is included in the configurations of Figs. 5(a) and 5(b).

III. CIRCUIT CONFIGURATIONS WHERE SRPTs ARE ABSENT

In this section, we show three superconducting circuit configurations where the absence of SRPTs can be confirmed by the semiclassical analysis explained in the previous section. Once we get an exact Hamiltonian of a circuit, we can examine the possibility of SRPTs following the semiclassical analysis or in other approaches [1,2,48–50]. However, in order to discuss a wide range of circuit structures, Hamiltonians of general forms are preferred, such as the minimal-coupling one for atomic systems.

Figure 1 shows a map of circuit configurations which we will discuss in this paper and the circuit structure proposed in Ref. [48]. We discuss the three circuit configurations depicted in Figs. 2–4 with treating artificial atoms as a black box (without specifying their circuit diagrams). The absence of SRPTs will be confirmed in an inductive-coupling configuration with an LC resonator in Sec. III A (Fig. 2), capacitive-coupling one with an LC resonator in Sec. III B (Fig. 3), and capacitive-coupling one with a transmission line in Sec. III C (Fig. 4). The two configurations depicted in Figs. 5 and 6, where the absence of SRPTs is not confirmed, will be discussed in the next section.
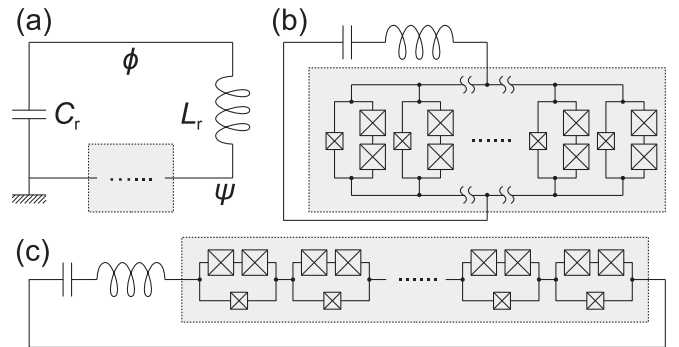

FIG. 2. (a) An LC resonator coupled inductively with a black box. (b), (c) Examples of circuits with artificial atoms. This circuit configuration does not show SRPTs by the coupling between the black box and the LC resonator.
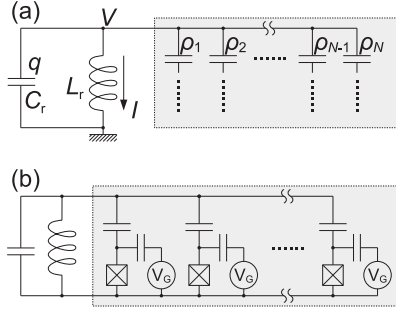
FIG. 3. (a) An LC resonator coupled capacitively with a black box. (b) Example of circuits with artificial atoms, which was already discussed in Ref. [47]. This circuit configuration does not show SRPTs by the coupling between the black box and the LC resonator.

A. Inductive coupling with an LC resonator

We first consider the circuit configuration depicted in Fig. 2(a) consisting of a black box and an LC resonator with inductance L_r and capacitance C_r. Following the flux-based quantization procedure in Ref. [58], we define two node fluxes ϕ, ψ and the ground as Fig. 2(a). A Lagrangian of this circuit is written as

$$\mathcal{L}_1 = \frac{C_r}{2}\dot{\phi}^2 - \frac{(\phi - \psi)^2}{2L_r} + \mathcal{L}_{\text{black}}(\psi, \dot{\psi}; \ldots). \quad (22)$$

The Lagrangian $\mathcal{L}_{\text{black}}$ represents the elements in the black box, and it is described by the flux ψ, its time derivative $\dot{\psi}$, and others inside the black box. The conjugate momenta (charges) of ϕ and ψ are derived, respectively, as

$$q \equiv \frac{\partial \mathcal{L}_1}{\partial \dot{\phi}} = C_r \dot{\phi}, \quad (23a)$$

$$\rho \equiv \frac{\partial \mathcal{L}_1}{\partial \dot{\psi}} = \frac{\partial \mathcal{L}_{\text{black}}}{\partial \dot{\psi}}. \quad (23b)$$

Then, we get a quantized Hamiltonian as

$$\hat{\mathcal{H}}_1 = \frac{\hat{q}^2}{2C_r} + \frac{(\hat{\phi} - \hat{\psi})^2}{2L_r} + \hat{\mathcal{H}}_{\text{black}}(\hat{\psi}, \hat{\rho}; \ldots), \quad (24)$$

where $\hat{\mathcal{H}}_{\text{black}}$ is the Hamiltonian of the black box derived from $\mathcal{L}_{\text{black}}$. The operators satisfy the following commutation

FIG. 5. An LC resonator coupled with a black box, where the absence of SRPTs cannot be confirmed by the analysis in this paper. It is because we could not derive a Hamiltonian for (a). For (b) and (d), their Hamiltonians can be derived, but they cannot be transformed as the minimal-coupling Hamiltonian. (c) The circuit proposed in Ref. [48].

relations:

$$[\hat{\phi}, \hat{q}] = i\hbar, \quad (25a)$$

$$[\hat{\psi}, \hat{\rho}] = i\hbar, \quad (25b)$$

and the other combinations are commutable. We consider the flux $\hat{\phi}$ and the charge $\hat{q}$ of the LC resonator as canonical variables of a photonic mode. Introducing the annihilation operator $\hat{a}$ of a photon and an impedance $Z_r = \sqrt{L_r/C_r}$, they are described as

$$\hat{\phi} = \sqrt{\frac{\hbar Z_r}{2}}(\hat{a} + \hat{a}^\dagger), \quad (26a)$$

$$\hat{q} = -i\sqrt{\frac{\hbar}{2Z_r}}(\hat{a} - \hat{a}^\dagger). \quad (26b)$$

The resonance frequency is expressed as

$$\omega_r = \frac{1}{\sqrt{L_r C_r}}. \quad (27)$$

In Eq. (24), the coupling between the LC resonator and the black box is described by the second term, the inductive energy at L_r. This expression corresponds to the Hamiltonian $\hat{\mathcal{H}}_{\text{dip}}$ of the length form in Eq. (20). Expanding the second term, we get $\hat{\phi}^2/(2L_r)$, $-\hat{\phi}\hat{\psi}/L_r$, and $\hat{\psi}^2/(2L_r)$ corresponding to the photonic flux energy, the interaction term, and the P^2 term, respectively.
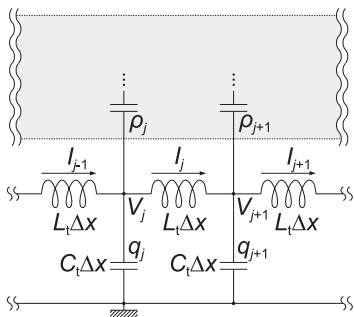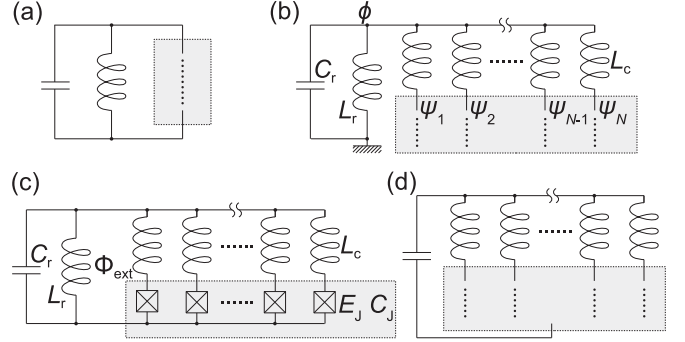

FIG. 4. A transmission line coupled capacitively with a long black box. This circuit configuration does not show SRPTs by the coupling between the black box and the transmission line.
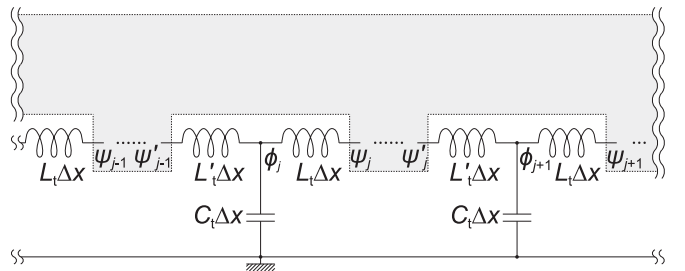

FIG. 6. A transmission line coupled inductively with a long black box. The absence of SRPTs cannot be confirmed by the analysis in this paper.

The no-go theorem for atomic systems starts from the minimal-coupling Hamiltonian in Eq. (1), where the light-matter coupling is described by the kinetic term of the charged particles. In order to describe the coupling by a part of the black box as similar as the minimal-coupling Hamiltonian, we transform Eq. (24) by a unitary operator

$$\hat{U}_1 = e^{-i\hat{q}\hat{\psi}/\hbar}. \tag{28}$$

Using this operator, we get

$$\hat{U}_1^{\dagger}\hat{\phi}\hat{U}_1 = \hat{\phi} + \hat{\psi}, \tag{29a}$$

$$\hat{U}_1^{\dagger}\hat{\rho}\hat{U}_1 = \hat{\rho} - \hat{q}, \tag{29b}$$

and the Hamiltonian is transformed to

$$\hat{\mathcal{H}}_{1'} \equiv \hat{U}_1^{\dagger}\hat{\mathcal{H}}_1\hat{U}_1 \tag{30a}$$

$$= \frac{\hat{q}^2}{2C_{\mathrm{r}}} + \frac{\hat{\phi}^2}{2L_{\mathrm{r}}} + \hat{\mathcal{H}}_{\mathrm{black}}(\hat{\psi},\hat{\rho} - \hat{q};\ldots) \tag{30b}$$

$$= \hbar\omega_{\mathrm{r}}(\hat{a}^{\dagger}\hat{a} + 1/2) + \hat{\mathcal{H}}_{\mathrm{black}}(\hat{\psi},\hat{\rho} - \hat{q};\ldots). \tag{30c}$$

This Hamiltonian has a similar form as the minimal-coupling Hamiltonian in Eq. (4). Specifying the black box and expanding the capacitive term depending on $\hat{\rho} - \hat{q}$, such as $(\hat{\rho} - \hat{q})^2/(2C)$ for a capacitance C, we get an interaction term $-\hat{\rho}\hat{q}/C$ and the A^2 term $\hat{q}^2/(2C)$. However, the following discussion does not depend on the detail of the black box.

Here, we suppose that there are many artificial atoms in the black box, for example, as Figs. 2(b) and 2(c), and the circuit satisfies *Assumptions 1 and 2, or A*. In the thermodynamic limit (infinite number of artificial atoms; $N \rightarrow \infty$), the partition function is written approximately as

$$\bar{Z}(T) = \int \frac{d^2\alpha}{\pi} \operatorname{Tr}[e^{-\beta\hat{\mathcal{H}}_{1'}'}], \tag{31}$$

where $\hat{a}$ is replaced by a c number α as

$$\hat{\mathcal{H}}_{1'}' = \hbar\omega_{\mathrm{r}}(|\alpha|^2 + 1/2) + \hat{\mathcal{H}}_{\mathrm{black}}(\hat{\psi},\hat{\rho} - q;\ldots), \tag{32}$$

and the operator $\hat{q}$ is also replaced by

$$q = -i\sqrt{\frac{\hbar}{2Z_{\mathrm{r}}}}(\alpha - \alpha^*). \tag{33}$$

Here, by substituting the c number also to the unitary operator as

$$\hat{U}_{1c} = e^{-iq\hat{\psi}/\hbar}, \tag{34}$$

the partition function is rewritten as

$$\bar{Z}(T) = \int \frac{d^2\alpha}{\pi} \operatorname{Tr}[e^{-\beta\hat{\mathcal{H}}_{1'}''}], \tag{35}$$

where

$$\hat{\mathcal{H}}_{1'}'' \equiv \hat{U}_{1c}\hat{\mathcal{H}}_{A'}'\hat{U}_{1c}^{\dagger} = \hbar\omega_{\mathrm{r}}(|\alpha|^2 + 1/2) + \hat{\mathcal{H}}_{\mathrm{black}}(\hat{\psi},\hat{\rho};\ldots). \tag{36}$$

In this way, the problem is reduced to the similar one discussed around Eq. (14) for atomic systems. Then, SRPTs originating from the coupling between the LC resonator and the black

box are absent in the circuit configuration of Fig. 2(a), if the circuits satisfy *Assumptions 1 and 2, or A*.

In Figs. 2(b) and 2(c), we suppose many flux qubits [62], which basically require an external magnetic flux in each loop consisting of three Josephson junctions for reaching the ideal two-level systems. Even in the presence of the external magnetic fluxes in these loops, the SRPTs are absent because the Lagrangian is still expressed as Eq. (22), while some phase transitions originating from the black box (not from the coupling with LC resonator) can exist. Of course, the SRPTs are absent also when the external magnetic fluxes are completely absent.

B. Capacitive coupling with an LC resonator

Next, we consider the circuit configuration depicted in Fig. 3(a). An LC resonator couples with a black box through capacitances inside the black box. Following the charge-based quantization procedure in Ref. [59], a Lagrangian is obtained as follows. We define the ground, voltage V, current I, charges q and $\{\rho_j\}$ for $j = 1,2,\ldots,N$ as in Fig. 3(a). The voltage V and charge q at capacitance C_{r} are related as

$$V = \frac{q}{C_{\mathrm{r}}}. \tag{37}$$

The current I through inductance L_{r}, charges $\{\rho_j\}$ at coupling capacitances in the black box, and q at C_{r} are related as

$$I = -\dot{q} - \sum_{j=1}^{N} \dot{\rho}_j. \tag{38}$$

Further, the voltage V and current I are related as

$$V = L_{\mathrm{r}}\dot{I}. \tag{39}$$

Then, we get an equation of motion as

$$\ddot{q} + \sum_{j=1}^{N} \ddot{\rho}_j = \frac{q}{L_{\mathrm{r}}C_{\mathrm{r}}}. \tag{40}$$

There are some other equations of motion describing the inside of the black box. A Lagrangian giving these equations is in general represented as

$$\mathcal{L}_2 = \frac{L_{\mathrm{r}}}{2}\left(\dot{q} + \sum_{j=1}^{N} \dot{\rho}_j\right)^2 - \frac{q^2}{2C_{\mathrm{r}}} + \mathcal{L}_{\mathrm{black}}(\{\rho_j\},\{\dot{\rho}_j\};\ldots). \tag{41}$$

The conjugate momenta are derived as

$$\phi \equiv \frac{\partial\mathcal{L}_2}{\partial\dot{q}} = L_{\mathrm{r}}\left(\dot{q} + \sum_{j=1}^{N} \dot{\rho}_j\right), \tag{42a}$$

$$\psi_j \equiv \frac{\partial\mathcal{L}_2}{\partial\dot{\rho}_j} = L_{\mathrm{r}}\left(\dot{q} + \sum_{j=1}^{N} \dot{\rho}_j\right) + \frac{\partial\mathcal{L}_{\mathrm{black}}}{\partial\dot{\rho}_j}. \tag{42b}$$

They satisfy

$$[\hat{q},\hat{\phi}] = i\hbar, \tag{43a}$$

$$[\hat{\rho}_j,\hat{\psi}_j] = i\hbar, \tag{43b}$$

and other combinations are commutable. The Hamiltonian is obtained as

$$\mathcal{H}_2 = \frac{\phi^2}{2L_r} + \frac{q^2}{2C_r} + \mathcal{H}_{\text{black}}(\{\rho_j\},\{\dot{\rho}_j\};\ldots), \qquad (44)$$

where the Hamiltonian of the black box is defined as

$$\mathcal{H}_{\text{black}}(\{\rho_j\},\{\dot{\rho}_j\};\ldots) \equiv \sum_{j=1}^{N} \dot{\rho}_j \frac{\partial \mathcal{L}_{\text{black}}}{\partial \dot{\rho}_j}$$
$$- \mathcal{L}_{\text{black}}(\{\rho_j\},\{\dot{\rho}_j\};\ldots). \qquad (45)$$

Let us rewrite this in terms of $\{\rho_j\}$, $\{\psi_j\}$, ... From Eqs. (42), we get

$$\frac{\partial \mathcal{L}_{\text{black}}}{\partial \dot{\rho}_j} = \psi_j - \phi. \qquad (46)$$

In the absence of the LC resonator, we simply get $\partial \mathcal{L}_{\text{black}}/\partial \dot{\rho}_j = \psi_j$, and the Hamiltonian is represented as $\mathcal{H}_{\text{black}}(\{\rho_j\},\{\psi_j\};\ldots)$. Then, in the presence of the LC resonator, ψ_j is replaced by $\psi_j - \phi$ in $\mathcal{H}_{\text{black}}$, and the Hamiltonian in Eq. (44) is rewritten in terms of $\{\rho_j\},\{\psi_j\},\ldots$ and in the quantized form as

$$\hat{\mathcal{H}}_2 = \frac{\hat{\phi}^2}{2L_r} + \frac{\hat{q}^2}{2C_r} + \hat{\mathcal{H}}_{\text{black}}(\{\hat{\rho}_j\},\{\hat{\psi}_j - \hat{\phi}\};\ldots). \qquad (47)$$

In this case, expanding an inductive energy depending on $\hat{\psi}_j - \hat{\phi}$ in the black box, we get an interaction term and the A^2 term, when we specify the black box in detail. In the same manner as the previous subsection, we rewrite $\hat{\phi}$ and $\hat{q}$ by annihilation operator $\hat{a}$ as

$$\hat{\phi} = -i\sqrt{\frac{\hbar Z_r}{2}}(\hat{a} - \hat{a}^\dagger), \qquad (48a)$$

$$\hat{q} = \sqrt{\frac{\hbar}{2Z_r}}(\hat{a} + \hat{a}^\dagger). \qquad (48b)$$

Then, we replace $\hat{a}$ by a c number α as

$$\hat{\mathcal{H}}_2' = \hbar\omega_r(|\alpha|^2 + 1/2) + \hat{\mathcal{H}}_{\text{black}}(\hat{\rho},\hat{\psi} - \phi;\ldots), \qquad (49)$$

where $\hat{\phi}$ is also replaced by

$$\phi = -i\sqrt{\frac{\hbar Z_r}{2}}(\alpha - \alpha^*). \qquad (50)$$

Using a unitary operator

$$\hat{U}_{2c} = \exp\left(\frac{i}{\hbar}\phi \sum_{j=1}^{N}\hat{\rho}_j\right), \qquad (51)$$

we get

$$\hat{U}_{2c}^\dagger \hat{\psi}_j \hat{U}_{2c} = \hat{\psi}_j + \phi, \qquad (52)$$

and the problem is reduced to the minimization of

$$\hat{\mathcal{H}}_2'' = \hat{U}_{2c}^\dagger \hat{\mathcal{H}}_2' \hat{U}_{2c}$$
$$= \hbar\omega_r(|\alpha|^2 + 1/2) + \hat{\mathcal{H}}_{\text{black}}(\{\hat{\rho}_j\},\{\hat{\psi}_j\};\ldots). \qquad (53)$$

In the same manner as discussed above, the SRPTs due to the coupling between the LC resonator and the black box are

absent in the circuit configuration of Fig. 3(a), if systems of interest satisfy *Assumptions 1 and 2, or A*. Then, for example, the SRPTs are absent in the circuit of Fig. 3(b), where the charge qubits couple capacitively with an LC resonator as already discussed in Ref. [47].

C. Capacitive coupling with a transmission line

We next consider a transmission line coupled capacitively with a long black box as depicted in Fig. 4. We can derive its Hamiltonian in the similar manner as the previous subsection.

In Fig. 4, C_t and L_t are, respectively, capacitance and inductance per unit length, and Δx is a short length for the discrete description of the transmission line. We define voltage V_j, current I_j, and charges q_j and ρ_j as in Fig. 4. The voltage V_j and the charge q_j at the jth capacitance $C_t\Delta x$ is related as

$$V_j = \frac{q_j}{C_t\Delta x}. \qquad (54)$$

The current I_j is related with the charges q_j and ρ_j as

$$I_j = I_{j-1} - \dot{q}_j - \dot{\rho}_j. \qquad (55)$$

Further, the voltage V_j and the current I_j are related as

$$V_{j+1} - V_j = -L_t\Delta x \dot{I}_j. \qquad (56)$$

From these relations, we get a difference equation as

$$\ddot{q}_j + \ddot{\rho}_j = \frac{q_{j+1} + q_{j-1} - 2q_j}{L_t C_t (\Delta x)^2}. \qquad (57)$$

This equation can be obtained by the following Lagrangian:

$$\mathcal{L}_3 = \sum_j \left[L_t\Delta x(\dot{q}_j + \dot{\rho}_j)^2 - \frac{(q_{j+1} - q_j)^2}{2C_t\Delta x} \right]$$
$$+ \mathcal{L}_{\text{black}}(\{\rho_j\},\{\dot{\rho}_j\};\ldots). \qquad (58)$$

The conjugate momenta are derived as

$$\phi_j \equiv \frac{\partial \mathcal{L}_3}{\partial \dot{q}_j} = L_t\Delta x(\dot{q}_j + \dot{\rho}_j), \qquad (59a)$$

$$\psi_j \equiv \frac{\partial \mathcal{L}_3}{\partial \dot{\rho}_j} = L_t\Delta x(\dot{q}_j + \dot{\rho}_j) + \frac{\partial \mathcal{L}_{\text{black}}}{\partial \dot{\rho}_j}. \qquad (59b)$$

Then, in the same manner as the previous subsection, the Hamiltonian is derived as

$$\mathcal{H}_3 = \sum_j \left[\frac{L_t\Delta x}{2}(\dot{q}_j + \dot{\rho}_j)^2 + \frac{(q_{j+1} - q_j)^2}{2C_t\Delta x} \right]$$
$$+ \mathcal{H}_{\text{black}}(\{\rho_j\},\{\dot{\rho}_j\};\ldots), \qquad (60)$$

$$\hat{\mathcal{H}}_3 = \sum_j \left[\frac{\hat{\phi}_j^2}{2L_t\Delta x} + \frac{(\hat{q}_{j+1} - \hat{q}_j)^2}{2C_t\Delta x} \right]$$
$$+ \hat{\mathcal{H}}_{\text{black}}(\{\hat{\rho}_j\},\{\hat{\psi}_j - \hat{\phi}_j\};\ldots). \qquad (61)$$

The first two terms are simply the Hamiltonian of the transmission line, in which a photon (microwave) propagates with a speed of $v = 1/\sqrt{L_t C_t}$ in the one-dimensional system. The boundary conditions of the transmission line do not affect the possibility of SRPTs in the semiclassical analysis relying on the c-number substitution.

In order to justify the c-number substitution performed in Eq. (7), let us discuss when the systems with the transmission line satisfy *Assumption A*. Here, we consider that the transmission line has a length of ℓ. The frequency of the photonic mode is $\omega_k = k(\pi v/\ell)$ for $k = 1, 2, \ldots$. Considering the minimum wavelength $\lambda_{\min}$ where the electromagnetic wave interacts sufficiently with the artificial atoms and is confined sufficiently in the one-dimensional transmission line, the effective number of the photonic modes is determined as $M = \ell/\lambda_{\min}$. The free energy per atom is in the same order as the characteristic frequency ω_a of the atomic transition, which gives a wavelength of $\lambda_a = 2\pi v/\omega_a$. Instead of the limit $N \to \infty$, we consider the limit of the number of atoms in the length of λ_a as $n = N\lambda_a/\ell \to \infty$. Then, *Assumption A* is rewritten as

$$\frac{1}{N} \frac{\hbar \pi v}{\ell} \frac{M(M+1)}{2} \ll \hbar \omega_a, \quad (62)$$

$$\frac{(\lambda_a/\lambda_{\min})^2}{4} \ll n. \quad (63)$$

In this way, the c-number substitution is justified when the number n of atoms in λ_a is much larger than $(\lambda_a/\lambda_{\min})^2$.

In the same manner as the previous subsections, when the c-number substitution is justified under the above condition, the SRPTs due to the coupling between the transmission line and the black box are absent in the circuit configuration of Fig. 4.

IV. CIRCUIT CONFIGURATIONS WHERE SRPTs CAN EXIST

Next, we show some circuit configurations where the absence of SRPTs cannot be confirmed by the analysis in this paper. In Sec. IV A (Fig. 5), we discuss another inductive-coupling configuration with an LC resonator. In Sec. IV B (Fig. 6), an inductive-coupling configuration with a transmission line is discussed. As shown in Fig. 1, these configurations include also the circuit structures that do not show SRPTs, while the configuration of Fig. 5 includes the circuit proposed in Ref. [48] that shows a SRPT.

A. Another inductive coupling with an LC resonator

Let us first consider the circuit configuration depicted in Fig. 5(a), which is generalized from the capacitive-coupling configuration in Fig. 3(a). We could not derive a Hamiltonian of this configuration in the flux- [58] or charge-based [59] quantization procedure. While other quantization procedures [63] might give a Hamiltonian, it in fact includes the circuit of Fig. 5(c) proposed in Ref. [48], which shows a SRPT in the presence of an external magnetic flux or π junctions. Then, even if we get a Hamiltonian of the circuit configuration in Fig. 5(a), the absence of SRPTs would not be confirmed by the semiclassical analysis.

For example, let us consider the configuration in Fig. 5(b), which is less general than Fig. 5(a) but includes the circuit of Fig. 5(c) proposed in Ref. [48]. Following the flux-based procedure, we define the ground and node fluxes ϕ and $\{\psi_j\}$ as in Fig. 5(b). In the same manner as Sec. III A, a Hamiltonian

can be derived as

$$\hat{\mathcal{H}}_4 = \frac{\hat{q}^2}{2C_r} + \frac{\hat{\phi}^2}{2L_r} + \sum_{j=1}^{N} \frac{(\hat{\phi} - \hat{\psi}_j)^2}{2L_c}$$
$$+ \hat{\mathcal{H}}_{\text{black}}(\{\hat{\psi}_j\}, \{\hat{\rho}_j\}; \ldots). \quad (64)$$

Let us derive the black-box Hamiltonian and roughly check the existence of the SRPT for the circuit proposed in Ref. [48] by specifying the detail inside the black box as Fig. 5(c). Each L_c is connected with a Josephson junction with Josephson energy E_J and capacitance C_J. A half of flux quantum $\Phi_0 = h/(2e)$ is applied to a loop as an external flux bias $\Phi_{\text{ext}} = \Phi_0/2$. For this circuit, the black-box Hamiltonian is derived as [48]

$$\hat{\mathcal{H}}_{\text{black}}^{\text{Ref. [48]}}(\{\hat{\psi}_j\}, \{\hat{\rho}_j\}) = \sum_{j=1}^{N} \left(\frac{\hat{\rho}_j^2}{2C_J} + E_J \cos \frac{2\pi \hat{\psi}_j}{\Phi_0} \right). \quad (65)$$

The sign of the last term (potential energy of the Josephson effect) is positive by the presence of the external flux bias $\Phi_{\text{ext}} = \Phi_0/2$. We can intuitively understand the existence of a SRPT by analyzing the minima of the inductive energy:

$$U(\phi, \psi) = \frac{\phi^2}{2L_r} + \sum_{j=1}^{N} \left[\frac{(\phi - \psi_j)^2}{2L_c} + E_J \cos \frac{2\pi \psi_j}{\Phi_0} \right]. \quad (66)$$

For $NL_r > [\Phi_0/(2\pi)]^2/E_J - L_c$, this function has two minima at $\phi = \pm\phi_0 \neq 0$ (and $\psi_j = \pm[1 + L_c/(NL_r)]\phi_0 \neq 0$). Since the potential barrier between the two minima becomes infinitely high in the thermodynamic limit $N \to \infty$, the symmetry (superposition of the two minima) in the ground state is broken spontaneously, and we get a coherent amplitude of the flux $\phi \approx \pm\phi_0$ below a critical temperature. In this way, SRPTs exist in superconducting circuits where the photonic harmonic energy [$\phi^2/(2L_r)$ minimized at $\phi = 0$] and the atomic anharmonic energy [$E_J \cos(2\pi \psi_j/\Phi_0)$ minimized at $\psi_j \neq 0$] competes through the coupling term [$(\phi - \psi_j)^2/(2L_c)$ minimized for $\phi = \psi_j$].

As we already found a counterexample above, we cannot get the no-go theorem for the Hamiltonian in Eq. (64) derived for the circuit in Fig. 5(b). In contrast to Sec. III A, we cannot relocate the photonic flux ϕ into $\hat{\mathcal{H}}_{\text{black}}$ by unitary transformations since there are N coupling terms $(\hat{\phi} - \hat{\psi}_j)^2/(2L_c)$, while the absence of SRPTs can be shown for $N = 1$ in the same manner as Sec. III A. On the other hand, if we consider the third term, the inductive energies at L_c, as a part of the black-box Hamiltonian as

$$\hat{\mathcal{H}}_{\text{black}}'(\hat{\phi}; \{\hat{\psi}_j\}, \{\hat{\rho}_j\}; \ldots) \equiv \sum_{j=1}^{N} \frac{(\hat{\phi} - \hat{\psi}_j)^2}{2L_c}$$
$$+ \hat{\mathcal{H}}_{\text{black}}(\{\hat{\psi}_j\}, \{\hat{\rho}_j\}; \ldots), \quad (67)$$

the coupling term is certainly included in the black box as

$$\hat{\mathcal{H}}_4 = \frac{\hat{q}^2}{2C_r} + \frac{\hat{\phi}^2}{2L_r} + \hat{\mathcal{H}}_{\text{black}}'(\hat{\phi}; \{\hat{\psi}_j\}, \{\hat{\rho}_j\}; \ldots). \quad (68)$$

However, we cannot remove the photonic flux ϕ from the black-box Hamiltonian even under the c-number substitution.

For example, by introducing a unitary operator as

$$\hat{U}_{4c} = \exp\left(-\frac{i}{\hbar}\phi\sum_{j=1}^{N}\hat{\rho}_j\right), \tag{69}$$

the Hamiltonian $\hat{\mathcal{H}}'_4$ under the c-number substitution is transformed to

$$\hat{U}_{4c}^{\dagger}\hat{\mathcal{H}}'_4\hat{U}_{4c} = \frac{q^2}{2C_{\mathrm{r}}} + \frac{\phi^2}{2L_{\mathrm{r}}} + \hat{\mathcal{H}}''_{\mathrm{black}}(\phi;\{\hat{\psi}_j\},\{\hat{\rho}_j\};\ldots), \tag{70}$$

where the black-box Hamiltonian is transformed as

$$\hat{\mathcal{H}}''_{\mathrm{black}}(\phi;\{\hat{\psi}_j\},\{\hat{\rho}_j\};\ldots)$$
$$= \sum_{j=1}^{N}\frac{\hat{\psi}_j{}^2}{2L_{\mathrm{c}}} + \hat{\mathcal{H}}_{\mathrm{black}}(\{\hat{\psi}_j + \phi\},\{\hat{\rho}_j\};\ldots). \tag{71}$$

In this way, the problem cannot be reduced to the minimization of the black-box Hamiltonian without the LC resonator. In other words, the Hamiltonian of the circuit configuration in Fig. 5(b) cannot be expressed as similar as the minimal-coupling Hamiltonian. Then, the absence of SRPTs cannot be confirmed by the same logic as the no-go theorem for atomic systems. This result is consistent with the proposal of a SRPT in Ref. [48].

In the similar manner, for the circuit configuration of Fig. 5(d), where L_{r} is eliminated, its Hamiltonian is simply derived as Eq. (64) without the second term. The absence of SRPTs cannot be confirmed also in this circuit configuration.

B. Inductive coupling with a transmission line

Finally, let us consider the circuit configuration depicted in Fig. 6. A transmission line couples with a long black box inductively, or we can instead consider small LC resonators coupled through the black box. Following the flux-based procedure, a Lagrangian is obtained as

$$\mathcal{L}_5 = \sum_j\left[\frac{C_{\mathrm{t}}\Delta x}{2}\dot{\phi}_j{}^2 - \frac{(\phi_j - \psi_j)^2}{2L_{\mathrm{t}}\Delta x} - \frac{(\phi_j - \psi'_{j-1})^2}{2L'_{\mathrm{t}}\Delta x}\right]$$
$$+ \mathcal{L}_{\mathrm{black}}(\{\psi_j\},\{\dot{\psi}_j\};\{\psi'_j\},\{\dot{\psi}'_j\};\ldots). \tag{72}$$

The conjugate momenta are derived as

$$q_j \equiv \frac{\partial\mathcal{L}_5}{\partial\dot{\phi}_j} = C_{\mathrm{t}}\Delta x\dot{\phi}_j, \tag{73a}$$

$$\rho_j \equiv \frac{\partial\mathcal{L}_5}{\partial\dot{\psi}_j} = \frac{\partial\mathcal{L}_{\mathrm{black}}}{\partial\dot{\psi}_j}, \tag{73b}$$

$$\rho'_j \equiv \frac{\partial\mathcal{L}_5}{\partial\dot{\psi}'_j} = \frac{\partial\mathcal{L}_{\mathrm{black}}}{\partial\dot{\psi}'_j}. \tag{73c}$$

Then, we get the Hamiltonian as

$$\hat{\mathcal{H}}_5 = \sum_j\left[\frac{\hat{q}_j{}^2}{2C_{\mathrm{t}}\Delta x} + \frac{(\hat{\phi}_j - \hat{\psi}_j)^2}{2L_{\mathrm{t}}\Delta x} + \frac{(\hat{\phi}_j - \hat{\psi}'_{j-1})^2}{2L'_{\mathrm{t}}\Delta x}\right]$$
$$+ \hat{\mathcal{H}}_{\mathrm{black}}(\{\hat{\psi}_j\},\{\hat{\rho}_j\};\{\hat{\psi}'_j\},\{\hat{\rho}'_j\};\ldots). \tag{74}$$

For this Hamiltonian, we cannot relocate the coupling terms into $\hat{\mathcal{H}}_{\mathrm{black}}$ as in the previous sections. For example, using a

unitary operator

$$\hat{U}_5 = \exp\left(-\frac{i}{\hbar}\sum_j\hat{q}_j\hat{\psi}_j\right), \tag{75}$$

we get

$$\hat{U}_5^{\dagger}\hat{\phi}_j\hat{U}_5 = \hat{\phi}_j + \hat{\psi}_j, \tag{76a}$$

$$\hat{U}_5^{\dagger}\hat{\rho}_j\hat{U}_5 = \hat{\rho}_j - \hat{q}_j, \tag{76b}$$

and

$$\hat{U}_5^{\dagger}\hat{\mathcal{H}}_5\hat{U}_5 = \sum_j\left[\frac{\hat{q}_j{}^2}{2C_{\mathrm{t}}\Delta x} + \frac{\hat{\phi}_j{}^2}{2L_{\mathrm{t}}\Delta x} + \frac{(\hat{\phi}_j + \hat{\psi}_j - \hat{\psi}'_{j-1})^2}{2L'_{\mathrm{t}}\Delta x}\right]$$
$$+ \hat{\mathcal{H}}_{\mathrm{black}}(\{\hat{\psi}_j\},\{\hat{\rho}_j - \hat{q}_j\};\{\hat{\psi}'_j\},\{\hat{\rho}'_j\};\ldots). \tag{77}$$

In this way, the coupling terms inevitably remain in the photonic Hamiltonian as far as we tried. Then, the absence of SRPTs in the transmission line of Fig. 6 cannot be confirmed by the analysis in this paper, while its Hamiltonian could be derived with treating the artificial atoms as a black box.

V. SUMMARY

Following the similar analysis as the no-go theorem for atomic systems [14,15], we examined the possibility of SRPTs in some configurations of superconducting circuits. By deriving Hamiltonians with treating artificial atoms as a black box, we show that three configurations depicted in Figs. 2–4 do not show SRPTs if the systems satisfy *Assumptions 1 and 2, or A* in the thermodynamic limit, which justify the c-number substitution performed in Eq. (7) and are essential in the no-go theorem for the atomic systems [14,15].

The absence of SRPTs cannot be confirmed for the circuit configurations in Figs. 5 and 6. It is because, for Fig. 5(a), we could not derive its Hamiltonian with treating artificial atoms as a black box. Concerning Figs. 5(b), 5(d), and 6, we can derive their Hamiltonians, but they cannot be transformed as the minimal-coupling Hamiltonian. Then, the absence of SRPTs cannot be confirmed in the analysis of this paper. In fact, Figs. 5(a) and 5(b) include the circuit in Ref. [48] depicted in Fig. 5(c), where a SRPT in the thermal equilibrium was proposed in the presence of an external magnetic flux or π junctions.

The analysis in this paper shows the absence of SRPTs originating from the coupling between the black box and the LC resonator or the transmission line. If the black box includes another resonator or transmission line, we must examine whether it can be reduced to the three circuit configurations in Figs. 2–4 or we must extend the discussion for circuits with multiple resonators or transmission lines. Further, there also remains the possibility of SRPTs in systems that do not satisfy *Assumption 1, 2, or A*, i.e., those SRPTs cannot be analyzed under the c-number substitution performed in Eq. (7).

In order to find SRPTs in the absence of an external magnetic flux or π junctions, we should explore the circuit configurations in Figs. 5 and 6 or others except Figs. 2–4, while the analysis in this paper does not basically depend on whether an external magnetic flux or π junctions exist or not.

ACKNOWLEDGMENTS

[1] W. R. Mallory, Solution of a multiatom radiation model using the bargmann realization of the radiation field, Phys. Rev. **188**, 1976 (1969).

[2] K. Hepp and E. H. Lieb, On the superradiant phase transition for molecules in a quantized radiation field: The Dicke maser model, Ann. Phys. (NY) **76**, 360 (1973).

[3] Y. K. Wang and F. T. Hioe, Phase transition in the dicke model of superradiance, Phys. Rev. A **7**, 831 (1973).

[4] M. Gross and S. Haroche, Superradiance: An essay on the theory of collective spontaneous emission, Phys. Rep. **93**, 301 (1982).

[5] E. Hanamura, Rapid radiative decay and enhanced optical nonlinearity of excitons in a quantum well, Phys. Rev. B **38**, 1228 (1988).

[6] K. Rzążewski, K. Wódkiewicz, and W. Żakowicz, Phase Transitions, Two-Level Atoms, and the A^2 Term, Phys. Rev. Lett. **35**, 432 (1975).

[7] K. Rzążewski and K. Wódkiewicz, Thermodynamics of two-level atoms interacting with the continuum of electromagnetic field modes, Phys. Rev. A **13**, 1967 (1976).

[8] M. Yamanoi, Influence of omitting the A^2 term in the conventional photon-matter-Hamiltonian on the photon-field equation, Phys. Lett. A **58**, 437 (1976).

[9] M. Yamanoi, On polariton instability and thermodynamic phase transition in a photon-matter system, J. Phys. A: Math. Gen. **12**, 1591 (1979).

[10] V. I. Emeljanov and Y. L. Klimontovich, Appearance of collective polarization as a result of phase transition in an ensemble of two-level atoms, interacting through electromagnetic field, Phys. Lett. A **59**, 366 (1976).

[11] M. Yamanoi and M. Takatsuji, Influence of omitting the P^2 term in the multipole photon-matter Hamiltonian on the stability and propagation, in *Coherence and Quantum Optics IV: Proceedings of the Fourth Rochester Conference on Coherence and Quantum Optics, University of Rochester*, edited by L. Mandel and E. Wolf (Plenum, New York, 1978), pp. 839–850.

[12] R. G. Woolley, Gauge invariance and the thermodynamics of the electromagnetic field, J. Phys. A: Math. Gen. **9**, L15 (1976).

[13] J. M. Knight, Y. Aharonov, and G. T. C. Hsieh, Are super-radiant phase transitions possible? Phys. Rev. A **17**, 1454 (1978).

[14] I. Bialynicki-Birula and K. Rzążewski, No-go theorem concerning the superradiant phase transition in atomic systems, Phys. Rev. A **19**, 301 (1979).

[15] K. Gawędzki and K. Rzążewski, No-go theorem for the superradiant phase transition without dipole approximation, Phys. Rev. A **23**, 2134 (1981).

[16] J. Keeling, Coulomb interactions, gauge invariance, and phase transitions of the Dicke model, J. Phys.: Condens. Matter **19**, 295213 (2007).

[17] A. Vukics, T. Grießer, and P. Domokos, Elimination of the A-Square Problem from Cavity QED, Phys. Rev. Lett. **112**, 073601 (2014).

[18] M. Bamba and T. Ogawa, Stability of polarizable materials against superradiant phase transition, Phys. Rev. A **90**, 063825 (2014).

[19] A. Vukics, T. Grießer, and P. Domokos, Fundamental limitation of ultrastrong coupling between light and atoms, Phys. Rev. A **92**, 043835 (2015).

[20] T. Grießer, A. Vukics, and P. Domokos, Depolarization shift of the superradiant phase transition, Phys. Rev. A **94**, 033815 (2016).

[21] C. Ciuti, G. Bastard, and I. Carusotto, Quantum vacuum properties of the intersubband cavity polariton field, Phys. Rev. B **72**, 115303 (2005).

[22] M. Devoret, S. Girvin, and R. Schoelkopf, Circuit-QED: How strong can the coupling between a Josephson junction atom and a transmission line resonator be? Ann. Phys. (Leipzig) **16**, 767 (2007).

[23] G. Gunter, A. A. Anappara, J. Hees, A. Sell, G. Biasiol, L. Sorba, S. de Liberato, C. Ciuti, A. Tredicucci, A. Leitenstorfer, and R. Huber, Sub-cycle switch-on of Sub-cycle switch-on of ultrastrong light-matter interaction, Nature (London) **458**, 178 (2009).

[24] A. A. Anappara, S. de Liberato, A. Tredicucci, C. Ciuti, G. Biasiol, L. Sorba, and F. Beltram, Signatures of the ultrastrong light-matter coupling regime, Phys. Rev. B **79**, 201303 (2009).

[25] Y. Todorov, A. M. Andrews, I. Sagnes, R. Colombelli, P. Klang, G. Strasser, and C. Sirtori, Strong Light-Matter Coupling in Subwavelength Metal-Dielectric Microcavities at Terahertz Frequencies, Phys. Rev. Lett. **102**, 186402 (2009).

[26] Y. Todorov, A. M. Andrews, R. Colombelli, S. de Liberato, C. Ciuti, P. Klang, G. Strasser, and C. Sirtori, Ultrastrong Light-Matter Coupling Regime with Polariton Dots, Phys. Rev. Lett. **105**, 196402 (2010).

[27] T. Niemczyk, F. Deppe, H. Huebl, E. P. Menzel, F. Hocke, M. J. Schwarz, J. J. Garcia-Ripoll, D. Zueco, T. Hummer, E. Solano *et al.*, Circuit quantum electrodynamics in the ultrastrong-coupling regime, Nat. Phys. **6**, 772 (2010).

[28] A. Fedorov, A. K. Feofanov, P. Macha, P. Forn-Díaz, C. J. P. M. Harmans, and J. E. Mooij, Strong Coupling of a Quantum Oscillator to a Flux Qubit at Its Symmetry Point, Phys. Rev. Lett. **105**, 060503 (2010).

[29] P. Forn-Díaz, J. Lisenfeld, D. Marcos, J. J. García-Ripoll, E. Solano, C. J. P. M. Harmans, and J. E. Mooij, Observation of the Bloch-Siegert Shift in a Qubit-Oscillator System in the Ultrastrong Coupling Regime, Phys. Rev. Lett. **105**, 237001 (2010).

[30] T. Schwartz, J. A. Hutchison, C. Genet, and T. W. Ebbesen, Reversible Switching of Ultrastrong Light-Molecule Coupling, Phys. Rev. Lett. **106**, 196405 (2011).

[31] M. Porer, J.-M. Ménard, A. Leitenstorfer, R. Huber, R. Degl'Innocenti, S. Zanotto, G. Biasiol, L. Sorba, and A. Tredicucci, Nonadiabatic switching of a photonic band structure:

Ultrastrong light-matter coupling and slow-down of light, Phys. Rev. B **85**, 081302(R) (2012).

[32] G. Scalari, C. Maissen, D. Turčinková, D. Hagenmüller, S. de Liberato, C. Ciuti, C. Reichl, D. Schuh, W. Wegscheider, M. Beck, and J. Faist, Ultrastrong coupling of the cyclotron transition of a 2D electron gas to a THz metamaterial, Science **335**, 1323 (2012).

[33] M. Goryachev, W. G. Farr, D. L. Creedon, Y. Fan, M. Kostylev, and M. E. Tobar, High-Cooperativity Cavity QED with Magnons at Microwave Frequencies, Phys. Rev. Appl. **2**, 054002 (2014).

[34] X. Zhang, C.-L. Zou, L. Jiang, and H. X. Tang, Strongly Coupled Magnons and Cavity Microwave Photons, Phys. Rev. Lett. **113**, 156401 (2014).

[35] Q. Zhang, M. Lou, X. Li, J. L. Reno, W. Pan, J. D. Watson, M. J. Manfra, and J. Kono, Collective non-perturbative coupling of 2D electrons with high-quality-factor terahertz cavity photons, Nat. Phys. **12**, 1005 (2016).

[36] P. Forn-Díaz, J. J. García-Ripoll, B. Peropadre, J.-L. Orgiazzi, M. A. Yurtalan, R. Belyansky, C. M. Wilson, and A. Lupascu, Ultrastrong coupling of a single artificial atom to an electromagnetic continuum in the nonperturbative regime, Nat. Phys. **13**, 39 (2017).

[37] F. Yoshihara, T. Fuse, S. Ashhab, K. Kakuyanagi, S. Saito, and K. Semba, Superconducting qubit-oscillator circuit beyond the ultrastrong-coupling regime, Nat. Phys. **13**, 44 (2017).

[38] S. Ashhab and F. Nori, Qubit-oscillator systems in the ultrastrong-coupling regime and their potential for preparing nonclassical states, Phys. Rev. A **81**, 042311 (2010).

[39] A. Quattropani, L. C. Andreani, and F. Bassani, Quantum theory of polaritons with spatial dispersion: Exact solutions, Il Nuovo Cimento D **7**, 55 (1986).

[40] F. Dimer, B. Estienne, A. S. Parkins, and H. J. Carmichael, Proposed realization of the Dicke-model quantum phase transition in an optical cavity QED system, Phys. Rev. A **75**, 013804 (2007).

[41] K. Baumann, C. Guerlin, F. Brennecke, and T. Esslinger, Dicke quantum phase transition with a superfluid gas in an optical cavity, Nature (London) **464**, 1301 (2010).

[42] K. Baumann, R. Mottl, F. Brennecke, and T. Esslinger, Exploring Symmetry Breaking at the Dicke Quantum Phase Transition, Phys. Rev. Lett. **107**, 140402 (2011).

[43] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Photons and Atoms: Introduction to Quantum Electrodynamics* (Wiley, New York, 1989).

[44] P. Nataf and C. Ciuti, No-go theorem for superradiant quantum phase transitions in cavity QED and counter-example in circuit QED, Nat. Commun. **1**, 72 (2010).

[45] O. Viehmann, J. von Delft, and F. Marquardt, Superradiant Phase Transitions and the Standard Description of Circuit QED, Phys. Rev. Lett. **107**, 113602 (2011).

[46] C. Ciuti and P. Nataf, Comment on "Superradiant Phase Transitions and the Standard Description of Circuit QED", Phys. Rev. Lett. **109**, 179301 (2012).

[47] T. Jaako, Z.-L. Xiang, J. J. Garcia-Ripoll, and P. Rabl, Ultrastrong-coupling phenomena beyond the Dicke model, Phys. Rev. A **94**, 033850 (2016).

[48] M. Bamba, K. Inomata, and Y. Nakamura, Superradiant Phase Transition in a Superconducting Circuit in Thermal Equilibrium, Phys. Rev. Lett. **117**, 173601 (2016).

[49] C. Emary and T. Brandes, Quantum Chaos Triggered by Precursors of a Quantum Phase Transition: The Dicke Model, Phys. Rev. Lett. **90**, 044101 (2003).

[50] C. Emary and T. Brandes, Chaos and the quantum phase transition in the Dicke model, Phys. Rev. E **67**, 066203 (2003).

[51] P. Nataf and C. Ciuti, Vacuum Degeneracy of a Circuit QED System in the Ultrastrong Coupling Regime, Phys. Rev. Lett. **104**, 023601 (2010).

[52] J. A. Hutchison, T. Schwartz, C. Genet, E. Devaux, and T. W. Ebbesen, Modifying chemical landscapes by coupling to vacuum fields, Angew. Chem. Int. Ed. **51**, 1592 (2012).

[53] J. A. Hutchison, A. Liscio, T. Schwartz, A. Canaguier-Durand, C. Genet, V. Palermo, P. Samori, and T. W. Ebbesen, Tuning the work-function via strong coupling, Adv. Mater. **25**, 2481 (2013).

[54] A. Canaguier-Durand, E. Devaux, J. George, Y. Pang, J. A. Hutchison, T. Schwartz, C. Genet, N. Wilhelms, J.-M. Lehn, and T. W. Ebbesen, Thermodynamics of molecules strongly coupled to the vacuum field, Angew. Chem. Int. Ed. **52**, 10533 (2013).

[55] S. Sen, K. S. Gupta, and J. M. D. Coey, Mesoscopic structure formation in condensed matter due to vacuum fluctuations, Phys. Rev. B **92**, 155115 (2015).

[56] J. A. Ćwik, P. Kirton, S. de Liberato, and J. Keeling, Excitonic spectral features in strongly coupled organic polaritons, Phys. Rev. A **93**, 033840 (2016).

[57] V. V. Ryazanov, V. A. Oboznov, A. Y. Rusanov, A. V. Veretennikov, A. A. Golubov, and J. Aarts, Coupling of Two Superconductors through a Ferromagnet: Evidence for a π Junction, Phys. Rev. Lett. **86**, 2427 (2001).

[58] M. H. Devoret, Quantum fluctuations in electrical circuits, in *Quantum Fluctuations, Les Houches LXIII, 1995*, edited by S. Reynaud, E. Giacobino, and J. Zinn-Justin (Elsevier, Amsterdam, 1997), Chap. 10, pp. 351–386.

[59] B. Yurke and J. S. Denker, Quantum network theory, Phys. Rev. A **29**, 1419 (1984).

[60] K. Hepp and E. H. Lieb, Equilibrium statistical mechanics of matter interacting with the quantized radiation field, Phys. Rev. A **8**, 2517 (1973).

[61] J. L. van Hemmen and K. Rzążewski, On the thermodynamic equivalence of the Dicke maser model and a certain spin system, Phys. Lett. A **77**, 211 (1980).

[62] T. P. Orlando, J. E. Mooij, L. Tian, C. H. van der Wal, L. S. Levitov, S. Lloyd, and J. J. Mazo, Superconducting persistent-current qubit, Phys. Rev. B **60**, 15398 (1999).

[63] F. Solgun, D. W. Abraham, and D. P. DiVincenzo, Black-box quantization of superconducting circuits using exact impedance synthesis, Phys. Rev. B **90**, 134504 (2014).