

Capability of local operations and classical communication to distinguish bipartite unitary operations

Lvzhou Li,^{1,2,*} Shenggen Zheng,¹ Haozhen Situ,³ and Daowen Qiu¹¹*Institute of Computer Science Theory, School of Data and Computer Science, Sun Yat-Sen University, Guangzhou 510006, China*²*The Key Laboratory of Machine Intelligence and Advanced Computing (Sun Yat-Sen University) Ministry of Education, Guangzhou 510006, China*³*College of Mathematics and Informatics, South China Agricultural University, Guangzhou 510642, China*

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The problem studied in this paper is, if the number of queries to unitary operations is fixed, say k , then when do local operations and classical communication (LOCC) suffice for optimally distinguishing bipartite unitary operations? We consider the above problem for two-qubit unitary operations in the case of $k = 1$, showing that for two two-qubit entangling unitary operations without local parties, LOCC achieves the same distinguishability as the global operations. Specifically, we obtain the following: (i) if such two unitary operations are perfectly distinguishable by global operations, then they are perfectly distinguishable by LOCC also, and (ii) if they are not perfectly distinguishable by global operations, then LOCC can achieve the same optimal discrimination probability as the global operations.

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I. INTRODUCTION

Distinguishability, as a fundamental concept, lies at the heart of quantum information theory, with a wide range of applications in quantum information and computation. While the distinguishability of quantum states has been intensively and extensively studied, it has also been extended to quantum evolution in various forms such as distinguishability of unitary operations [1–10], measurements [11], Pauli channels [12], oracle operators [13], and quantum operations [14–18]. In this paper, we focus on the distinguishability of unitary operations.

The discrimination of unitary operations is generally transformed to the discrimination of quantum states by preparing a probe state and then discriminating the output states generated by different unitary operations. Two unitary operations U and V are said to be perfectly distinguishable (with a single query) if there exists a state $|\psi\rangle$ such that $U|\psi\rangle \perp V|\psi\rangle$. It was shown that U and V are perfectly distinguishable if and only if $\Theta(U^\dagger V) \geq \pi$, where $\Theta(W)$ denotes the length of the smallest arc containing all the eigenvalues of W on the unit circle [1,2]. The situation changes dramatically when multiple queries are allowed since any two different unitary operations are perfectly distinguishable in this case. Specifically, it was shown that for any two different unitary operations U and V , there exist a finite number N and a suitable state $|\varphi\rangle$ such that $U^{\otimes N}|\varphi\rangle \perp V^{\otimes N}|\varphi\rangle$ [1,2]. Such a discriminating scheme is intuitively called a *parallel scheme*. Note that in the parallel scheme, an N -partite entangled state as an input is required and plays a crucial role. The result was further refined in Ref. [3] by showing that the entangled input state is not necessary for perfect discrimination of unitary operations. Specifically, the authors of Ref. [3] showed that for any two different unitary operations U and V there exist an input state $|\varphi\rangle$ and auxiliary unitary operations w_1, \dots, w_N such that $U w_N U, \dots, w_1 U |\varphi\rangle \perp V w_N V, \dots, w_1 V |\varphi\rangle$. Such a discriminating scheme is generally called a *sequential scheme*.

Note that in the research mentioned above, it was assumed by default that the unitary operations to be discriminated are under the complete control of a single party who can perform any physically allowed operations to achieve an optimal discrimination. Actually, a more complicated case is that the unitary operations to be discriminated are shared by several spatially separated parties. In this case a reasonable constraint on the discrimination is that each party can only make local operations and classical communication (LOCC). Despite this constraint, it has been shown that any two bipartite unitary operations can be perfectly discriminated by LOCC when multiple queries to the unitary operations are allowed [4–7]. More specifically, the authors of Refs. [4,5] independently proved this result with tools from the universality of quantum gates [19] and analysis of numerical range [20], respectively. However, the authors of Refs. [4,5] generally required a complicated network combining the sequential and the parallel schemes to achieve a perfect discrimination, where one of the two parties who share the bipartite unitary operations must prepare a multipartite entangled state.

A further result was obtained by the authors of Ref. [6], who asserted that any two bipartite unitary operations acting on $d \otimes d$ (i.e., a two-qudit system) with multiple queries allowed, in principle, can be perfectly discriminated by LOCC with merely a sequential scheme. Note that a sequential scheme usually represents the most economic strategy for discrimination since it does not require any entanglement and saves the spatial resources. Nevertheless, the result in Ref. [6] has two limitations: (i) the unitary operations to be discriminated were limited to act on $d \otimes d$, and (ii) the inverses of the unitary operations were assumed to be accessible, although this assumption may be unrealizable in the experiment. Therefore, the first author improved the result in Ref. [7] by showing that any two bipartite unitary operations acting on $d_A \otimes d_B$ can be locally discriminated with a sequential scheme, without using the inverses of the unitary operations.

Following these works, we have a relatively comprehensive understanding on local discrimination of unitary operations.

*lilvzh@mail.sysu.edu.cn

The above results imply that LOCC and global operations can achieve the same distinguishability for unitary operations, that is, perfect discrimination for both cases, when the unitary operations can be queried multiple times. However, note that for achieving a perfect discrimination, the two situations may require different numbers of queries to the unitary operations. Let N be the optimal number of queries to U and V for a perfect discrimination between them in the case of global operations (similarly, N' is the one for the case of LOCC). Then it is obvious that $N' \geq N$. However, what the condition for $N' = N$ is seemed unknown until now. Thus, this inspires us to consider such a question: if the number of queries to unitary operations is fixed, say k , then when do LOCC suffice for optimally distinguishing bipartite unitary operations?

In this paper we consider the above problem for distinguishing two-qubit unitary operations in the case of $k = 1$ (i.e., the unitary operations can be queried only once). We show that if two two-qubit entangling unitary operations without local parties can be queried only once, then LOCC achieve the same distinguishability as the global operations. More specifically, we obtain the following: (i) if the two unitary operations are perfectly distinguishable by global operations, then they are perfectly distinguishable by LOCC as well, and (ii) if they are not perfectly distinguishable by global operations, then LOCC can achieve the same optimal discrimination probability as the global operations. We hope these discussions about this elementary case shed some light on the more generalized cases.

The main idea of our method is described as follows. First, the error probability of discriminating between U_1 and U_2 is given by

$$P_E(U_1, U_2) = \frac{1}{2}(1 - \sqrt{1 - 4p_1 p_2 F(U_1, U_2)^2}),$$

where $F(U_1, U_2) = \min_{|\psi\rangle} |\langle \psi | U_1^\dagger U_2 | \psi \rangle|$. If U_1 and U_2 are acting on a two-qubit system AB , and we want to discriminate them by LOCC, then the probe state $|\psi\rangle$ should be a product state, that is, $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$. Note that for discriminating two multipartite states, it has been shown that LOCC can achieve the same distinguishability that the global operations would have [22,23]. Therefore, if we can find a product state $|\psi\rangle$ such that $F(U_1, U_2) = |\langle \psi | U_1^\dagger U_2 | \psi \rangle|$, then it can be asserted that LOCC is as powerful as the global operations in discriminating U_1 and U_2 . We will prove this point by using some simple geometric knowledge. Note that the authors of Ref. [9] obtained a similar result. However, if one carefully checks the result there, it can be found that the result in this paper seems more generalized and different ideas are used to derive the results.

The rest of this paper is organized as follows. Section II recalls the canonical decomposition of two-qubit unitary operations. The main result is presented in Sec. III. A conclusion is made in Sec. IV.

II. DECOMPOSITION OF TWO-QUBIT UNITARY OPERATIONS

Concerning decomposition of two-qubit unitary operations and related notations, one can refer to Ref. [21] and references therein for details, and here we only recall some necessary results. Any unitary operation U acting on two qubits A and

B has the following canonical decomposition:

$$U = (U_A \otimes U_B)U_d(V_A \otimes V_B), \quad (1)$$

where $U_A, U_B, V_A,$ and V_B are single-qubit unitary operations and U_d has the following form:

$$U_d = e^{-i(\alpha_x \sigma_x \otimes \sigma_x + \alpha_y \sigma_y \otimes \sigma_y + \alpha_z \sigma_z \otimes \sigma_z)}. \quad (2)$$

Here, $\sigma_x, \sigma_y,$ and σ_z are Pauli operators, and the vector $d = (\alpha_x, \alpha_y, \alpha_z)$ has real entries satisfying

$$0 \leq \alpha_z \leq \alpha_y \leq \alpha_x \leq \frac{\pi}{4}.$$

If $\alpha_x = \alpha_y = \alpha_z = 0$, then $U_d = I$. Otherwise if $\alpha_x = \alpha_y = \alpha_z = \frac{\pi}{4}$, then U_d is the SWAP operation. Otherwise, it is *entangling*, that is, it can create entanglement between two qubits initially in a product state. Thus, U_d is called the *entangling part* of U in this paper. Denote the Bell states by $|\Phi^\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}, |\Psi^\pm\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$. The following states form the so-called magic basis:

$$|\Phi_1\rangle = |\Phi^+\rangle, |\Phi_2\rangle = -i|\Phi^-\rangle,$$

$$|\Phi_3\rangle = |\Psi^+\rangle, |\Phi_4\rangle = -i|\Psi^-\rangle.$$

Then, U_d is diagonal in the magic basis, and it can be written as

$$U_d = \sum_{j=1}^4 e^{-i\lambda_j} |\Phi_j\rangle\langle\Phi_j|, \quad (3)$$

where

$$\lambda_1 = \alpha_x - \alpha_y + \alpha_z, \lambda_2 = -\alpha_x + \alpha_y + \alpha_z,$$

$$\lambda_3 = -\alpha_x - \alpha_y - \alpha_z, \lambda_4 = \alpha_x + \alpha_y - \alpha_z.$$

It is easily seen that $\lambda_4 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3$.

Any two-qubit state $|\psi\rangle$ can be represented as $|\psi\rangle = \sum_k u_k |\Phi_k\rangle$, where $\sum_k |u_k|^2 = 1$. We can use the so-called *concurrence* C to measure the entanglement of a pure state of two qubits, which is defined by

$$C(|\psi\rangle) = |\langle \psi | \sigma_y \otimes \sigma_y | \psi^* \rangle|,$$

where $|\psi^*\rangle$ denotes the complex conjugate of $|\psi\rangle$ in the computational basis. Writing $|\psi\rangle$ in the magic basis, we get

$$C(|\psi\rangle) = \left| \sum_k u_k^2 \right|.$$

Thus, $C(|\psi\rangle) = 0$, that is, $|\psi\rangle$ is a product state, iff $\sum_k u_k^2 = 0$.

III. DISCRIMINATION OF ENTANGLING TWO-QUBIT UNITARY OPERATIONS

Note that when considering discrimination of quantum states or quantum operations, there are several discriminating fashions such as minimum-error discrimination, unambiguous discrimination, and minimax discrimination. Here we consider minimum-error discrimination between two unitary operations U_1 and U_2 . The problem can be reformulated into the problem of finding a probe state $|\psi\rangle$ such that the error probability in discriminating between the output states $U_1|\psi\rangle$ and $U_2|\psi\rangle$

is minimum. Denote by $P_E(U_1, U_2)$ the error probability of discriminating between U_1 and U_2 . Then we have

$$\begin{aligned} P_E(U_1, U_2) &= \min_{|\psi\rangle} \frac{1}{2} (1 - \sqrt{1 - 4p_1 p_2 |\langle \psi | U_1^\dagger U_2 | \psi \rangle|^2}) \\ &= \frac{1}{2} (1 - \sqrt{1 - 4p_1 p_2 \min_{|\psi\rangle} |\langle \psi | U_1^\dagger U_2 | \psi \rangle|^2}) \\ &= \frac{1}{2} (1 - \sqrt{1 - 4p_1 p_2 F(U_1, U_2)^2}), \end{aligned}$$

where p_1 and p_2 are the *a priori* probabilities for U_1 and U_2 , respectively. Here the minimum value is taken over all states $|\psi\rangle$ with $\| |\psi\rangle \| = 1$, and

$$F(U_1, U_2) \equiv \min_{|\psi\rangle} |\langle \psi | U_1^\dagger U_2 | \psi \rangle|$$

is called the fidelity of U_1 and U_2 . If $F(U_1, U_2) = 0$, then $P_E(U_1, U_2) = 0$, and in this case U_1 and U_2 are said to be perfectly distinguishable. Otherwise, they can be distinguished with some error probability.

Now suppose that U_1 and U_2 are acting on a two-qubit system AB , and consider how to discriminate them by LOCC. In this case, the probe state $|\psi\rangle$ should be a product state, that is, $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$. Note that for discriminating two multipartite states, it has been shown that LOCC can achieve the same distinguishability that the global operations would have. In other words, if two multipartite states are perfectly distinguishable by global operations, then they are also perfectly distinguishable by LOCC [22]; if they are distinguishable with some error probability, then they can be distinguished by LOCC with the same error probability [23]. Therefore, if we can find a product state $|\psi\rangle$ such that $F(U_1, U_2) = |\langle \psi | U_1^\dagger U_2 | \psi \rangle|$, then it can be asserted that LOCC is as powerful as the global operations in discriminating U_1 and U_2 .

In the following, we consider discrimination between two entangling unitary operations in Eq. (3) [equivalently, in Eq. (2)]. Given two unitary operations U_1 and U_2 in Eq. (3), the product $U_1^\dagger U_2$ also has the diagonal form of Eq. (3):

$$U_1^\dagger U_2 = \sum_{j=1}^4 e^{-i\omega_j} |\Phi_j\rangle \langle \Phi_j|. \quad (4)$$

Note that it does not necessarily hold that $\omega_4 \geq \omega_1 \geq \omega_2 \geq \omega_3$. Let $|\psi\rangle = \sum_k u_k |\Phi_k\rangle$ with $\sum_{k=1}^4 |u_k|^2 = 1$. Then we get

$$\begin{aligned} F(U_1, U_2) &= \min_{|\psi\rangle} |\langle \psi | U_1^\dagger U_2 | \psi \rangle| \\ &= \min \left\{ \left| \sum_{k=1}^4 |u_k|^2 e^{-i\omega_k} \right| : \sum_{k=1}^4 |u_k|^2 = 1 \right\}. \end{aligned}$$

We will show below that the value of $F(U_1, U_2)$ can be achieved by a product state $|\psi\rangle$ (that is, $|\psi\rangle$ satisfies $\sum_k u_k^2 = 0$). First, by letting $S = \{e^{-i\omega_k}\}_{k=1}^4$, we get

$$\text{conv}(S) = \left\{ \sum_{k=1}^4 |u_k|^2 e^{-i\omega_k} : \sum_{k=1}^4 |u_k|^2 = 1 \right\}, \quad (5)$$

where $\text{conv}(S)$ denotes the convex hull of S . Then, $F(U_1, U_2)$ corresponds to the minimum distance from the original point

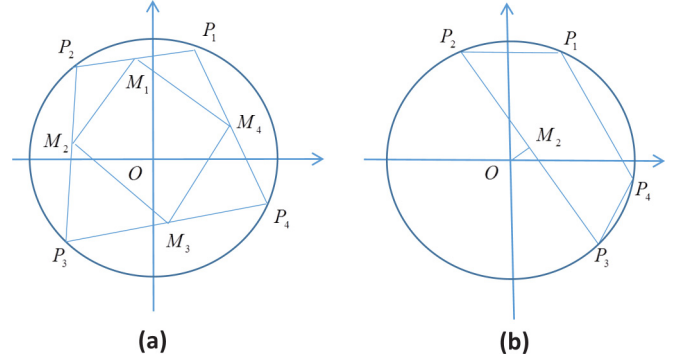


FIG. 1. The convex hull of Eq. (5). P_k corresponds to $e^{-i\omega_k}$ with $i = 1, \dots, 4$, where without loss of generality, we assume these points on the unit circle are P_1, P_2, P_3, P_4 in the counterclockwise order. M_i denotes the midpoint of P_i and $P_{(i+1) \bmod 4}$ for $i = 1, \dots, 4$. In (a), the convex hull contains the original point O , which implies that $F(U_1, U_2) = 0$, and then a perfect discrimination is achievable. In (b), the convex hull does not contain O . Then $F(U_1, U_2)$ is equal to the distance between O and M_2 .

O to the convex hull $\text{conv}(S)$, that is,

$$F(U_1, U_2) = \min_{P \in \text{conv}(S)} \|O - P\|,$$

where it is readily seen that $O \in \text{conv}(S)$, iff $F(U_1, U_2) = 0$, which means a perfection discrimination is achievable.

In geometry, each $e^{-i\omega_k}$ stands for a point on the unit circle in the complex plane. As shown in Fig. 1, let P_k denote the point $e^{-i\omega_k}$ with $k = 1, \dots, 4$. Without loss of generality, assume the counterclockwise order of these points on the unit circle is P_1, P_2, P_3, P_4 . Denote by $\square P_1 P_2 P_3 P_4$ the region enclosed by the convex polygon with endpoints P_1, P_2, P_3, P_4 . Then $\square P_1 P_2 P_3 P_4$ is the convex hull $\text{conv}(S)$.

In the following we show that the value of $F(U_1, U_2)$ is always achievable by a product state, by discussing two cases.

Case (i): $F(U_1, U_2) = 0$. In this case, we have $O \in \text{conv}(S)$, or equivalently, $O \in \square P_1 P_2 P_3 P_4$ as shown in Fig. 1(a).

Let M_i denote the midpoint of P_i and $P_{(i+1) \bmod 4}$ with $i = 1, \dots, 4$, that is,

$$M_1 = \frac{1}{2}(P_1 + P_2) = \frac{1}{2}(e^{-i\omega_1} + e^{-i\omega_2}),$$

$$M_2 = \frac{1}{2}(P_2 + P_3) = \frac{1}{2}(e^{-i\omega_2} + e^{-i\omega_3}),$$

$$M_3 = \frac{1}{2}(P_3 + P_4) = \frac{1}{2}(e^{-i\omega_3} + e^{-i\omega_4}),$$

$$M_4 = \frac{1}{2}(P_4 + P_1) = \frac{1}{2}(e^{-i\omega_4} + e^{-i\omega_1}).$$

First, it is not difficult to show that $O \in \square P_1 P_2 P_3 P_4$ implies $O \in \square M_1 M_2 M_3 M_4$, according to some geometric properties. As a result, there exist a set of positive coefficients $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ satisfying $\sum_j \alpha_j = 1$ such that $\sum \alpha_j M_j = O$, that is,

$$\begin{aligned} \alpha_1 \frac{(e^{-i\omega_1} + e^{-i\omega_2})}{2} + \alpha_2 \frac{(e^{-i\omega_2} + e^{-i\omega_3})}{2} \\ + \alpha_3 \frac{(e^{-i\omega_3} + e^{-i\omega_4})}{2} + \alpha_4 \frac{(e^{-i\omega_4} + e^{-i\omega_1})}{2} = 0. \end{aligned}$$

It can be rewritten as

$$\begin{aligned} & \frac{(\alpha_1 + \alpha_4)}{2} e^{-i\omega_1} + \frac{(\alpha_1 + \alpha_2)}{2} e^{-i\omega_2} \\ & + \frac{(\alpha_2 + \alpha_3)}{2} e^{-i\omega_3} + \frac{(\alpha_3 + \alpha_4)}{2} e^{-i\omega_4} = 0. \end{aligned} \quad (6)$$

Let

$$\begin{aligned} u_1 &= \sqrt{(\alpha_1 + \alpha_4)/2}, \quad u_2 = i\sqrt{(\alpha_1 + \alpha_2)/2}, \\ u_3 &= \sqrt{(\alpha_2 + \alpha_3)/2}, \quad u_4 = i\sqrt{(\alpha_3 + \alpha_4)/2}. \end{aligned}$$

Then Eq. (6) means $\sum_k |u_k|^2 e^{-i\omega_k} = 0$, and one can check that $\sum_k u_k^2 = 0$. Therefore, we have shown that there exists a product state $|\psi\rangle = \sum_k u_k |\Phi_k\rangle$ such that $0 = F(U_1, U_2) = |\langle\psi|U_1^\dagger U_2|\psi\rangle|$.

Case (ii): $F(U_1, U_2) \neq 0$. In this case, $O \notin \text{conv}(S)$, or equivalently, $O \notin \square P_1 P_2 P_3 P_4$ as shown in Fig. 1(b). Then the minimum distance from the original point O to $\square P_1 P_2 P_3 P_4$ is the distance from O to the line $P_2 P_3$, which is equal to the distance between O and the midpoint M_2 of P_2 and P_3 . Therefore, we obtain

$$F(U_1, U_2) = |OM_2| = \|\frac{1}{2}(e^{-i\omega_2} + e^{-i\omega_3})\|.$$

Now let

$$u_1 = 0, \quad u_2 = \frac{1}{\sqrt{2}}, \quad u_3 = i\frac{1}{\sqrt{2}}, \quad u_4 = 0.$$

Then we have $F(U_1, U_2) = |\langle\psi|U_1^\dagger U_2|\psi\rangle|$ for $|\psi\rangle = \sum_k u_k |\Phi_k\rangle$ with $\sum_k u_k^2 = 0$, that is, $F(U_1, U_2)$ is achieved by a product state.

In summary, we show that for any two entangling two-qubit unitary operations U_1 and U_2 in the form of Eq. (3), their fidelity $F(U_1, U_2) \equiv \min_{|\psi\rangle} |\langle\psi|U_1^\dagger U_2|\psi\rangle|$ can be achieved by

a product state $|\psi\rangle = |\psi\rangle_A \otimes |\psi\rangle_B$, and as a result, U_1 and U_2 can be optimally discriminated by LOCC.

IV. CONCLUSION

In this paper we show that LOCC are as powerful as the global operations in discriminating two two-qubit entangling unitary operations without local parties, when they can be queried only once. More specifically, we obtained the following: (i) if such two unitary operations are perfectly distinguishable by global operations, then they are perfectly distinguishable by LOCC also, and (ii) if they are not perfectly distinguishable by global operations, then LOCC can achieve the same optimal discrimination probability as the global operations. Therefore, LOCC suffice for optimally discriminating the mentioned unitary operations with only one query allowed. We hope these discussions about this elementary case will shed some light on the following general problem: if the number of queries to unitary operations is fixed, say k , then when do LOCC suffice for optimally discriminating bipartite unitary operations?

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