Logarithmic coherence: Operational interpretation of ℓ_1-norm coherence

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We show that the distillable coherence—which is equal to the relative entropy of coherence—is, up to a constant factor, always bounded by the ℓ_1-norm measure of coherence (defined as the sum of absolute values of off diagonals). Thus the latter plays a similar role as logarithmic negativity plays in entanglement theory and this is the best operational interpretation from a resource-theoretic viewpoint. Consequently the two measures are intimately connected to another operational measure, the robustness of coherence. We find also relationships between these measures, which are tight for general states, and the tightest possible for pure and qubit states. For a given robustness, we construct a state having minimum distillable coherence.

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I. INTRODUCTION

A single quantum system, where the notion of entanglement is meaningless, nonetheless, differs in many ways from a classical system. Indeed, the inception of quantum mechanics itself was triggered by phenomena such as interference, wave-particle duality, observed on a single system. In such cases, the figure of merit is attributed to the superposition principle, i.e., the characteristic of quantum mechanics which allows superposition of prefixed basis states as a valid state. This coherence, one of the fundamental reasons for many counterintuitive features of quantum mechanics, allows precise description at mesoscopic scales. In general, coherence is an important physical resource in single-particle interferometry [1–3], quantum thermodynamics [4–10], spin systems [11,12], nanoscience [13–15], quantum algorithms [16–18], and even some biomolecular processes [19–23]. With such an ample usefulness, it is desirable to have a modern resource-theoretic approach to coherence. Recently, one such framework has been put forward [24,25], which has been subsequently developed [26] and advanced further [17,27–33]. For many other models, applications, and further details of coherence theory see the review in Ref. [34].

Undoubtedly, the monotones are an important aspect of any resource theory. On one hand, they certify impossibility of converting resources, while on the other they induce a partial order among the resource states. In the framework proposed in Ref. [25], among the most interesting coherence monotones are the ℓ_1-norm-based coherence (C_{ℓ_1}) [25], the relative entropy of coherence (C_r) [25], and the robustness of coherence (C_R) [35,36], which are formally defined as follows:

$$
\begin{aligned}
C_{\ell_1}(\rho) &:= \sum_{i\neq j} |\rho_{i,j}|,\\
C_r(\rho) &:= \min_{\delta\in\mathscr{I}} S(\rho\|\delta) = S(\rho\|\operatorname{diag}(\rho)) = H(d) - H(\lambda),\\
C_R(\rho) &:= \min_{\sigma}\left\{ s\geqslant 0 \,\middle|\, \frac{\rho+s\,\sigma}{1+s}\in\mathscr{I}\right\}\\
&= \min_{\tau\in\mathscr{I}}\{s\geqslant 0|\,\rho\leqslant(1+s)\tau\},
\end{aligned}
\tag{1}
$$

where $\mathscr{I}$ is the set of incoherent states (i.e., states which are diagonal with respect to the chosen basis), $S(x\|y) = \operatorname{Tr}[x(\log_2 x - \log_2 y)]$ is the relative entropy, d and λ are the vectors of diagonal elements and eigenvalues of ρ, respectively, and $H(p) = -\sum p_i \log_2(p_i)$ is the Shannon entropy of p. Both C_r and C_R have exact analogs in entanglement theory, both are operational quantities, and have direct physical significance. In contrast, C_{ℓ_1} is peculiar in the sense that it has no explicit prominent role in any other known resource theory so far (entanglement, nonlocality, discord, purity, etc.), presumably due to its explicit dependence on the chosen basis. However, C_{ℓ_1} captures the simple intuitive idea that on the level of density matrix description of quantum mechanical states, superposition corresponds to off-diagonal matrix elements (always with respect to the selected basis). In fact, the ℓ_1 norm has been used in a necessary condition for separability known as computable-cross-norm criterion [37], and in quantification of a discordlike quantity named negativity of quantumness [38]. Physically, for instance, C_{ℓ_1} is responsible for the duality between fringe-visibility and which-path information in a two-path interferometer [39]; more generally, it also captures the which-path information about a particle inside a multipath interferometer [40].

Motivated by this evident usefulness of C_{ℓ_1}, in this work, we aim to give it an operational interpretation. Based on the facts that for pure states C_{ℓ_1} is equal (up to a factor of 2) to negativity of the corresponding bipartite pure state, and both measures satisfy strong monotonicity, we have surmised in Ref. [41] that C_{ℓ_1} is analogous to negativity in entanglement theory; we argued that if true, it would be one of the best operational interpretations of C_{ℓ_1}. We show in this work that this is indeed the case. Our primary aim is thus to establish the sharpest possible interrelations between C_r and C_{ℓ_1}. Keeping this in mind, we develop our results in steps, starting from the simplest qubit case, then pure states, and finally general states. Conditions for equality as well as interrelations with other operational monotones (mainly C_R) will be mentioned along the way.

II. QUBIT CASE

Using an inequality between Holevo information and trace norm, it was shown in Ref. [41] that all qubit states satisfy

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$C_r(\rho) \leqslant C_{\ell_1}(\rho)$. Several proofs of this fact will be given throughout this article. However, this is not the sharpest possible interrelation, as strict inequality occurs for almost all states. The following result represents the sharpest interrelations.

Proposition 1. All qubit states ρ with a given coherence $C_{\ell_1}(\rho) = 2b$ satisfy

$$1 - H_2\left(\frac{1-2b}{2}\right) \leqslant C_r(\rho) \leqslant H_2\left(\frac{1-\sqrt{1-4b^2}}{2}\right) \leqslant C_{\ell_1}(\rho), \tag{2}$$

where $H_2(0 \leqslant x \leqslant 1) := -x\,\log_2(x) - (1-x)\log_2(1-x)$ is the binary entropy function. The lower and upper bounds on C_r are saturated by a unique state for each b (up to incoherent unitaries). Equality holds in all the inequalities iff ρ is either an incoherent or a maximally coherent state; otherwise $C_r(\rho) < C_{\ell_1}(\rho)$.

The proof uses convexity of C_r and is given in Appendix A. It also uses the following well-known inequality for binary entropy:

$$2\,\min\{x,1-x\} \leqslant H_2(x) \leqslant 2\sqrt{x(1-x)}, \quad \forall x \in [0,1]. \tag{3}$$

But this inequality alone does not yield even the crude bound $C_r(\rho) \leqslant C_{\ell_1}(\rho)$.

III. PURE STATES

We showed in [41] that all pure states also satisfy $C_r \leqslant C_{\ell_1}$. An independent proof was also given in [42]. We first characterize the equality conditions.

Proposition 2. All pure states satisfy $C_{\ell_1}(\rho) \geqslant C_r(\rho)$. Equality holds iff the diagonal elements are (up to permutation) either $\{1,0,\ldots,0\}$ or $\{1/2,1/2,0,\ldots,0\}$.

Proof. Using the recursive property [43] of entropy function $H(\lambda)$, we have

$$\begin{aligned}C_{\ell_1}\left(|\psi\rangle = \sum_{i=1}^{d}\sqrt{\lambda_i}|i\rangle\right) - C_r(|\psi\rangle) &= 2\sum_{i=1}^{d-1}\sqrt{\lambda_i}\sum_{j=i+1}^{d}\sqrt{\lambda_j} - H(\lambda) \geqslant 2\sum_{i=1}^{d-1}\sqrt{\lambda_i}\sqrt{\sum_{j=i+1}^{d}\lambda_j} - H(\lambda)\\ &= \sum_{i=1}^{d-1}\left[\left(\sum_{k=i}^{d}\lambda_k\right)\left(2\sqrt{\frac{\lambda_i}{\sum_{k=i}^{d}\lambda_k}\left(1-\frac{\lambda_i}{\sum_{k=i}^{d}\lambda_k}\right)} - H_2\left(\frac{\lambda_i}{\sum_{k=i}^{d}\lambda_k}\right)\right)\right].\end{aligned} \tag{4}$$

By inequality (3), each term in the above sum is non-negative, so for vanishing of the sum, each term should vanish. For equality in Eq. (4), only two of the λ_i's could be nonzero. Vanishing of the first term yields $\lambda_1 = 1,0,1/2$. ■

Now we give an upper bound for the difference $C_{\ell_1} - C_r$ and present an alternative proof, arguably the simplest one, for its lower bound.

Proposition 3: For all pure states $|\psi\rangle$ with $\mathrm{rank}[\mathrm{diag}(|\psi\rangle\langle\psi|)] = d > 2$,

$$0 \leqslant C_{\ell_1}(|\psi\rangle) - C_r(|\psi\rangle) \leqslant d - 1 - \log_2 d. \tag{5}$$

The proof uses Schur concavity of $C_{\ell_1}(|\psi\rangle) - C_r(|\psi\rangle)$ in $\mathrm{diag}(|\psi\rangle\langle\psi|)$ and is detailed in Appendix B.

These bounds are rough in the sense that they do not require knowledge of either $C_r(\rho)$ or $C_{\ell_1}(\rho)$. A better bound follows below, whose proof is given in Appendix C.

Proposition 4. For a given ℓ_1-norm coherence $C_{\ell_1}(|\psi\rangle) = b$, $C_r(|\psi\rangle)$ is bounded by

$$\frac{\sqrt{2}b^2}{d(d-1)} \leqslant C_r \leqslant \log_2(1+b), \tag{6}$$

where $d = \mathrm{rank}[\mathrm{diag}(|\psi\rangle\langle\psi|)]$. The lower bound is saturated only for incoherent states while the upper bound is saturated by incoherent and maximally coherent states.

The lower bound indicates that for a given C_{ℓ_1}, the value of C_r probably could be made arbitrarily small for high dimension. In contrast, the upper bound does not depend on the dimension d. Thus, for a fixed $C_{\ell_1} = b$, we cannot increase C_r beyond $\log_2(1+b)$ even by increasing the dimension arbitrarily (but keeping it finite).

This motivates the following question: what could be the sharpest (maximum and minimum) values of C_r given only the knowledge of C_{ℓ_1}? Fortunately, we are able to give the precise answer in the following.

Theorem 5. All pure states $|\psi\rangle$ with a given $C_{\ell_1}(|\psi\rangle) = b$ satisfy

$$\begin{aligned}H_2(\alpha) + (1-\alpha)\log_2(d-1) &\leqslant C_r(|\psi\rangle) \leqslant H_2(\beta) + (1-\beta)\log_2(n-1),\\ \text{where } \alpha &= \frac{2+(d-2)(d-b)+2\sqrt{(b+1)(d-1)(d-1-b)}}{d^2},\\ \beta &= \frac{2+(n-2)(n-b)-2\sqrt{(b+1)(n-1)(n-1-b)}}{n^2},\\ d &= \mathrm{rank}[\mathrm{diag}(|\psi\rangle\langle\psi|)], \quad n = \begin{cases} b+1 & \text{if } b \text{ is integer},\\ [b]+2 & \text{otherwise},\end{cases}\end{aligned} \tag{7}$$

with $[x]$ denoting the integer part of x.

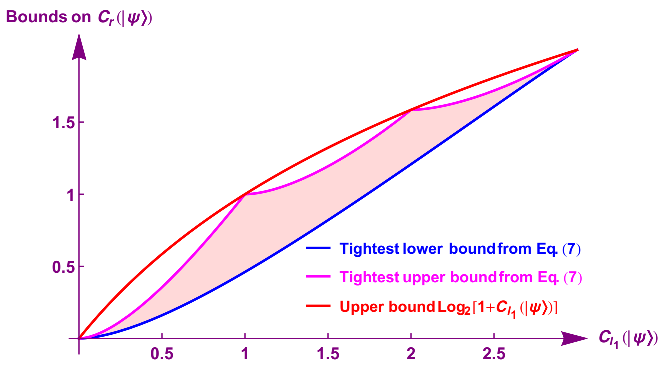

FIG. 1. $C_r(|\psi\rangle)$ vs $C_{\ell_1}(|\psi\rangle)$ for (normalized) $|\psi\rangle \in \mathbb{C}^4$: given only C_{ℓ_1} and d, the bounds in Eq. (7) are the tightest possible. For any point (x,y) inside the pink region (including the boundary curves), there is a $|\psi\rangle$ such that $x = C_{\ell_1}(|\psi\rangle)$ and $y = C_r(|\psi\rangle)$.

Each of the bounds is satisfied by a unique state, up to permutation the diagonal elements of the state with minimum C_r are given by $\{\alpha,(1-\alpha)/(d-1),(1-\alpha)/(d-1),\ldots,(1-\alpha)/(d-1)\}$ and that with maximum C_r are $\{\beta,(1-\beta)/(n-1),(1-\beta)/(n-1),\ldots,(1-\beta)/(n-1)\}$.

The proof is based on Lagrange multipliers, and uses some techniques recently employed in Refs. [44,45]. The complete proof is given in Appendix D. In Fig. 1, we show the several bounds on C_r as a function of C_{ℓ_1}.

We note that for any fixed b, as $d \to \infty$ $\alpha \to 1$ and the lower bound of $C_r \to 0$. Thus, for any fixed value of $C_{\ell_1} = b$, there is a $|\psi\rangle \in \mathbb{C}^d$ (for sufficiently high d) with $C_{\ell_1}(|\psi\rangle) = b$ and arbitrary small $C_r(|\psi\rangle)$. In contrast, we cannot increase C_r beyond the upper bound (which depends on b but is independent of dimension). An explanation is that, given more and more components, the probability could be made more biased but not more uniform than the initial one.

This result for pure states has an interesting aspect: since $C_{\ell_1} = C_R$ for all pure states [[36], Theorem 6], Theorem 5 also gives the sharpest bounds on C_r, for a given robustness $C_R = b$. Note also that unless C_{ℓ_1} has an integral value, no pure state saturates the inequality $C_r \leqslant \log_2(1 + C_{\ell_1})$.

IV. ARBITRARY STATES

As usual, the case of mixed states is more demanding, since in this case C_r depends on the eigenvalues, which are implicit functions of the matrix elements. Another difficulty is that the quantity C_{ℓ_1} is not unitarily invariant. So, we have to resort to different techniques. But, before dealing with general mixed states, let us mention that the result $C_r \leqslant C_{\ell_1}$ holds for the following simple class of states.

Proposition 6. Any pseudopure state of the form $\rho = p|\psi\rangle\langle\psi| + (1-p)\delta$ with $p \in [0,1]$ and $\delta \in \mathcal{I}$ satisfies $C_r(\rho) \leqslant C_{\ell_1}(\rho)$. This gives an alternative proof for the validity of the same relation for any qubit state.

Proof. From the convexity of C_r, we have

$$\begin{aligned} C_r(\rho) &\leqslant pC_r(|\psi\rangle\langle\psi|) \leqslant pC_{\ell_1}(|\psi\rangle\langle\psi|) \\ &= C_{\ell_1}(p|\psi\rangle\langle\psi| + (1-p)\delta) = C_{\ell_1}(\rho). \end{aligned}$$

Since every (mixed) qubit state can be expressed as the above pseudomixture, the result follows. ■

We have seen that for pure states when $C_{\ell_1} > 1$, C_{ℓ_1} is too high compared to C_r and so $\log_2(1 + C_{\ell_1})$ is reasonably a better upper bound for C_r. It turns out that the same upper bound holds also for all (mixed) states.

Theorem 7. For any state ρ,

$$C_r(\rho) \leqslant \log_2[1 + C_R(\rho)] \leqslant \log_2[1 + C_{\ell_1}(\rho)]. \tag{8}$$

The proof uses operator monotonicity of log function and the details are presented in Appendix E. Here we give an alternative proof for $C_r \leqslant \log_2[1 + C_{\ell_1}]$, highlighting the similarity of C_{ℓ_1} with negativity. Recalling that distillable entanglement E_d is upper bounded by the logarithmic negativity [46] and, for any state, coherent information is upper bounded by one-way distillable entanglement $E_\to$ (by the so-called hashing inequality, [[47], Theorem 10]), we get

$$S(\sigma^A) - S(\sigma^{AB}) \leqslant E_\to(\sigma) \leqslant E_d(\sigma) \leqslant \log_2[1 + 2\mathcal{N}(\sigma)]. \tag{9}$$

For any given $\rho = \sum a_{ij}|i\rangle\langle j|$, consider the state $\sigma^{AB} = \sum a_{ij}|ii\rangle\langle jj|$. One immediately verifies that $S(\sigma^A) - S(\sigma^{AB}) = C_r(\rho)$. The eigenvalues of partial transposition of σ^{AB} are a_{ii} for $i = 1,2,\ldots,d$, and $\pm|a_{ij}|$ for $1 \leqslant i < j \leqslant d$ [[48], Lemma 6.3]. Therefore, $2\mathcal{N}(\sigma) = 2\sum_{i<j}|a_{ij}| = C_{\ell_1}(\rho)$. Substituting these values in Eq. (9), we get the desired result. ■

Yet another method to prove the same inequality is to use the monotonicity of sandwiched α-Rényi relative entropy

$$S_\alpha(A\|B) := \frac{1}{\alpha - 1}\log_2 \mathrm{Tr}\left[B^{\frac{1-\alpha}{2\alpha}} A B^{\frac{1-\alpha}{2\alpha}}\right]^\alpha$$

in $\alpha > 0$ [49].

Since $\log_2(1 + x) \leqslant x$ for all $x \geqslant 1$, we have by Eq. (8)

$$C_r(\rho) \leqslant \begin{cases} C_{\ell_1}(\rho), & \text{if } C_{\ell_1}(\rho) \geqslant 1, \\ C_{\ell_1}(\rho)\log_2 e, & \text{if } C_{\ell_1}(\rho) < 1. \end{cases} \tag{10}$$

Thus $C_r \leqslant C_{\ell_1}$ holds for all states, at most up to a multiplicative constant of $1/\ln 2$. Unfortunately, we could not resolve the conjecture $C_r \leqslant C_{\ell_1}$ made in Ref. [41] in full generality. However, employing perturbative techniques, we could prove it when C_{ℓ_1} is very small (see Appendix F). Note, on the other hand, that if $C_r(\rho) \leqslant C_{\ell_1}(\rho)$ is true, then it is the sharpest possible upper bound on C_r when $C_{\ell_1}(\rho) \leqslant 1$.

Proposition 8. For any $0 < b < 1$ and $d \geqslant 3$, there is a d-dimensional state ρ with $C_r(\rho) = C_{\ell_1}(\rho) = b$.

Proof. One such state is given by

$$\rho = \begin{pmatrix} b/2 & b/2 \\ b/2 & b/2 \end{pmatrix} \oplus (1-b)\delta,$$

with any $(d-2)$-dimensional diagonal state $\delta \in \mathcal{I}$. ■

It is desirable to sharpen Eq. (8) to something like Eq. (7). However, we are not aware of any sharper bounds. Our numerical study suggests that, for a given C_{ℓ_1}, the state with max C_r is generally a mixed one, unless we put restriction also on the dimension (it is a pure state, if additionally $d \leqslant [C_{\ell_1}] + 2$).

Nonetheless, we have completely characterized the sharpest lower bound of C_r for a given C_R. The next result guarantees the minimum amount of distillable coherence from a resource state given only the dimension d and C_R.

Theorem 9. All states ρ with a given $C_R(\rho) = b$ satisfy

$$C_r(\rho) \geqslant \log_2 d - H_2(\alpha) - (1-\alpha)\log_2(d-1),$$
$$\text{where } d = \text{rank}[\text{diag}(\rho)] \text{ and } \alpha = \frac{1+b}{d}. \tag{11}$$

Equality occurs for isotropic-like states $\rho = p|\Psi\rangle\langle\Psi| + (1-p)\mathbb{1}/d$, $p = b/(d-1)$, and $|\Psi\rangle$ being the maximally coherent state.

The full proof is presented in Appendix G. Appendix H contains an unsuccessful attempt to prove $C_r \leqslant C_{\ell_1}$ via convex roofs [50,51]. Nevertheless, it could be of independent interest because of its close connection with the convex roof of negativity for maximally correlated states.

V. LOGARITHMIC COHERENCE: A STRONG MONOTONE WHICH IS NOT CONVEX

Similar to logarithmic negativity $E_{\mathcal{N}}$ [46,52], we can define $C_{\log_2}(\rho) := \log_2[1 + C_{\ell_1}(\rho)]$. The addition by 1 not only makes $C_{\log_2} \geqslant 0$, but also yields the additivity under tensor products, $C_{\log_2}(\rho\otimes\sigma) = C_{\log_2}(\rho) + C_{\log_2}(\sigma)$, just like C_r and $E_{\mathcal{N}}$. The strong monotonicity follows easily from that of C_{ℓ_1}, using concavity and monotonicity of the logarithm:

$$\begin{aligned}\sum_i p_i \log_2[1 + C_{\ell_1}(\rho_i)] &\leqslant \log_2\left[\sum_i p_i[1 + C_{\ell_1}(\rho_i)]\right]\\ &= \log_2\left[1 + \sum_i p_i C_{\ell_1}(\rho_i)\right]\\ &\leqslant \log_2[1 + C_{\ell_1}(\rho)],\end{aligned}$$

the last inequality due to strong monotonicity of C_{ℓ_1}. Due to the concavity of log function, however, $C_{\log_2}$ is not convex:

$$C_{\log_2}\left(\tfrac{1}{2}\rho + \tfrac{1}{2}\sigma\right) > \tfrac{1}{2}C_{\log_2}(\rho) + \tfrac{1}{2}C_{\log_2}(\sigma),$$

iff $C_{\ell_1}(\rho)C_{\ell_1}(\sigma)[C_{\ell_1}(\rho) - C_{\ell_1}(\sigma)] \neq 0$. Note that the above arguments show that for any strong monotone C, the logarithmic version $C_{\log_2} = \log_2(1 + C)$ is also a (nonconvex) strong monotone; there is nothing special about C_{ℓ_1}—except that in this case $C_{\log_2}$ is additive under tensor products.

VI. RELEVANCE

The main importance of this work is that it gives operational interpretation to C_{ℓ_1} in a completely quantitative way, namely it is similar to negativity in entanglement theory, and indeed the logarithmic coherence defined here, though not convex, is a better motivated one. The latter plays the exact role of logarithmic negativity in entanglement theory, giving a tight upper bound on distillable resource. Once this is established, we can seamlessly browse all instances of usefulness of (logarithmic) negativity as an entanglement monotone from entanglement theory to coherence theory. For example, Theorem 7 is just a manifestation of known interrelations between relative entropy of entanglement, (logarithmic) negativity, and robustness of entanglement. Thus C_{ℓ_1}, though arguably one of the simplest monotones which has apparently no conspicuous role in entanglement theory, is significant for most relevant operational quantities in coherence theory. Later we will mention relevance of our results beyond a particular resource theory.

In many practical scenarios, the density matrix depends on some parameters (e.g., the entries are functions of time—in time-dependent evolution; temperature, or other relevant parameters—in thermometry or metrology). In such cases, the density matrix cannot be diagonalized and hence C_r becomes uncomputable. The precise bounds given in this work are the best from the knowledge of the entries.

We would also like to mention possible applications of our results to some related fields, namely information theory and matrix analysis. First note that $C_{\ell_1}(|\psi\rangle)$ is the Rényi entropy

$$R_\alpha(\lambda) := \frac{1}{1-\alpha}\log_2\left[\sum_{i=1}^{d}\lambda_i^\alpha\right] = \frac{\alpha}{1-\alpha}\log_2(\|\lambda\|_\alpha),$$

of order $\alpha = 1/2$ in disguise. Thus the relation between $C_r(|\psi\rangle)$ and $C_{\ell_1}(|\psi\rangle)$ is actually optimizing R_α ($\alpha \to 1$) subject to the given fixed value of $R_{1/2}$. The upper bound in Eq. (6) is just a consequence of nonincreasing property of R_α. The optimization technique employed in Appendix D could also be applicable to other values of α. Indeed, it is easy to find sharpest bounds on $R_{\alpha\to 1}$ subjected to a fixed R_2, which reproduce the result from [53].

Lastly, finding trade-off relations between diagonals, eigenvalues, singular values, etc., are standard problems in matrix analysis [[54], Ch. 9]. Our main quest here was a small part, finding exact trade-off between diagonals and eigenvalues (via entropy function), having the knowledge of the sum of absolute values of the entries. One such independent relation is Eq. (11) (it is worth mentioning that the same matrix maximizes the determinant [55], a log-concave function). More precisely, our problem is exactly similar to finding sharpest Fannes-Audenaert bound [56,57] for a single state and our results are independent of similar bounds [58,59].

VII. DISCUSSION AND CONCLUSION

We have shown that in the coherence theory [25], C_{ℓ_1} operationally plays the exact role of negativity in entanglement theory. Since there is no bound coherence [26] (analogous to no bound entanglement in maximally correlated states), C_{ℓ_1} is intimately connected to any operationally relevant quantity or process. For example, the sharpest bounds on C_r from Theorem 5 remain the same even if we replace C_{ℓ_1} by C_R. Thus our approach here supports the idea that coherence theory is a subclass of entanglement theory for maximally correlated states. Nonetheless, similar to entanglement theory, we showed that the requirement of convexity, although a desirable property, should be relaxed for coherence monotones.

Given their similar operational meaning, it would be interesting to compare Eq. (7) with its entanglement-analog $E_d = \log_2[1 + 2\mathcal{N}]$, especially since in contrast to $C_d = C_r$, E_d is a noncomputable quantity. Note that for the NPT bound entangled states [proof of whose (non)existence is an open problem in quantum information theory, with all conjectures in literature claiming the existence [60]], the bound on E_d is worst as it gives absolutely no information. However, the relation for C_d always gives some nonzero bound, thereby the inequality has more to offer in coherence theory. From quantitative perspectives, both bounds are quite rough as almost all the states never achieve equality. Our results in Theorem 5 and Theorem 9 are the best possible in this regard, as they give the optimal bound on one quantity from the knowledge of the other.

It is worth remarking that the relation $C_{\ell_1} \geqslant C_r$ does not hold for normalized quantities. The normalized quantities, being dimension dependent, need not be monotone. Also, C_r and C_{ℓ_1} do not give the same ordering of state space. For example, there are states ρ and σ such that $C_{\ell_1}(\rho) > C_{\ell_1}(\sigma) > C_r(\sigma) > C_r(\rho)$.

Before concluding, we would like to mention that over the past two years many alternative frameworks of coherence theory have been proposed [61–65], stemming mainly from different notions of incoherent (free) operations. In some of these models, C_{ℓ_1} is not a monotone and arguably there is no maximally coherent state [[34], Table II], thereby lacking the interpretation of C_r as distillable coherence. However, both C_r and C_R are not only monotones, but also operational quantities even in the most general (reversible) resource theory [66,67]. Most of our results, as could also be seen as relations between C_r and C_R, are thus applicable to more general scenarios. Pertinent to coherence, the most general framework by Åberg [24], where incoherent states are block-diagonal of any block size, allows an interrelation analogous to Eq. (8); we have to replace C_{ℓ_1} by the sum of trace norm of all off-diagonal blocks [49].

Note added. The operational interpretation of C_{ℓ_1} presented in this work has been complemented in Ref. [68].

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APPENDIX A: PROOF OF PROPOSITION 1

Proof. Without loss of generality, let

$$\rho = \begin{pmatrix} a & b \\ b & 1-a \end{pmatrix}$$

be a state with given ℓ_1-norm coherence $2b > 0$. For positivity of ρ, we must have

$$\frac{1-\sqrt{1-4b^2}}{2} \leqslant a \leqslant \frac{1+\sqrt{1-4b^2}}{2}, \quad 0 < b \leqslant \frac{1}{2}. \tag{A1}$$

We will now show that for fixed b, $C_r(\rho) := H_2(a) - H_2(\lambda)$ is a convex function of a over the entire region (A1). To this end, the double derivative of C_r with respect to a is given by

$$\frac{b^2\left[\sqrt{1-4(a(1-a)-b^2)}(8a^2-8a+4b^2+1)+8a(1-a)(a(1-a)-b^2)\ln\left(\frac{1+\sqrt{1-4(a(1-a)-b^2)}}{1-\sqrt{1-4(a(1-a)-b^2)}}\right)\right]}{a(1-a)[a(1-a)-b^2][1-4(a(1-a)-b^2)]^{3/2}\ln 2}.$$

Applying the inequality $\ln[(1+x)/(1-x)] \geqslant 2x$ for $x \in [0,1]$, the numerator is bounded below by $b^2(1-2a)^2[1-4(a(1-a)-b^2)]^{3/2}$, a non-negative quantity. Therefore, $C_r(\rho)$ is convex and hence the maximum value will be attained at the extreme values of a (and the corresponding state is a pure state). Thus for a given fixed b, we have

$$C_r(\rho) \leqslant H_2\left(\frac{1-\sqrt{1-4b^2}}{2}\right). \tag{A2}$$

The upper bound on $H_2(x)$ from Eq. (3) gives the rightmost inequality of Eq. (2),

$$C_r(\rho) \leqslant H_2\left(\frac{1-\sqrt{1-4b^2}}{2}\right) \leqslant 2b = C_{\ell_1}(\rho). \tag{A3}$$

Note that for any given b there is a ρ (indeed a pure state) such that equality occurs in the first inequality of Eq. (A3), while except for incoherent states and maximally coherent states, the last inequality is always strict.

The expression of C_r remains unchanged if we interchange a and $(1-a)$, i.e., C_r is symmetric about $a = 1/2$. Also, from Eq. (A1) the allowed range of a is symmetric about $a = 1/2$. Therefore, C_r being a symmetric convex function would have a

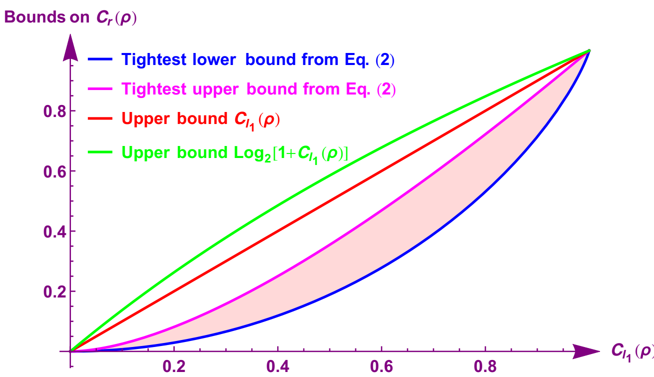

FIG. 2. $C_r(\rho)$ vs $C_{\ell_1}(\rho)$ for qubit ρ: the bounds given by Eq. (2) are the tightest possible. For any point (x,y) inside the pink region (or over the boundary curves), there is a qubit state ρ such that $x = C_{\ell_1}(\rho)$ and $y = C_r(\rho)$. Note that for a given C_{ℓ_1}, there is a unique pure state, whose C_r is given by the tightest upper bound (the magenta colored curve).

unique global minimum at $a = 1/2$. Hence the first inequality in Eq. (2). All the bounds for qubit systems are depicted in Fig. 2. ■

APPENDIX B: PROOF OF PROPOSITION 3

Proof. Without loss of generality, let $|\psi\rangle := \sum_{i=1}^{d} \sqrt{\lambda_i}|i\rangle$, with $\lambda_i > 0$ and $\sum \lambda_i = 1$. We will now show that the function

$$f(\lambda) := C_{\ell_1}(|\psi\rangle) - C_r(|\psi\rangle) = \left(\sum_{i=1}^{d}\sqrt{\lambda_i}\right)^2 - 1 + \sum_{i=1}^{d}\lambda_i \log_2 \lambda_i$$

is Schur concave in λ, which will complete the proof. One verifies that

$$\frac{\partial f}{\partial \lambda_1} - \frac{\partial f}{\partial \lambda_2} = \frac{(\sqrt{\lambda_2}-\sqrt{\lambda_1})}{\sqrt{\lambda_1\lambda_2}}\left(\sum_{i=1}^{d}\sqrt{\lambda_i}\right) + \log_2\left(\frac{\lambda_1}{\lambda_2}\right)$$
$$= (\sqrt{\lambda_2}-\sqrt{\lambda_1})\left[\frac{\sum_{i=1}^{d}\sqrt{\lambda_i}}{\sqrt{\lambda_1\lambda_2}} - \frac{\log_2\left(\sqrt{\frac{\lambda_1}{\lambda_2}}\right)}{2(\sqrt{\lambda_1}-\sqrt{\lambda_2})}\right].$$

Thus it suffices to show that the quantity inside the brackets is non-negative. Using the geometric-logarithmic-mean inequality [[54], p. 141], we get

$$-\frac{\log_2\left(\sqrt{\frac{\lambda_1}{\lambda_2}}\right)}{2(\sqrt{\lambda_1}-\sqrt{\lambda_2})} \geqslant -\frac{1}{2\,\ln 2\,(\lambda_1\lambda_2)^{1/4}} > -\frac{1}{(\lambda_1\lambda_2)^{1/4}},$$

and hence the quantity inside the brackets is non-negative. ■

A sufficient (but not necessary) condition for a Schur-concave function ϕ to satisfy $\phi(x) > \phi(y)$ whenever $x \preceq y$ and y is not a permutation of x is that ϕ is strictly Schur concave [[54], p. 83]. Although both $C_{\ell_1}(\lambda)$ and $C_r(\lambda)$ are strictly Schur concave, $f(\lambda)$ is not. This makes it difficult to characterize the equality conditions in Eq. (5). Nonetheless, saturation of the lower bound has been fully characterized in Proposition 2. It is tempting to think that the upper bound will be saturated only by maximally coherent states if $d > 2$. Although it could be true for $d \geqslant 4$, there are many λ's giving the same maximum of f, with $\lambda = (2/3,1/6,1/6)$ being an example for $d = 3$.

APPENDIX C: PROOF OF PROPOSITION 4

Using the inequality

$$-x\,\log_2 x \geqslant \sqrt{2}x(1-x) \quad \forall x \in [0,1], \tag{C1}$$

we get

$$\begin{aligned} C_r(|\psi\rangle) &= \sum_{i=1}^{d} -\lambda_i \log_2 \lambda_i \\ &\geqslant \sqrt{2}\left[1-\sum_{i=1}^{d}\lambda_i^2\right] = \sqrt{2}\sum_{i\neq j}\lambda_i\lambda_j \\ &\geqslant \frac{\sqrt{2}}{d(d-1)}\left(\sum_{i\neq j}\sqrt{\lambda_i\lambda_j}\right)^2 \\ &= \frac{\sqrt{2}b^2}{d(d-1)}, \end{aligned}$$

where, in the last inequality, we have used the fact that, for a d-dimensional vector x and for $0 < p < q < \infty$, $\|x\|_p \leqslant d^{1/p-1/q}\|x\|_q$.

One weakness of this bound is that equality holds for incoherent states only. A lower bound on C_r, which is saturated by all incoherent and maximally coherent states, can also be derived easily. For example, using the following bound on entropy [69]:

$$H(\lambda) \geqslant \log_2 d - \frac{1}{\ln 2}\left[d\left(\sum_{i=1}^{d}\lambda_i^2\right) - 1\right],$$

we get

$$C_r \geqslant \log_2 d - \frac{(d-1)^2 - b^2}{(d-1)\ln 2}. \tag{C2}$$

Note that this lower bound is useful only when $b > \sqrt{(d-1)[(d-1)-\ln d]}$.

Now, to get the upper bound, we use concavity of logarithm,

$$\begin{aligned} C_r(|\psi\rangle) = H(\lambda) &= 2\sum_{i=1}^{d}\lambda_i \log_2(1/\sqrt{\lambda_i}) \\ &\leqslant \log_2\left[\left(\sum_{i=1}^{d}\sqrt{\lambda_i}\right)^2\right] \\ &= \log_2(1+b). \end{aligned}$$

To prove inequality (C1), note that, for $x \in (0,1)$, $\ln x = \ln[1-(1-x)] = -(1-x) - (1-x)^2/2 - \cdots \leqslant -(1-x)$. Multiplying by $x/\ln 2$ and noticing that $1/\ln 2 > \sqrt{2}$, the inequality follows. ■

APPENDIX D: PROOF OF THEOREM 5

To prove the bounds we will optimize the entropy function with respect to the two equality constraints. The objective function being continuous, bounded (over the probability simplex Δ_d, for a given dimension d), and the constraints describing compact sets, there is a maximum and a minimum. The optimum points should be either at interior or at the boundary of Δ_d. As $\lambda_i = 0$ neither affects the constraint nor the objective function, if the optimum occurs on the boundary of Δ_d, it should occur in the interior of Δ_n for some $n < d$. So, without loss of generality, we can assume that the optimum occurs in the interior of some Δ_n, and use Lagrange's multiplier method to get the possible stationary points. For simplicity, we can consider the natural-logarithm-based entropy (as it is a constant multiple of the binary-based entropy) and the Lagrange's function is set to be

$$\mathcal{L}(\lambda,\mu,\nu) := -\sum_{i=1}^{n} \lambda_i \ln \lambda_i + \mu\left[\sum_{i=1}^{n} \sqrt{\lambda_i} - \sqrt{1+b}\right] + \nu\left[\sum_{i=1}^{n} \lambda_i - 1\right].$$

Vanishing of the gradient ($\nabla\mathcal{L} = 0$) gives

$$1 + \ln \lambda_i - \frac{\mu}{2\sqrt{\lambda_i}} - \nu = 0. \tag{D1}$$

Solving Eq. (D1) analytically is difficult. Instead, let us show that when seen as an equation in a particular $\lambda_i \in (0,1)$, it can have at most two (non-negative) solutions. The equation can be written as

$$z \ln z = a, \quad z = \sqrt{\lambda_i e^{1-\nu}}, \quad a = \frac{\mu\sqrt{e^{1-\nu}}}{4}.$$

The function $z \ln z$ is strictly convex in $(0,\infty)$ with a unique global minimum at $z = 1/e$. So for any given $a > 0$, the equation $z \ln z = a$ has a unique solution in $(1,\infty)$. However, for any $a \in (-1/e,0)$ there are two solutions, one in $(0,1/e)$ and the other in $(1/e,1)$. Thus overall there are at most two solutions to Eq. (D1) for each $\lambda_i \in (0,1)$. Therefore, all possible stationary points of $\mathcal{L}$ must have two distinct values of λ's, thereby potentially k number of λ_1 and $(n-k)$ number of λ_2 with $0 < \lambda_1 < \lambda_2$, $k = 1,2,\dots,n$, $n \leqslant d$. Note that we could exclude the case $k = n$, as this is the case only when $b = n-1$, so there is only one pure state and no optimization is required. For a (given) finite d, this gives finite number of stationary points. However, as we will show below, we do not need to check all these points. Writing x for λ_1, we get from normalization $\lambda_2 = (1-kx)/(n-k)$ and $x \leqslant 1/n$. Thus the problem becomes

$$\begin{aligned} &\text{Optimize} \quad f(x,k) := -kx \ln x - (1-kx)\ln\left[\frac{1-kx}{n-k}\right] \\ &\text{such that} \quad g(x,k) := k\sqrt{x} + \sqrt{(n-k)(1-kx)} = \sqrt{1+b} \end{aligned}$$

over $0 < x \leqslant 1/n$, $k \in \{1,2,\dots,n-1\}$, $n \leqslant d$. We will now employ the approach from [[44], Lemma 15]. To show that for a fixed $g(x,k)$, $f(x,k)$ is a decreasing function in k, let us temporally remove the integral restriction on k and consider it as a real variable in $[0,n)$. Due to the constraint on $g(x,k)$, changing k will also change x. So, to keep $g(x,k)$ fixed $(= \sqrt{1+b})$, let $x(k)$ be the function of k implicitly given by $g(x(k),k) = \sqrt{1+b}$. Then $\frac{dg}{dk} = 0 = \frac{\partial g}{\partial k} + \frac{\partial g}{\partial x}\cdot\frac{\partial x}{\partial k}$ gives

$$\frac{\partial x}{\partial k} = -\frac{\partial g}{\partial k}\Big/\frac{\partial g}{\partial x} = \frac{x}{k}\left[\sqrt{\frac{1-kx}{(n-k)x}} - 1\right].$$

Therefore,

$$\begin{aligned} \frac{df}{dk} &= \frac{\partial f}{\partial k} + \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial k} \\ &= -\frac{1-nx}{n-k} + x\sqrt{\frac{1-kx}{(n-k)x}} \ln\left[\frac{1-kx}{(n-k)x}\right] \\ &\leqslant -\frac{1-nx}{n-k} + x\left[\frac{1-kx}{(n-k)x} - 1\right] \\ &= 0, \end{aligned}$$

where in the inequality we have used the fact that $\sqrt{y}\ln y \leqslant y-1$ for all $y \geqslant 1$. Thus $f(x,k)$ is a decreasing function of k, and the minimum of f is obtained for $k = n-1$. Finally, a global optimization over $n \leqslant d$ is required. Similar to the above method, we will show that for a fixed $g(n-1,x(n))$, $f(n-1,x(n))$ is a decreasing function in n. Solving $0 = \frac{dg}{dn} = \frac{\partial g}{\partial n} + \frac{\partial g}{\partial x}\cdot\frac{\partial x}{\partial n}$,

$$\frac{\partial x}{\partial n} = \frac{x(\sqrt{x} - 2\sqrt{1+x-nx})}{(n-1)(\sqrt{1+x-nx} - \sqrt{x})}.$$

Substituting into $\frac{df}{dn}$, we get

$$\begin{aligned} \frac{df}{dn} &= \frac{x\left[\sqrt{1+x-nx} - \sqrt{x} + \sqrt{1+x-nx}\ln\left(\frac{x}{1+x-nx}\right)\right]}{\sqrt{1+x-nx} - \sqrt{x}} \leqslant \frac{x\left[\sqrt{1+x-nx} - \sqrt{x} + 2\sqrt{1+x-nx}\left(\sqrt{\frac{x}{1+x-nx}} - 1\right)\right]}{\sqrt{1+x-nx} - \sqrt{x}} \\ &= -x, \end{aligned}$$

where in the inequality we have used the fact that $\ln y \leqslant y-1$ for all $y > 0$. Thus f is a decreasing function of the dimension d, and hence the minimum of f is attained at $n = d$. The minimum is obtained at λ all of whose $(d-1)$ components are equal and the rest one is at least $1/d$. Assuming this larger component to be α, the unique $\alpha \geqslant 1/d$ is obtained from the constraint $g(\alpha,d) = \sqrt{1+b}$ as

$$\alpha = \frac{2 + (d-2)(d-b) + 2\sqrt{(b+1)(d-1)(d-1-b)}}{d^2}.$$

For easy verification of the constraint, we note that

$$\sqrt{\alpha} = \frac{\sqrt{(d-1)(d-1-b)} + \sqrt{b+1}}{d},$$
$$\sqrt{1-\alpha} = \frac{\sqrt{(b+1)(d-1)} - \sqrt{d-1-b}}{d}.$$

This gives the lower bound of C_r in Eq. (7). Since $f(x,k)$ is decreasing in k and n, for a given $C_{\ell_1} = b$, if $n-2 < b \leqslant n-1$ then the maximum of f occurs inside Δ_n and the corresponding λ will have one component $\beta \leqslant 1/n$ and all the other $(n-1)$ components larger than β. Solving the constraint $g(x,n) = \sqrt{1+b}$ then gives the unique $\beta \leqslant 1/n$ as

$$\beta = \frac{2 + (n-2)(n-b) - 2\sqrt{(b+1)(n-1)(n-1-b)}}{n^2}.$$

We note that

$$\sqrt{\beta} = \frac{\sqrt{b+1} - \sqrt{(n-1)(n-1-b)}}{n},$$
$$\sqrt{1-\beta} = \frac{\sqrt{(b+1)(n-1)} + \sqrt{n-1-b}}{n}. \qquad \blacksquare$$

APPENDIX E: PROOF OF THEOREM 7

Let $\tau \in \mathscr{I}$ be an optimal state for $C_R(\rho)$. Then from the definitions in Eq. (1),

$$C_r(\rho) \leqslant S(\rho\|\tau) = \mathrm{Tr}\left[\rho\left(\log_2\rho - \log_2\frac{(1+s)\tau}{(1+s)}\right)\right]$$
$$= \log_2(1+s) + \mathrm{Tr}[\rho(\log_2\rho - \log_2(1+s)\tau)].$$

Since $\rho \leqslant (1+s)\tau$ and log is operator monotone, the trace term is nonpositive and the first inequality follows. The last inequality follows from $C_R \leqslant C_{\ell_1}$ [35,36], for which an independent proof is given below.

The dual of C_R from Eq. (1) gives

$$1 + C_R(\rho) = \max_{T\geqslant 0,\ \mathrm{diag}(T)=\mathbb{1}} \mathrm{Tr}[\rho T]$$
$$= \sum T_{ij}\rho_{ji} = 1 + 2\sum_{i<j}\mathrm{Re}(T_{ij}\rho_{ji})$$
$$\leqslant 1 + 2\sum_{i<j}|T_{ij}\rho_{ji}| = 1 + C_{\ell_1},$$

where we have used the inequality $|T_{ij}| \leqslant 1$ which follows from $\begin{pmatrix} 1 & \bar{T}_{ij} \\ T_{ij} & 1 \end{pmatrix} \geqslant 0$. $\blacksquare$

APPENDIX F: EVIDENCES FOR THE CONJECTURE $C_r(\rho) \leqslant C_{\ell_1}(\rho)$

Here we present two propositions to support our conjecture that $C_r(\rho) \leqslant C_{\ell_1}(\rho)$ also holds for $C_{\ell_1}(\rho) \leqslant 1$. Let us consider $\rho = r + \delta$, with diagonal r and off-diagonal δ, and the family of states $\rho(\epsilon) = r + \epsilon\delta$ for $0 \leqslant \epsilon \leqslant 1$. Then both $C_r[\rho(\epsilon)]$ and $C_{\ell_1}[\rho(\epsilon)]$ are analytic functions of ϵ.

Proposition 10. For a given ρ, consider the family of states $\rho(\epsilon)$. For $\epsilon \to 0$,

$$C_r[\rho(\epsilon)] = O(\epsilon^2) \leqslant C_{\ell_1}[\rho(\epsilon)] = \epsilon C_{\ell_1}[\rho(1)] = O(\epsilon).$$

This shows that the conjecture holds when the coherences are infinitesimally small.

Proof. We have $C_r[\rho(0)] = C_{\ell_1}[\rho(0)] = 0$. Moreover,

$$\frac{d}{d\epsilon}C_{\ell_1}[\rho(\epsilon)]|_{\epsilon=0} = C_{\ell_1}[\rho(1)] > 0,$$

since we assume $\delta \neq 0$. Denoting the eigenvectors and eigenvalues of $\rho(\epsilon)$ by $|\lambda_i(\epsilon)\rangle$ and $\lambda_i(\epsilon)$, respectively, we have $C_r[\rho(\epsilon)] = H(r) - H[\lambda(\epsilon)]$. The Hellmann-Feynman theorem [70,71] states

$$\frac{d}{d\epsilon}\lambda_i(\epsilon) = \left\langle\lambda_i(\epsilon)\middle|\frac{d}{d\epsilon}\rho(\epsilon)\middle|\lambda_i(\epsilon)\right\rangle = \langle\lambda_i(\epsilon)|\delta|\lambda_i(\epsilon)\rangle,\ \forall i. \tag{F1}$$

Since $\mathrm{Tr}[\rho(\epsilon)] = 1$, we also have

$$\sum_i \frac{d}{d\epsilon}\lambda_i(\epsilon) = 0. \tag{F2}$$

Using Eqs. (F1) and (F2) we get

$$\frac{d}{d\epsilon}C_r[\rho(\epsilon)]|_{\epsilon=0} = \sum_i\langle\lambda_i(0)|\delta|\lambda_i(0)\rangle\log_2[\lambda_i(0)]$$
$$= 0. \qquad \blacksquare$$

We now give a rough bound on allowed ϵ.

Proposition 11. If

$$\int_0^\epsilon \log_2\left[\frac{\lambda_{\max}(\epsilon')}{\lambda_{\min}(\epsilon')}\right]d\epsilon' \leqslant C_{\ell_1}[\rho(\epsilon)] = \epsilon C_{\ell_1}[\rho(1)],$$

then $C_r[\rho(\epsilon')] \leqslant C_{\ell_1}[\rho(\epsilon')]$ for all $0 \leqslant \epsilon' \leqslant \epsilon$. This shows that the conjecture is correct for some states, even if their $C_{\ell_1}[\rho(\epsilon)]$ is smaller than one.

Proof. As in the previous proposition,

$$\frac{d}{d\epsilon}C_r[\rho(\epsilon)] = \sum_i\langle\lambda_i(\epsilon)|\delta|\lambda_i(\epsilon)\rangle\log_2[\lambda_i(\epsilon)].$$

We observe that $\langle\lambda_i(\epsilon)|r+\delta|\lambda_i(\epsilon)\rangle \geqslant \lambda_{\min}(1)$ and $\log_2[\lambda_i(\epsilon)] \leqslant 0$, so that

$$\frac{d}{d\epsilon}C_r[\rho(\epsilon)] \leqslant \sum_i[\lambda_{\min}(1) - \langle\lambda_i(\epsilon)|r|\lambda_i(\epsilon)\rangle]\log_2[\lambda_i(\epsilon)]$$
$$\leqslant d\lambda_{\min}(1)\log_2[\lambda_{\max}(\epsilon)]$$
$$- \sum_i\langle\lambda_i(\epsilon)|r|\lambda_i(\epsilon)\rangle\log_2[\lambda_{\min}(\epsilon)]$$
$$\leqslant \log_2\left[\frac{\lambda_{\max}(\epsilon)}{\lambda_{\min}(\epsilon)}\right]. \tag{F3}$$

The proposition follows by integrating over ϵ' from zero to ϵ. $\blacksquare$

Examples. To illustrate usefulness of the above propositions, let us consider families of states with $r = \mathbb{1}/d$. Then

$$\lambda_{\max}(\epsilon) = \max_{|\psi\rangle}\langle\psi|(1-\epsilon)r + \epsilon(r+\delta)|\psi\rangle$$
$$= (1-\epsilon)/d + \epsilon\lambda_{\max}(1) \leqslant \lambda_{\max}(1),$$

as $1/d \leqslant \lambda_{\max}(1)$. Similarly, $\lambda_{\min}(\epsilon) \geqslant \lambda_{\min}(1)$. Substituting the bounds in Eq. (F3) and integrating over ϵ' from zero to ϵ, we get $C_r[\rho(\epsilon)] \leqslant \epsilon \log_2[\lambda_{\max}(1)/\lambda_{\min}(1)]$. Thus

$$C_{\ell_1}[\rho(1)] \geqslant \log_2\left[\frac{\lambda_{\max}(1)}{\lambda_{\min}(1)}\right]$$
$$\Rightarrow\ C_r[\rho(\epsilon)] \leqslant C_{\ell_1}[\rho(\epsilon)], \quad \forall 0 \leqslant \epsilon \leqslant 1. \tag{F4}$$

The set of states fulfilling condition (F4) is not empty—in particular, all matrices for which $\log_2[\lambda_{\max}(1)/\lambda_{\min}(1)] \leqslant 1$, i.e., $\lambda_{\max}(1)/\lambda_{\min}(1) \leqslant 2$, typically fulfill this condition if $C_{\ell_1}(\rho(1))$ is not too large. Examples of such matrices are easily constructed with all off-diagonal elements equal.

APPENDIX G: PROOF OF THEOREM 9

First note that the following two optimization problems are equivalent (solving one equally solves the other):

$$t(b) = \min C_r(\rho) \quad \text{such that } C_R(\rho) \geqslant b, \tag{G1}$$

$$b(t) = \max C_R(\rho) \quad \text{such that } C_r(\rho) \leqslant t. \tag{G2}$$

Due to convexity of the functions involved, in each case the optimum will occur on the equality condition.

Now, using dual form of C_R [35,36],

$$\begin{aligned} 1 + C_R(\rho) &= \max \mathrm{Tr}[D] \quad \text{such that } \rho \leqslant D,\ D = \mathrm{diag}(D) \\ &= \max \mathrm{Tr}[\rho B] \quad \text{such that } B \geqslant 0,\ \mathrm{diag}(B) = \mathbb{1} \\ &\geqslant \mathrm{Tr}[\rho J], \end{aligned} \tag{G3}$$

where $J = d|\Psi\rangle\langle\Psi|$ is the matrix having all entries 1. Invoking Eqs. (G1) and (G2), our problem reduces to finding

$$1 + b(t) \geqslant \max\ \mathrm{Tr}[\rho J] \quad \text{such that } C_r(\rho) \leqslant t. \tag{G4}$$

Note that both quantities $\mathrm{Tr}[\rho J]$ and $C_r(\rho)$ in Eq. (G4) remain invariant under permutations of rows and columns of ρ. Replacing

$$\rho \longmapsto \frac{1}{d!}\sum_{\pi} U_\pi \rho U_\pi^\dagger,$$

where the sum runs over the permutations, shows that the maximum is achieved at $\rho = p|\Psi\rangle\langle\Psi| + (1-p)\mathbb{1}/d$. The value of p is determined by the condition $C_R = C_{\ell_1}(\rho) = b$. ■

Our numerical study indicates that the same state may have minimum C_r for a given C_{ℓ_1} as well. Unfortunately, the above method is not applicable in this case.

APPENDIX H: COULD C_{ℓ_1} BE SMALLER THAN ITS CONVEX ROOF?

As usual, let us define the convex roof extension of C_{ℓ_1} as (see also [50,51])

$$C_1(\rho) := \min_{\{p_i,|\psi_i\rangle\}} \left\{\sum_i p_i C_{\ell_1}(|\psi_i\rangle)\,|\,\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|\right\}. \tag{H1}$$

Then $C_{\ell_1}(\rho) \leqslant C_1(\rho)$. Note that if equality occurs for a state ρ, then by convexity of C_r and Proposition 2, it must satisfy $C_r(\rho) \leqslant C_{\ell_1}(\rho)$. The result below gives another proof that all qubits fulfill our conjecture.

Proposition 12. All qubit states ρ satisfy $C_{\ell_1}(\rho) = C_1(\rho)$.

Proof. It suffices to show that for given a,b there are p,λ,μ, such that

$$\begin{pmatrix} a & b \\ b & 1-a \end{pmatrix} = p\begin{pmatrix} \lambda & \sqrt{\lambda(1-\lambda)} \\ \sqrt{\lambda(1-\lambda)} & 1-\lambda \end{pmatrix} + (1-p)\begin{pmatrix} \mu & \sqrt{\mu(1-\mu)} \\ \sqrt{\mu(1-\mu)} & 1-\mu \end{pmatrix},$$

where we may assume without loss of generality $a \in (0,1/2]$, $b \in (0,1/2)$, with $a(1-a) > b^2$. This (positivity) demands that $(1-\sqrt{1-4b^2})/2 < a \leqslant 1/2$. Since we require each of the pure states to have $C_{\ell_1} = 2b$, λ,μ are necessarily the two roots of $x(1-x) = b^2$. Setting $\lambda = (1+\sqrt{1-4b^2})/2$, $\mu = 1-\lambda$, and comparing the first diagonal element we get a unique solution

$$p = \frac{1}{2} - \frac{1}{2}\frac{1-2a}{\sqrt{1-4b^2}}.$$

Since $(1-2a)^2 = 1-4a(1-a) \leqslant 1-4b^2$, hence $p \in (0,1/2]$. Note that the standard spectral decomposition does not help, as each of the eigenprojectors have $C_{\ell_1} = 2b/\sqrt{1-4[a(1-a)-b^2]} > 2b$. ■

As mentioned in the proof of Theorem 7, $C_{\ell_1}(\rho)$ is exactly double of the negativity of the corresponding maximally correlated state σ:

$$\begin{aligned} &C_{\ell_1}(\rho) = 2\mathcal{N}(\sigma), \\ \text{where } &\rho = \sum a_{ij}|i\rangle\langle j|, \\ \text{and } &\sigma = \sum a_{ij}|ii\rangle\langle jj|. \end{aligned}$$

Therefore, denoting the convex roof of negativity by $\mathcal{N}_c$,

$$C_{\ell_1}(\rho) = C_1(\rho) \tag{H2}$$

$$\Leftrightarrow \mathcal{N}(\sigma) = \mathcal{N}_c(\sigma). \tag{H3}$$

It is known [72] that equality occurs in Eq. (H3) for isotropic states while strict inequality for Werner states (in $d > 2$). Unfortunately, none of those states is maximally correlated for $d > 2$; hence we cannot browse the results directly into the coherence scenario. It was observed in Ref. [50] that equality holds in Eq. (H2) for all pseudopure states defined in Proposition 6. Also, it was shown in Ref. [[51], Theorem 3] that strict inequality occurs for a similar quantity.

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