

Continuous-variable entanglement of two bright coherent states that never interactedDavid Barral,^{1,*} Nadia Belabas,¹ Lorenzo M. Procopio,¹ Virginia D'Auria,² Sébastien Tanzilli,²
Kamel Bencheikh,¹ and Juan Ariel Levenson¹¹*Centre de Nanosciences et de Nanotechnologies C2N, CNRS, Université Paris-Saclay, Route de Nozay, 91460 Marcoussis, France*²*Université Côte d'Azur, Institut de Physique de Nice (INPHYNI), CNRS UMR 7010, Parc Valrose, 06108 Nice Cedex 2, France*

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We study continuous-variable entanglement of bright quantum states in a pair of evanescently coupled nonlinear $\chi^{(2)}$ waveguides operating in the regime of degenerate down conversion. We consider the case where only the energy of the nonlinearly generated fields is exchanged between the waveguides while the pump fields stay independently guided in each original waveguide. We show that this device, when operated in the depletion regime, entangles the two noninteracting bright pump modes due to a nonlinear cascade effect. It is also shown that two-color quadripartite entanglement can be produced when certain system parameters are appropriately set. This device works in the traveling-wave configuration, such that the generated quantum light shows a broad spectrum. The proposed device can be easily realized with current technology and therefore stands as a good candidate for a source of bipartite or multipartite entangled states for the emerging field of optical continuous-variable quantum information processing.

DOI: [10.1103/PhysRevA.96.053822](https://doi.org/10.1103/PhysRevA.96.053822)**I. INTRODUCTION**

In recent years there has been an increasing interest in quantum information processing (QIP) with continuous variables (CV) [1,2]. In contrast to optical QIP with discrete variables, where qubits are coded by using discrete photon observables, CV-based quantum information is encoded in the fluctuations of the field quadratures that can take a continuous spectrum of eigenvalues. Entanglement of such quadratures for Gaussian states constitutes the primary resource of CV-QIP protocols. Interestingly, the latter take advantage from deterministic resources and unconditional operations as well as highly efficient homodyne detection. These features have made optical CV a thriving area of research: teleportation [3], three-color entanglement [4], one-way quantum computation [5], long-distance quantum key distribution [6], and entanglement distillation and swapping [7] are some examples of breakthrough demonstrations reached in this framework.

These achievements have been mostly accomplished with bulk optics in table-top experiments. However, integrated optics (IO) is one of the strongest candidates for transferring these systems to real-world light-based QIP technologies [8]. Both discrete and continuous-variable quantum states of light can be generated, processed, and measured in practical, low-cost, interlinked, and reconfigurable optical chips. IO displays features out of reach from bulk-optics analogs such as miniaturization, subwavelength stability, generation, and manipulation of quantum light by means of enhanced nonlinear effects and the thermo-, electro- or strain-optic properties of the substrates, and the ability to integrate detectors on chip [9–11]. Nevertheless, IO-based CV is a relatively young area of research. The progressive advance of materials technology pushes this field through the development of highly nonlinear and low-loss materials such as, among others, lithium niobate, potassium titanyl phosphate, and silicon. Over the last few years important on-chip demonstrations have been reported,

such as continuous-wave single-mode squeezing up to 1.8 dB in a traveling-wave configuration [12] and up to 2.9 dB in a cavity resonator architecture [13,14], or the entanglement of a pair of squeezed states remotely produced in cavities via integrated configurable directional couplers [15].

Beyond integrated $\chi^{(2)}$ single-waveguide performances, the on-chip integration of different functionalities is of primary importance. In this regard, the directional coupler with a built-in nonlinearity is an emblematic device that deserves special attention [16]. Actually, the first demonstrations of two-photon NOON-state generation in nonlinear directional couplers have been recently shown [17,18]. The purpose of this work is to study the CV quantum properties of light propagated in this device in both spontaneous parametric down-conversion (SPDC) and optical amplification regimes. Our first prediction is the entanglement of the pump fields. This is particularly interesting since we consider independent pumps for each waveguide and no direct energy transfer between them, i.e., no evanescent coupling nor cavity feedback. We unveil the physical origin of such entanglement as a cascading nonlinear phase mediated, through pump depletion, by the interacting signal waves. We further demonstrate that entanglement between pumps can coexist with signals' entanglement leading to two-color quadripartite entanglement of the four propagating modes for a specific set of parameters. Strikingly, there is no bulk-optics analog to this IO device or, in other words, a nonlinear beam splitter does not exist. Other theoretical proposals dealing with the entanglement of bright waves in couplers have been reported over the last years [19–24]. Let us stress, however, that from a fundamental point of view, in these studies a linear coupling between the pump fields is considered, whereas in our proposal these modes never interact directly. Moreover, from a practical point of view, these devices are based on doubly resonant optical cavities, which are narrowband stationary-wave-based devices with a limited bandwidth of entanglement. This is in contrast with our proposal which is based on traveling waves and shows much broader bandwidths in the continuous-wave regime, only limited by the phase-matching acceptance (up to 10 THz in

*Corresponding author: david.barral@c2n.upsaclay.fr

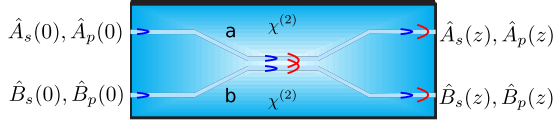


FIG. 1. Sketch of proposed nonlinear directional coupler made of two identical waveguides a and b with second-order susceptibilities $\chi^{(2)}$. Phase matching is fulfilled only in the coupling region. In red are the signal waves, evanescently coupled. In blue are the noninteracting pump waves.

periodically poled lithium niobate (PPLN) waveguides [9]). Therefore, our approach stands as a good option for practical CV-based optical QIP. Likewise, another strength of our proposal is its simplicity. As will be shown, CV entanglement is generated over a large range of propagation lengths by only controlling the input powers and phases. Finally, we stress that this device can be realized with current technology, notably on lithium niobate [9].

The article is organized as follows: In Sec. II, we introduce the device under investigation and the propagation equations describing its operation. In Sec. III we present the underlying cascade effect due to linear coupling of the signal modes that appears in the undepleted-pump approximation and yields entanglement oscillations. In Sec. IV we study the evolution of the covariance matrix, which describes the full system in the depleted-pump regime, and show how bipartite and quadripartite entanglement arise. In Sec. V we present the numerical results obtained for a specific example and discuss their importance. Finally, the main results of this work are summarized in Sec. VI.

II. THE NONLINEAR DIRECTIONAL COUPLER

The nonlinear directional coupler, sketched in Fig. 1, is made of two identical $\chi^{(2)}$ waveguides in which degenerate SPDC, or optical amplification when seeded, can take place. In each waveguide, a pump photon (p) at frequency ω_p is down-converted into indistinguishable idler (i) and signal (s) photons with equal frequencies $\omega_s = \omega_i = \omega_p/2$ and the same polarization modes (fully degenerate process). The efficiency of this process is maximum when the mismatch of the propagation constants between the pump and signal photons caused by dispersion is negligible, so that $\Delta k \equiv k(\omega_p) - 2k(\omega_s) = 0$. In the following we consider that this phase-matching condition is fulfilled only in the coupling zone. The energy of the signal modes generated in each waveguide is exchanged between the waveguides through the evanescent waves, resulting into a linear coupling of the signal waves, whereas the interplay of the higher-frequency pumps is negligible for the propagation lengths considered. This approximation is made since the guided-mode fields are more confined into the guiding region as the mode wavelength decreases. As a result, the coupling constants for the pump modes are smaller than those corresponding to the signal modes [25]. The relevant operator which describes this system is the interaction momentum [26,27]

$$\hat{M} = \hbar \{ g \hat{A}_p \hat{A}_s^{\dagger 2} + g \hat{B}_p \hat{B}_s^{\dagger 2} + C \hat{A}_s \hat{B}_s^{\dagger} + \text{H.c.} \}, \quad (1)$$

where \hat{A} and \hat{B} are slowly varying amplitude annihilation operators of signal (s) and pump (p) photons corresponding to the upper (a) and lower (b) waveguides, respectively, g is the nonlinear constant proportional to $\chi^{(2)}$, C is the linear coupling constant, \hbar is Planck's constant, and H.c. stands for Hermitian conjugate. From this momentum operator, the following Heisenberg equations are obtained:

$$\frac{d\hat{A}_s}{dz} = iC\hat{B}_s + 2ig\hat{A}_p\hat{A}_s^{\dagger}, \quad (2)$$

$$\frac{d\hat{A}_p}{dz} = ig\hat{A}_s^2, \quad (3)$$

$$\frac{d\hat{B}_s}{dz} = iC\hat{A}_s + 2ig\hat{B}_p\hat{B}_s^{\dagger}, \quad (4)$$

$$\frac{d\hat{B}_p}{dz} = ig\hat{B}_s^2, \quad (5)$$

where z is the coordinate corresponding to the direction of propagation, and C has been taken as real without loss of generality. Notably, the factor 2 arising in both equations (2) and (4) does not appear in the classically derived equations [26].

III. UNDEPLETED-PUMP APPROXIMATION

The production of CV entangled states in a nonlinear $\chi^{(2)}$ directional coupler working in a traveling- and continuous-wave configuration through SPDC has been thoroughly studied in Ref. [28]. In this case, the undepleted-pump approximation can in principle be safely assumed if strong coherent pumps are used, i.e., $|\alpha_p|^2, |\beta_p|^2 \gg 1$, where $\hat{A}_p|\alpha_p\rangle = \alpha_p|\alpha_p\rangle$ and $\hat{B}_p|\beta_p\rangle = \beta_p|\beta_p\rangle$. This previous work was focused on the analysis of the nonclassicality and entanglement of the quantum states related to the signal modes for different input parameters such as pump powers and phases or coupling constants. It was then demonstrated that, if the linear coupling dominates over the nonlinear one ($C > 2g$) for $\alpha_p = \beta_p$ (equal amplitudes and phases), a measure of entanglement, the logarithmic negativity $E_{\mathcal{N}}$ (see Sec. IV for a detailed discussion of this concept), shows an oscillatory evolution, periodically shifting between a maximum and zero values. This intriguing behavior caught our attention. We solved analytically the system of equations (2)–(5) in the undepleted-pump approximation, obtaining [16]

$$\begin{aligned} \hat{A}_s(z) = & \cos\left(\frac{\pi z}{2L_{ab}}\right)\hat{A}_s(0) \\ & + i\frac{2L_{ab}}{\pi}\sin\left(\frac{\pi z}{2L_{ab}}\right)[2\eta\hat{A}_s^{\dagger}(0) + C\hat{B}_s(0)], \end{aligned} \quad (6)$$

$$\begin{aligned} \hat{B}_s(z) = & \cos\left(\frac{\pi z}{2L_{ab}}\right)\hat{B}_s(0) \\ & + i\frac{2L_{ab}}{\pi}\sin\left(\frac{\pi z}{2L_{ab}}\right)[2\eta\hat{B}_s^{\dagger}(0) + C\hat{A}_s(0)], \end{aligned} \quad (7)$$

where $L_{ab} = \pi/[2(C^2 - 4\eta^2)^{1/2}]$ is the beat length, i.e., the transfer distance of signal energy from waveguides a to b and from b to a, and $\eta \equiv g|\alpha_p| = g|\beta_p|$ acts as a nonlinear correction to the linear coupling constant. To know if the absence of entanglement was related to energy conservation,

we calculated the mean number of photons of the signal modes from the solutions given by Eqs. (6) and (7), obtaining the following in each waveguide:

$$N_s(z) = \left(\frac{4\eta L_{ab}}{\pi} \right)^2 \sin^2 \left(\frac{\pi z}{2L_{ab}} \right). \quad (8)$$

Note that, in the case of no coupling, the usual $N_s(z) = \sinh^2(2\eta z)$ is found. Likewise, following Ref. [28] and using Eqs. (6) and (7), we calculated the evolution of the logarithmic negativity corresponding to the case under study, which is given by

$$E_{\mathcal{N}}(z) = -2 \log_2(\sqrt{\sigma(z)} - 1/2 - \sqrt{\sigma(z)} + 1/2), \quad (9)$$

with $\sigma(z) = \{1 + [(C/\eta)N_s(z)]^2\}^{1/2}/2$. Both maximum entanglement and number of signal photons are obtained at $z = (2n+1)L_{ab}$, with $n = 0, 1, \dots$, whereas both signal photons and entanglement fully disappear at $z = 2nL_{ab}$ [29]. Since the intrinsic nature of the undepleted-pump approximation violates conservation of energy, this result indicates that the signal photons generated between $z = 0$ and $z = L_{ab}$ are periodically transferred back to the pump in the interval $z = \{L_{ab}, 2L_{ab}\}$, but this behavior is not accounted for in the undepleted-pump model [28].

After this connection between the entanglement evolution and the conservation of energy, the next step is to identify the origin of this effect. Since perfect phase matching is considered in the system, we wonder if a dynamical phase is showing up during the propagation. To tackle this question, we calculate the classical phase accumulated by the signal modes. Optical parametric amplification, i.e., seeding the signal modes, is considered since SPDC is not suitable to this calculation. Taking identical input seeds and calculating their propagation through Eqs. (6) and (7), the following phase mismatch is obtained:

$$\Delta\Phi(C, \eta, z) = -2 \arctan \left\{ \sqrt{\frac{C+2\eta}{C-2\eta}} \tan \left(\frac{\pi z}{2L_{ab}} \right) \right\}, \quad (10)$$

where $\Delta\Phi \equiv \Phi_p - 2\Phi_s = -2\Phi_s$ with Φ_s being the classical phase related to each signal mode and Φ_p being a constant phase corresponding to the pump taken as a reference ($\Phi_p = 0$). This nonlinear phase mismatch drives two cascaded nonlinear optical processes, down conversion followed by up conversion, mediated by the linear coupling of the signal modes. This evolving phase mismatch periodically switches the system from an efficient pump-to-signal conversion to an efficient signal-to-pump conversion. At the beat lengths, the cascaded phases are $\Delta\Phi(C, \eta, (2n+1)L_{ab}) = \pi$ or $\Delta\Phi(C, \eta, 2nL_{ab}) = 2\pi$. Applying these phases in Eq. (1) is equivalent to switching the nonlinear coupling constant from g to $-g$ at the odd multiples of L_{ab} and keeping its sign positive at the even multiples. Therefore, this phase mismatch $\Delta\Phi$ is at the origin of the transfer of photons from pumps to signals and from signals to pumps. Note that, in the case of uncoupled waveguides ($C = 0$), only down conversion is produced in each waveguide and the above phase disappears due to the phase matching.

In light of these results a question arises: if down conversion within the nonlinear coupler produces entanglement of the signal modes, does up conversion entangle the pump modes

which do not directly interact in the system? To answer this question we need to include the quantum character of the pump modes. This is carried out in the following section by means of the linearization method in the depleted-pump regime.

IV. DEPLETED-PUMP REGIME

Unfortunately, there is no known exact analytical solution to Eqs. (2)–(5) when the quantum nature of the pump fields is taken into account. We thus implement the linearization of the equations by means of quantum-fluctuation operators with zero mean value and the same variances as the input operators \hat{A}_j, \hat{B}_j , with $j = s, p$, which is a standard method in quantum optics [30]. Under this approximation, we first need to solve the propagation of the classical fields $\alpha_s(\alpha_p)$ and $\beta_s(\beta_p)$ in order to obtain the evolution of the quantum fluctuations. Let us define the following dimensionless amplitudes and phases related to the classical fields for each waveguide [31]:

$$u_s(z) = \frac{|\alpha_s(z)|}{\sqrt{P}}, \quad v_s(z) = \frac{|\beta_s(z)|}{\sqrt{P}}, \quad (11)$$

$$\theta_s(z) = \arg\{\alpha_s(z)\}, \quad \phi_s(z) = \arg\{\beta_s(z)\}, \quad (12)$$

$$u_p(z) = \sqrt{\frac{2}{P}} |\alpha_p(z)|, \quad v_p(z) = \sqrt{\frac{2}{P}} |\beta_p(z)|, \quad (13)$$

$$\theta_p(z) = \arg\{\alpha_p(z)\}, \quad \phi_p(z) = \arg\{\beta_p(z)\}, \quad (14)$$

and the normalized propagation coordinate $\zeta = \sqrt{2P}gz$, with P being a constant of propagation related to the conservation of energy and the power of the whole system, which reads

$$P = |\alpha_s(z)|^2 + |\beta_s(z)|^2 + 2|\alpha_p(z)|^2 + 2|\beta_p(z)|^2, \quad \forall z \geq 0. \quad (15)$$

Introducing these variables into the classical version of Eqs. (2)–(5), we obtain [31]

$$\frac{du_s}{d\zeta} = -\kappa v_s \sin(\phi_s - \theta_s) - u_s u_p \sin(\Delta\theta), \quad (16)$$

$$\frac{d\theta_s}{d\zeta} = \kappa \frac{v_s}{u_s} \cos(\phi_s - \theta_s) + u_p \cos(\Delta\theta), \quad (17)$$

$$\frac{du_p}{d\zeta} = u_s^2 \sin(\Delta\theta), \quad (18)$$

$$\frac{d\theta_p}{d\zeta} = \frac{u_s^2}{u_p} \cos(\Delta\theta), \quad (19)$$

with $\Delta\theta \equiv \theta_p - 2\theta_s$ and $\kappa = C/(\sqrt{2P}g)$. The other four equations can be obtained by exchanging $u \leftrightarrow v$ and $\theta \leftrightarrow \phi$. κ is the governing parameter of the system. It acts as an effective coupling which relates the linear and nonlinear couplings as well as the total input power. It is important to notice that we restrict the calculations to values of $\kappa > 1$, since for $\kappa \leq 1$ the fluctuations start to grow exponentially and the linearization approximation is no longer valid [21]. This situation corresponds to the experimental regime available with the current technology. Moreover, when considering identical input energy at each waveguide, the following initial

conditions are chosen:

$$u_s(0) = v_s(0) = \operatorname{sech}(-\delta_0)/\sqrt{2}, \quad (20)$$

$$u_p(0) = v_p(0) = -\tanh(-\delta_0)/\sqrt{2}, \quad (21)$$

such that $\sum_{j=s,p}(u_j^2 + v_j^2) = 1$ at any normalized plane ζ . The initial conditions are related to the measurable signal and pump input powers, P_s and P_p , through the parameter $\delta_0 = \operatorname{arcsinh}(\sqrt{P_p/P_s})$. Note that, in the case of different input powers in each waveguide, a different set of initial conditions should be used. Furthermore, the behavior of the light propagating in the device depends on the input phases $(\theta_s(0), \theta_p(0), \phi_s(0), \phi_p(0))$. However, we found that, if all initial phases are set equal, the pump-field entanglement is maximized. Therefore, $\theta_s(0) = \theta_p(0) = \phi_s(0) = \phi_p(0)$ is taken along the paper.

The solutions of this classical system of equations are then fed into first-order equations in the quantum fluctuations, keeping only the linear terms. Since we are interested in CV entanglement, it is more convenient to deal with quadratures of the field $\hat{X}_{(s,p)}^{(A,B)}$, $\hat{Y}_{(s,p)}^{(A,B)}$, where $\hat{X}^O = (\hat{O} + \hat{O}^\dagger)/\sqrt{2}$ and $\hat{Y}^O = i(\hat{O}^\dagger - \hat{O})/\sqrt{2}$ are the orthogonal quadratures corresponding to an optical mode O . In terms of dimensionless variables, the propagation of the quantum field quadratures are given by [31]

$$\begin{aligned} \frac{d\hat{X}_s^A}{d\zeta} &= -u_p \sin(\theta_p) \hat{X}_s^A + u_p \cos(\theta_p) \hat{Y}_s^A - \kappa \hat{Y}_s^B \\ &+ \sqrt{2}u_s \sin(\theta_s) \hat{X}_p^A - \sqrt{2}u_s \cos(\theta_s) \hat{Y}_p^A, \end{aligned} \quad (22)$$

$$\begin{aligned} \frac{d\hat{Y}_s^A}{d\zeta} &= u_p \cos(\theta_p) \hat{X}_s^A + u_p \sin(\theta_p) \hat{Y}_s^A + \kappa \hat{X}_s^B \\ &+ \sqrt{2}u_s \cos(\theta_s) \hat{X}_p^A + \sqrt{2}u_s \sin(\theta_s) \hat{Y}_p^A, \end{aligned} \quad (23)$$

$$\frac{d\hat{X}_p^A}{d\zeta} = -\sqrt{2}u_s \sin(\theta_s) \hat{X}_s^A - \sqrt{2}u_s \cos(\theta_s) \hat{Y}_s^A, \quad (24)$$

$$\frac{d\hat{Y}_p^A}{d\zeta} = \sqrt{2}u_s \cos(\theta_s) \hat{X}_s^A - \sqrt{2}u_s \sin(\theta_s) \hat{Y}_s^A, \quad (25)$$

and the other four equations are obtained by exchanging again $u \leftrightarrow v$, $\theta \leftrightarrow \phi$, and $A \leftrightarrow B$. This system of equations can be rewritten in compact form as $d\hat{\xi}/d\zeta = \Delta(\zeta)\hat{\xi}$, where $\Delta(\zeta)$ is an 8×8 matrix of coefficients of Eqs. (22)–(25), and $\hat{\xi} = (\hat{X}_s^A, \hat{Y}_s^A, \hat{X}_p^A, \hat{Y}_p^A, \hat{X}_s^B, \hat{Y}_s^B, \hat{X}_p^B, \hat{Y}_p^B)^T$. The formal solution of this equation is given by

$$\hat{\xi}(\zeta) = \mathbf{S}(\zeta)\hat{\xi}(0), \quad (26)$$

with $\mathbf{S}(\zeta) = \exp\{\int_0^\zeta \Delta(\zeta')d\zeta'\}$. This is a linear canonical transformation between the input and output quadratures of the four fields which contains the full evolution of our quantum system. Remarkably, Eqs. (24) and (25) provide information about the propagation of the quantum fluctuations of the pumps in each waveguide which, as we will see, are entangled when the system is appropriately set.

Now, a suitable and experimentally accessible observable of the system has to be selected. Since the input states we deal with are Gaussian—actually coherent states—and since we are interested in CV entanglement properties of the four modes interacting within the system, we choose the second-order

moments of the quadrature operators, properly arranged in the covariance matrix \mathbf{V} with elements defined as [32,33]

$$V(\xi_j, \xi_k) = \frac{1}{2}(\langle \Delta\hat{\xi}_j \Delta\hat{\xi}_k \rangle + \langle \Delta\hat{\xi}_k \Delta\hat{\xi}_j \rangle), \quad (27)$$

where $\Delta\hat{\xi} \equiv \hat{\xi} - \langle \hat{\xi} \rangle$. This is a real symmetric matrix that contains all the useful information about the quantum states propagating in the device. The covariance matrix corresponding to the input four-mode state, where the pumps are coherent states and the signals vacuum (spontaneous down conversion) or coherent (optical amplification), is proportional to the identity matrix $\mathbf{V}(0) = (1/2)\mathbf{I}_8$, where $1/2$ corresponds to the shot noise in our notation. Likewise, the covariance matrix at any normalized propagation plane ζ is given by

$$\mathbf{V}(\zeta) = \mathbf{S}(\zeta)\mathbf{V}(0)\mathbf{S}^T(\zeta). \quad (28)$$

The positivity of \mathbf{V} indicates that it is a *bona fide* covariance matrix. This feature is mathematically given by the condition $\mathbf{V}(\zeta) + i\Omega/2 \geq 0$, equivalent to the Heisenberg uncertainty principle, where Ω is the symplectic form given by $\Omega = \bigoplus_{k=1}^4 \sigma$ with $\sigma \equiv \operatorname{adiag}[1, -1]$. We have checked that this condition is satisfied in all the cases studied in this paper.

The covariance matrix can be efficiently measured by means of homodyne detection [33,34]. In these works, the covariance matrices of bipartite systems are considered. Once \mathbf{V} is known, the amount of CV entanglement in bipartite systems is easily quantified through the logarithmic negativity E_N [35]. This entanglement witness is based on the Peres–Horodecki–Simon criterion, which establishes that a quantum state is entangled if the partially transposed density matrix is nonpositive. The logarithmic negativity is obtained from the symplectic spectrum $\{v_k\}_{k=1}^4$ of the partial transpose of the covariance matrix with respect to a subsystem j , \mathbf{V}^{T_j} , computed as the standard eigenvalues spectrum of the matrix $|i\Omega\mathbf{V}^{T_j}|$. E_N is then given by [35]

$$E_N = \sum_{k=1}^4 F(v_k) \text{ with } F(v) = \begin{cases} 0 & \text{for } v \geq 1/2 \\ -\log_2(2v) & \text{for } v < 1/2, \end{cases} \quad (29)$$

such that any value $E_N > 0$ indicates entanglement. Furthermore, in addition to entanglement quantification, this function also presents other appealing properties such as additivity. It also represents an upper bound on the efficiency of distillation, measuring to what extent a quantum state is useful for a certain QIP protocol or the number of resources needed to generate it [35].

In the regime under study, the system is composed of four quantum fields. Then, it would be also interesting to investigate if there is any collection of parameters for which multipartite entanglement is also displayed. The logarithmic negativity is limited to the measurement of bipartite entanglement, where each party is composed by one or more modes, but van Loock and Furusawa introduced a set of conditions to be simultaneously fulfilled for detecting multipartite full inseparability in CV systems [36]. This criterion leads to genuine multipartite entanglement when pure states are considered. In the case of mixed states a more general criterion has been devised [37]. Since we deal with pure states, we use the criterion established in Ref. [36]. Transposed to our four-mode quantum state, genuine quadripartite entanglement is present if the following

inequalities are simultaneously violated:

$$\begin{aligned} \langle [\Delta(\hat{X}_s^A - \hat{X}_p^A)]^2 \rangle + \langle [\Delta(\hat{Y}_s^A + \hat{Y}_p^A + r_3\hat{Y}_s^B + r_4\hat{Y}_p^B)]^2 \rangle &\geq 2, \\ \langle [\Delta(\hat{X}_p^A - \hat{X}_s^B)]^2 \rangle + \langle [\Delta(r_1\hat{Y}_s^A + \hat{Y}_p^A + \hat{Y}_s^B + r_4\hat{Y}_p^B)]^2 \rangle &\geq 2, \\ \langle [\Delta(\hat{X}_s^B - \hat{X}_p^B)]^2 \rangle + \langle [\Delta(r_1\hat{Y}_s^A + r_2\hat{Y}_p^A + \hat{Y}_s^B + \hat{Y}_p^B)]^2 \rangle &\geq 2, \end{aligned} \quad (30)$$

where r_j ($j = 1, \dots, 4$) are arbitrary real parameters used for optimization. These equations can be easily rewritten in terms of the elements of the covariance matrix, being therefore calculated from Eq. (28) [22].

Consequently, from the covariance matrix \mathbf{V} related to the multimode quantum system under study, we can measure the bipartite entanglement and assess if quadripartite entanglement is present with the tools introduced above.

V. NUMERICAL RESULTS AND DISCUSSION

We now present the numerical results obtained for a specific example with the theory developed in Sec. IV. We display a detailed study of the CV bipartite entanglement in two regimes: negligible depletion (or undepletion) and high depletion, and quantify it by means of the logarithmic negativity given by Eq. (29). Then we assess the potential of producing quadripartite entanglement with this device by means of the van Loock and Furusawa inequalities given by Eqs. (30), and finally we present a discussion about the feasibility and possible applications of the investigated device with current technology.

As device substrate we choose lithium niobate due to its appealing properties such as low propagation losses, large conversion efficiencies, and broad bandwidth when operated in continuous wave [9]. The conversion efficiency of the nonlinear process can be maximized mainly in two ways: First, the input pumps are set as strong coherent fields generating type-0 down conversion, which couples pump and signal modes polarized along the extraordinary (e) axis of the crystal through the higher component of the second-order nonlinear tensor ($d_{33} \approx 34$ pm/V). Second, a periodically poled grating is engineered along the substrate (PPLN) to obtain quasi phase matching of the propagation constants. The periodicity of the grating Λ is such that the mismatch Δk is compensated by the wave vector associated with the grating $\Delta k \approx 2\pi/\Lambda$. Perfect or partial quasi phase matching has been considered at the quantum level in different theoretical articles, where it was demonstrated that the linearization procedure also applies for PPLN [38,39]. Typical PPLN chip lengths are on the order of a few centimeters.

We have solved numerically the classical mean value equations (16)–(19), the quantum quadrature equations (22)–(25), and the propagation of the covariance matrix (28) for the following waveguide parameters: $C = 8 \times 10^{-2} \text{ mm}^{-1}$ and $g = 25 \times 10^{-4} \text{ mm}^{-1} \text{ mW}^{-1/2}$, which will be used in the remainder of the article. These are standard values in PPLN waveguides [9,12]. Particularly, the value of linear coupling is chosen sufficiently low in order to obtain large nonlinear effects in the considered PPLN lengths, but similar results are obtained with other realistic values of C and g .

A. Bipartite entanglement

In Figs. 2 and 3 we show, respectively, for the negligible and high depleted-pump-field regimes, the dimensionless classical powers for each mode in each waveguide along the propagation [Figs. 2(a) and 3(a)], relevant elements of the correlation matrix, $V(X_{(s,p)}^A, X_{(s,p)}^B)$ and $V(Y_{(s,p)}^A, Y_{(s,p)}^B)$ (see Appendix), related to the signal (s) and pump (p) subsystems [Figs. 2(b) and 3(b) and 2(c) and 3(c), respectively], and the logarithmic negativity $E_{\mathcal{N}}$ corresponding to the signals and pumps which quantifies bipartite entanglement [Figs. 2(d) and 3(d)]. The following effective coupling is chosen to be $\kappa = 1.13$, in such a way that $\zeta = 1$ stands for $z \approx 14$ mm. Likewise, the ratio between the signal and pump powers at each waveguide is taken as $P_s/P_p = 10^{-20}$ for the negligible-depletion case [40] and as $P_s/P_p = 1$ for the high-depletion case, fixing δ_0 for each case.

1. Negligible depletion

In the negligible depleted-pump-field regime (Fig. 2), the classical dimensionless signal powers $u_s^2 = v_s^2$ [Fig. 2(a)], the covariances of the signal waves [Fig. 2(b)] and the logarithmic negativity related to the signal modes [Fig. 2(d)], are driven by the difference of classical phases $\Delta\theta$ ($\Delta\phi$), as expected from the simplified calculation of Sec. III, where vertical lines at each subfigure stand for multiples of π . In more detail, coming back to Eqs. (16)–(19) it is observed that, due to the symmetry of the system, $\theta(\zeta) = \phi(\zeta)$ (numerically checked), so that the amplitudes depend only on the phase difference $\Delta\theta(\zeta)$ but not on the single phases. It is remarkable that the classical amplitudes are periodic oscillatory functions with $\Delta\theta$ independent of the depletion, as shown in Figs. 2(a) and 3(a).

Besides, by closely inspecting Eqs. (22)–(25), it can be easily seen that, for $u_s(\zeta) \ll u_p(\zeta)$ (equivalent to getting rid of the pump fluctuations), $u_p(\zeta) \approx 1/\sqrt{2}$ and $\theta_p(\zeta) \approx 0$. Under this approximation, the solutions for the propagation of the signal quadratures lead to analytical expressions equivalent to Eqs. (6) and (7) of Sec. III. Then, the elements of the covariance matrix under scrutiny for the signals read

$$V(X_s^A, X_s^B) = -V(Y_s^A, Y_s^B) \approx -\frac{2^{3/2}\kappa\tilde{L}_{ab}^2}{\pi^2} \sin^2\left(\frac{\pi\zeta}{2\tilde{L}_{ab}}\right), \quad (31)$$

with $\tilde{L}_{ab} = \pi/\{2[\kappa^2 - (1/2)]^{1/2}\}$ being the normalized beat length. This equation shows the strong anticorrelations appearing in the signals subsystem. For SPDC, $\zeta/\tilde{L}_{ab} \approx z/L_{ab}$, such that the covariances present the same periodicity as that corresponding to the number of photons given by Eq. (8).

Conversely, the covariance matrix elements related to the pump fields are very low, leading to negligible values of entanglement, and do not show the same periodicity [Fig. 2(c)]. These values are zero in the approximation introduced above [$u_s(\zeta) \ll u_p(\zeta)$], as well as by using the theory of Sec. III. To get the required accuracy on the covariance matrix elements (10^{-19}), the fluctuations of the pump fields have to be taken into account, allowing only a numerical solution of Eqs. (22)–(25). Both the delay and different periodicity shown in Fig. 2(c) are due to the inclusion of the pump fluctuations (\hat{X}_p, \hat{Y}_p) in the calculation.

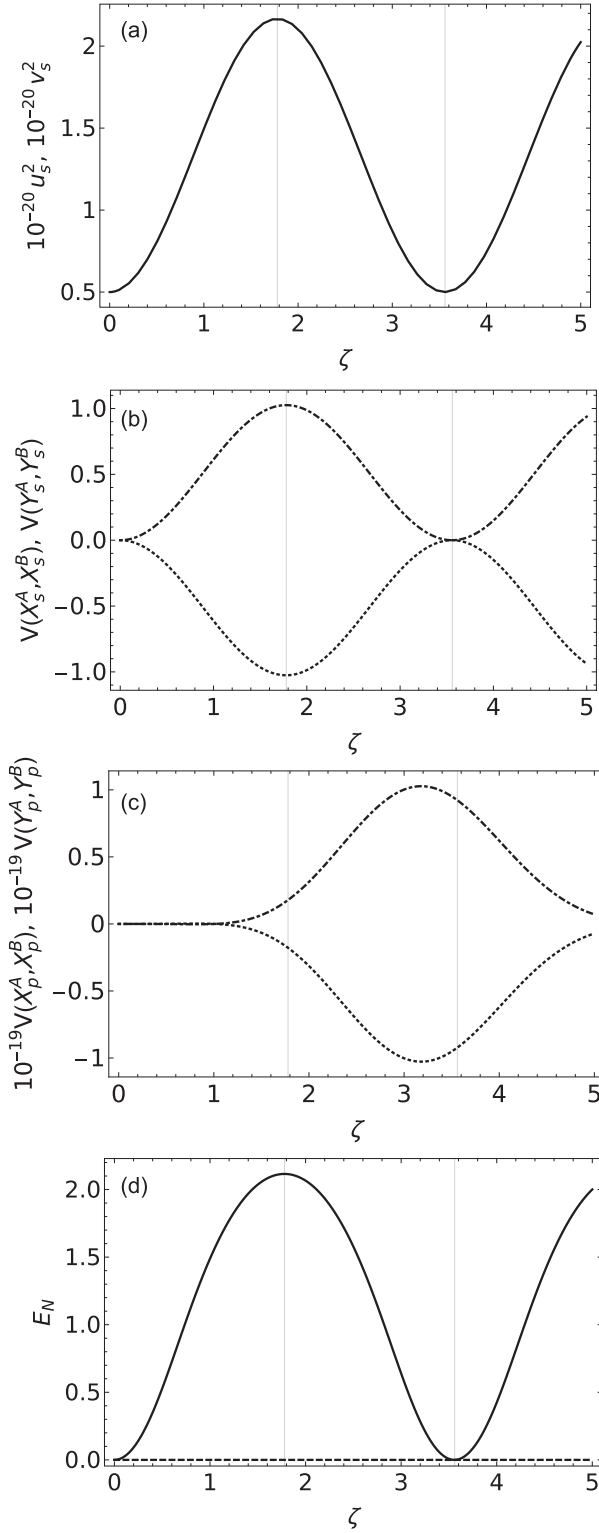


FIG. 2. Negligible depleted-pump-field regime. From top to bottom: (a) dimensionless signal powers (solid), the elements of the correlation matrix $V(X^A, X^B)$ (dotted), $V(Y^A, Y^B)$ (dot-dashed) for the (b) signal and (c) pump fields, respectively, and (d) the logarithmic negativity $E_{\mathcal{N}}$ corresponding to the subsystem of signals (solid) and pumps (dashed), respectively. $\kappa = 1.13$ and $P_s/P_p = 10^{-20}$. ζ is the normalized propagation coordinate. The vertical lines show the planes where the phase difference of the classical waves $\Delta\theta$ ($\Delta\phi$) are multiples of π .

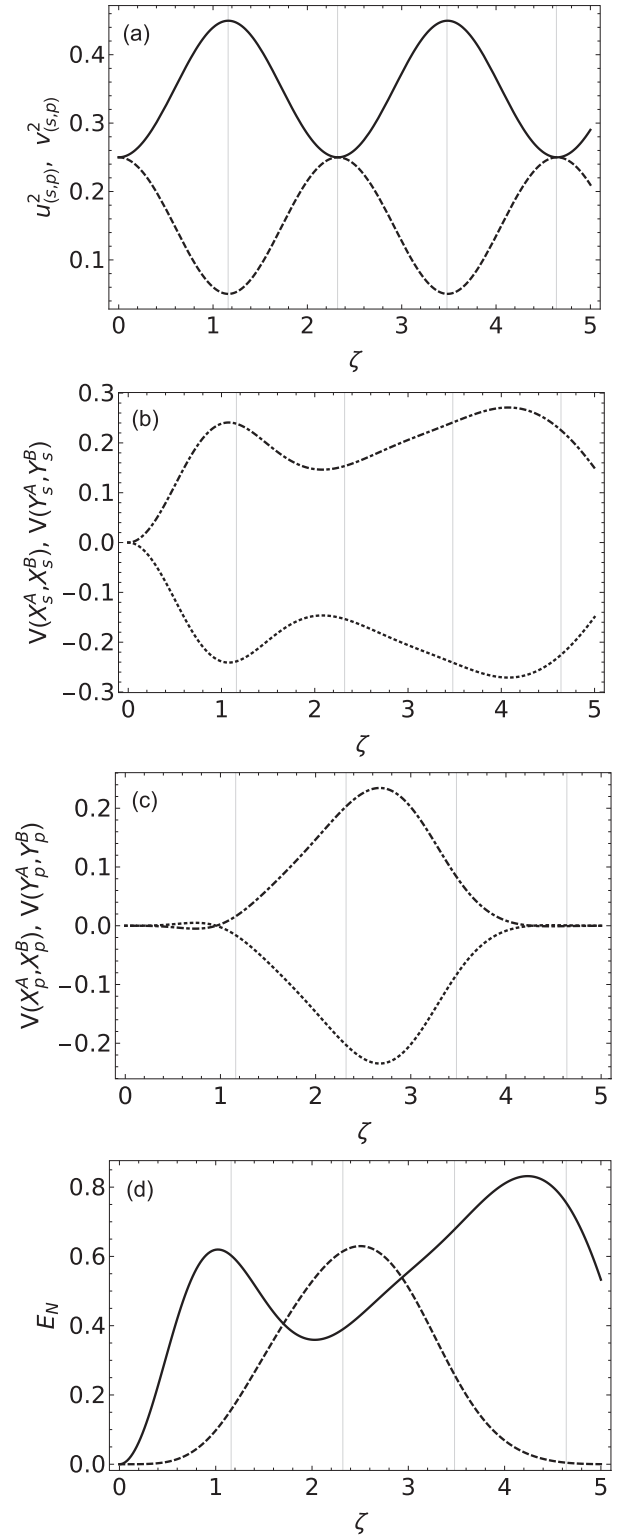


FIG. 3. Highly-depleted-pump-field regime. From top to bottom: (a) dimensionless signal (solid) and pump (dash) powers, the elements of the correlation matrix $V(X^A, X^B)$ (dotted), $V(Y^A, Y^B)$ (dot-dashed) for the (b) signal and (c) pump fields, respectively, and (d) the logarithmic negativity $E_{\mathcal{N}}$ corresponding to the subsystem of signals (solid) and pumps (dash), respectively. $\kappa = 1.13$ and $P_s/P_p = 1$. ζ is the normalized propagation coordinate. The vertical lines show the planes where the phase difference of the classical waves $\Delta\theta$ ($\Delta\phi$) are multiples of π .

Finally, notable values of logarithmic negativity for the signal as high as 2 can be obtained in devices of less than 30 mm [21].

2. High depletion

Let us now analyze the highly-depleted-pump-field regime (Fig. 3). The classical dimensionless powers oscillate periodically above (u_s^2, v_s^2) and below (u_p^2, v_p^2) the normalized initial value $1/4$ [Fig. 3(a)]. Moreover, the signal covariances decrease as the pump covariances increase, reaching measurable values $V(\xi_j, \xi_k) > 10^{-3}$ [Figs. 3(b) and 3(c)] [33,34]. This rise brings as a consequence the entanglement of both the signal and, remarkably, the entanglement of the pump modes which have not directly interacted [Fig. 3(d)]. Actually, the seeding of the signal waves acts as an entanglement switch for the pump fields. It is important to outline that there are distances where values of entanglement as high as $1/2$ are found in both signal and pump subsystems. These values are on the order of those reported with schemes of optical cavities [21]. A very interesting consequence of this effect is that the measurement of entanglement on one subsystem, signal or pump field, can be used as a nonperturbative measure of entanglement on the other.

It should be noted that in this regime, the quantum features of the system are no longer fully driven by the classical phase difference $\Delta\theta$: a depletion-based phase mismatch drives the entanglement evolution (Fig. 3). In this case the relation of the cascade effect with the correlations gets more complex. Since both the pump and signal amplitudes and phases are of the same order, the auto-interaction of the signal quadratures given by the $u_p \sin(\theta_p)$ factors in Eqs. (22) and (23) are no longer negligible. They are at the origin of the phase mismatch between classical and quantum propagation shown in Fig. 3. Furthermore, note that, unlike the signals subsystem, where $V(X_s^A, X_s^B)$ and $V(Y_s^A, Y_s^B)$ are the only relevant elements, in the pumps subsystem other elements like $V(X_p^A, Y_p^B)$ and $V(Y_p^A, X_p^B)$ (not shown) yield also non-negligible correlations which $E_{\mathcal{N}}$ also takes into account.

Let us now analyze more deeply the entanglement of the pump modes. Figure 4(a) shows the values of logarithmic negativity $E_{\mathcal{N}}$ versus propagation for different ratio of signal-pump input powers P_s/P_p and a fixed value of the effective coupling κ . All the curves display a maximum close to $\zeta = 3$. However, the global maximum is obtained for $P_s/P_p = 1/4$, which optimizes the pump depletion. It is important to outline that the entanglement is obtained over a broad range of values of ζ , in such a way that the length of the coupler is not a critical parameter. This is a fact of great practical importance when designing a sample. In Fig. 4(b), we set $P_s/P_p = 1/4$ and the different curves are obtained for different values of κ , i.e., the total power per waveguide. It is shown that the entanglement negativity follows the input power ($\kappa \rightarrow 1$), reaching its maximum for $\kappa = 1.01$, which corresponds to the largest value of total input power considered here. In other words, for a given nonlinearity g and fixed total input power P , the lower the linear coupling C , the higher the entanglement strength. The reason is that the rate of down or up conversion scales with the $\sqrt{2Pg}$ factor whereas the linear coupling controls the speed of cascading, such that, if the coupling is low, more down-converted signal photons are available to

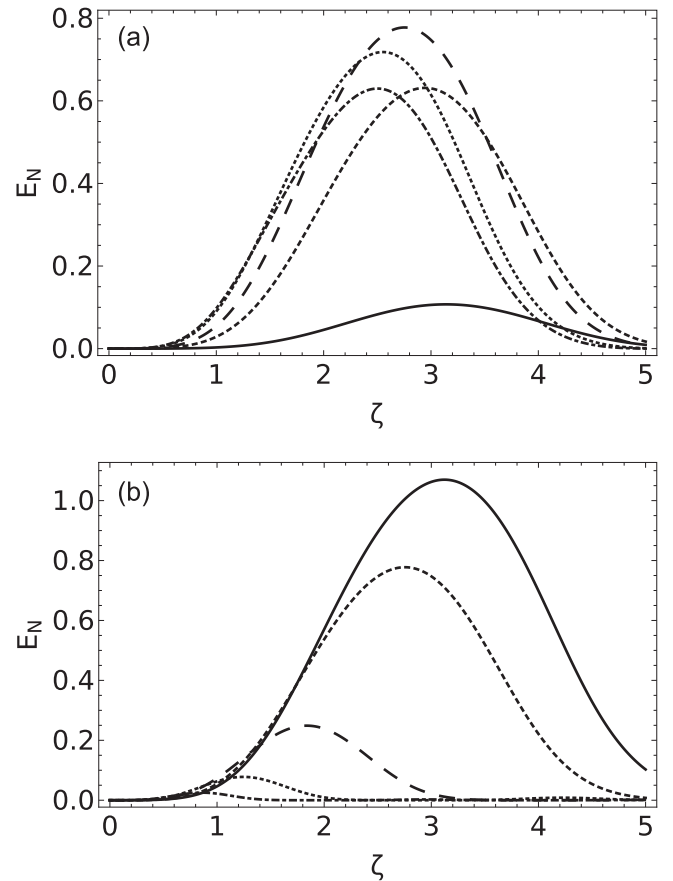


FIG. 4. Pump entanglement for different values of κ and ratio P_s/P_p . (a) Logarithmic negativity $E_{\mathcal{N}}$ for a set value of $\kappa = 1.13$ and different ratios P_s/P_p : $1/10$ (solid), $1/9$ (small-dashed), $1/4$ (large-dashed), $2/3$ (dotted), and 1 (dot-dashed). (b) Logarithmic negativity $E_{\mathcal{N}}$ for a set value of the ratio $P_s/P_p = 1/4$ and different values of κ : 1.01 (solid), 1.13 (small-dashed), 1.6 (large-dash), 2.26 (dotted), and 3.2 (dot-dashed). ζ is the normalized propagation coordinate. Note that, in the panel (b), ζ is different for each plot.

be up converted back to the pump modes and get entangled. Note that the normalized coordinate ζ is different for each plot, due to its dependence on the input power. In terms of physical lengths, the maxima of entanglement are obtained for lengths $z = 34$ mm with $\kappa = 3.2$ (dot-dash) and $z = 39$ mm with $\kappa = 1.01$ (solid), both values being easily attainable in lithium niobate.

B. Quadripartite entanglement

Eventually, we demonstrate the existence of quadripartite entanglement in the system under study. Figure 5 shows the van Loock and Furusawa inequalities given by Eqs. (30) for an effective coupling $\kappa = 2.26$ and $P_s/P_p = 1$. The normalized propagation coordinate is such that $\zeta = 1$ is equivalent to $z \approx 28$ mm. The parameters r_j ($j = 1, \dots, 4$) have been optimized to maximize the violation of Eqs. (30). Due to the symmetry of the system, the violation of the first and third inequalities are equal (solid). Notably, there are lengths over which all the inequalities are violated, therefore showing two-color quadripartite entanglement within the system (Fig. 5, gray area). Note that the degree of violation of the three inequalities

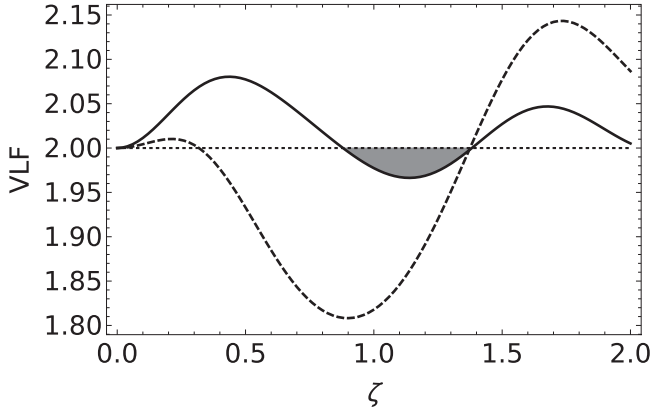


FIG. 5. Optimized van Loock–Furusawa inequalities. Simultaneous values under the threshold value imply CV quadripartite entanglement. Solid line shows the first and third inequalities. Dashed line shows the second inequality. Dotted line shows quadripartite entanglement threshold. In gray the area where the violation of the three inequalities is obtained. $\kappa = 2.26$ and $P_s/P_p = 1$. ζ is the normalized propagation coordinate.

is much lower than that obtained in cavities [22]. However, as highlighted above, our nonlinear directional coupler does not couple the pump fields through the evanescent coupling, so the mere appearance of this effect, even though weak, is outstanding. Furthermore, it is obtained for relatively low input powers and interaction lengths of about 30 mm.

C. Feasibility and applications

We would like to outline that losses could be straightforwardly included in our analysis by following Ref. [28], where it was shown that entanglement is maintained even under strong damping. Typical values of losses in PPLN waveguides are on the order of 0.1 dB cm^{-1} [9] which, under the assumptions of Ref. [28], would lead to a reduction of approximately 6% (12%) in the 2-wave (4-wave) interaction strength after 30 mm of propagation within the device. These figures guarantee the suitable operation of the proposed device with the waveguide parameters C and g considered above. Further optimization of these parameters for a requested operation mode, leading to either bipartite or quadripartite entanglement, can maximize the performance of the device.

Overall, all the results presented in this article could be obtained for reasonably low pump and seed powers by using already existing technology. Note that strong depletion in the traveling-wave regime usually requires high levels of pump power. Here it is not the case due to two main reasons: First, the depletion can be governed by the ratio of signal-pump input powers P_s/P_p . This property was discussed at the classical level in Ref. [41]. Second, the periodical funneling of the signal waves between the waveguides increases the depletion effect.

As closure, we outline that bright sources of bipartite entanglement can be of primary importance in several CV-based quantum protocols such as quantum communications [42,43], quantum key distribution [44,45], optomechanical entanglement [46], and quantum imaging [47], among others. Multipartite entanglement of bright beams opens up additional interesting avenues, such as multipartite EPR steering [48].

VI. CONCLUSION

We have studied the CV entanglement in a nonlinear $\chi^{(2)}$ directional coupler and have shown that two noninteracting bright pump fields become entangled during the propagation due to a nonlinear cascade effect. Likewise, we found that there are distances where both subsystems, pump and signal modes, show large values of entanglement, in such a way that the measurement of entanglement on one subsystem can be used as a measure of entanglement on the other, without destroying it. Moreover, we have shown that two-color quadripartite entanglement is also present in the system under certain conditions. Finally, we have demonstrated that the device proposed here can be realized with current technology. Consequently, it stands as a good candidate for a source of multicolor and/or multipartite entangled states for complex continuous-variable quantum information processing protocols.

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APPENDIX

To clarify the origin of the covariance matrix elements shown in Figs. 2 and 3, we include here the complete covariance matrix related to our system:

$$\mathbf{V} = \begin{pmatrix} V(X_s^A, X_s^A) & V(X_s^A, Y_s^A) & V(X_s^A, X_p^A) & V(X_s^A, Y_p^A) & \mathbf{V}(X_s^A, X_s^B) & V(X_s^A, Y_s^B) & V(X_s^A, X_p^B) & V(X_s^A, Y_p^B) \\ & V(Y_s^A, Y_s^A) & V(Y_s^A, X_p^A) & V(Y_s^A, Y_p^A) & V(Y_s^A, X_s^B) & \mathbf{V}(Y_s^A, Y_s^B) & V(Y_s^A, X_p^B) & V(Y_s^A, Y_p^B) \\ & & V(X_p^A, X_p^A) & V(X_p^A, Y_p^A) & V(X_p^A, X_s^B) & V(X_p^A, Y_s^B) & \mathbf{V}(X_p^A, X_p^B) & V(X_p^A, Y_p^B) \\ & & & V(Y_p^A, Y_p^A) & V(Y_p^A, X_s^B) & V(Y_p^A, Y_s^B) & V(Y_p^A, X_p^B) & \mathbf{V}(Y_p^A, Y_p^B) \\ & & & & V(X_s^B, X_s^B) & V(X_s^B, Y_s^B) & V(X_s^B, X_p^B) & V(X_s^B, Y_p^B) \\ & & & & & V(Y_s^B, Y_s^B) & V(Y_s^B, X_p^B) & V(Y_s^B, Y_p^B) \\ & & & & & & V(X_p^B, X_p^B) & V(X_p^B, Y_p^B) \\ & & & & & & & V(Y_p^B, Y_p^B) \end{pmatrix},$$

with the relevant elements in bold. Repetitive entries have been omitted since $\mathbf{V} = \mathbf{V}^T$.

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