Broadcasting quantum coherence via cloning

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Quantum coherence has recently emerged as a key candidate for use as a resource in various quantum information processing tasks. Therefore, it is of utmost importance to explore the possibility of creating a greater number of coherent states from an existing coherent pair. In other words, we start with an initial incoherent pair and induce coherence via quantum cloning. More specifically, we start with a genuinely incoherent state that remains incoherent with the change of basis and make it a coherent state at the end. This process is known as broadcasting of coherence via cloning, which can either be optimal or nonoptimal. Interestingly, in this work, we are able to give a method by which we can introduce coherence in the genuinely incoherent state. We use the computational basis representation of the most general two-qubit mixed state, shared between Alice and Bob, as the input state for the universal symmetric optimal Buzek-Hillery cloner. First of all we show that it is impossible to ensure optimal broadcast of coherence. Second, in the case of nonoptimal broadcasting, we show that the coherence introduced in the output states of the cloner will always be lesser than the initial coherence of the input state. Finally, we take the examples of statistical mixture of the most coherent state and most incoherent state, and then the Bell-diagonal states, to obtain their respective ranges of nonoptimal broadcasting in terms of their input state parameters.

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I. INTRODUCTION

A. Quantum coherence

From the perspective of quantum information theory, it is always interesting to ask what can be classified as a useful resource for information processing tasks. In last few decades, we have witnessed that quantum correlations [1,2] and entanglement [3], in particular, have lent a helping hand in revolutionizing many information processing tasks. In recent years, another well-known quantum phenomenon, known as quantum coherence, has been identified as a candidate to act as a resource in quantum computation and quantum information processing.

Recently, a rigorous framework to quantify coherence has been proposed by Baumgratz *et al.* [4]. Some of the functionals which satisfy Baumgratz's framework include the relative entropy of coherence, the l_1 norm of coherence, the Wigner-Yanase-Dyson skew information [5], and robustness of coherence [6,7].

Moreover, just like entanglement, a significant amount of research has been carried out to propose an operational resource theory of coherence [8–13]. However, due to the basis-dependent nature of quantum coherence, some researchers felt the need to address the notion of *genuine quantum coherence*, thereby suggesting classes of genuinely incoherent operations [14]. Apart from the research on single-party systems, prospects of multipartite coherence, such as frozen quantum coherence and monogamy, have been explored in Refs. [15,16].

B. Quantum cloning and broadcasting

The impossibility to clone quantum states is regarded as one of the most fundamental restriction that comes from nature [17]. Even though we cannot copy an unknown quantum state perfectly, quantum mechanics never rules out the possibility of cloning it approximately. It allows probabilistic cloning, as one can always clone an arbitrary quantum state perfectly with some nonzero probability of success. In 1996, Buzek and Hillery went beyond the idea of perfect cloning and introduced the concept of approximate state independent cloning [18]. Later, in another work by Gisin, this cloner was shown to be optimal [19,20].

In a recent work, it has been shown that it is impossible to optimally broadcast quantum correlation across different laboratories [21]. But if we consider nonoptimal broadcasting, it is possible to broadcast quantum correlation between separated parties [22]. Apart from cloning, there are different ways to broadcast quantum correlation. Quite recently, many authors showed by using sophisticated methods that correlations in a single bipartite state can be locally or unilocally broadcast if and only if the states are classical (i.e., having classical correlation) or classical-quantum, respectively [23–27].

C. Motivation for broadcasting coherence

Unlike quantum entanglement, quantum coherence is a basis-dependent quantity. As an example, let us take the case of a two-qubit state, $\rho_{12} = |00\rangle\langle 00|$. Clearly, ρ_{12} is an incoherent state in the computational basis $\{|00\rangle,|01\rangle,|10\rangle,|11\rangle\}$. But, when we write the same state in the Bell basis $\{|\Phi^+\rangle,|\Phi^-\rangle,|\Psi^+\rangle,|\Psi^-\rangle\}$, then, $\rho_{12} = \frac{1}{2}(|\Phi^+\rangle\langle\Phi^+| + |\Phi^+\rangle\langle\Phi^-| + |\Phi^-\rangle\langle\Phi^+| + |\Phi^-\rangle\langle\Phi^-|)$ becomes a coherent state because in the Bell-basis representation, ρ_{12} has off-diagonal elements. So now we have a very relevant question: What is the need for broadcasting coherence via cloning if we can always make coherence appear or disappear in the given state by a mere change of basis?

In order to answer this question, let us take the example of the genuinely incoherent two-qubit state, $\rho = \frac{I}{4}$, where

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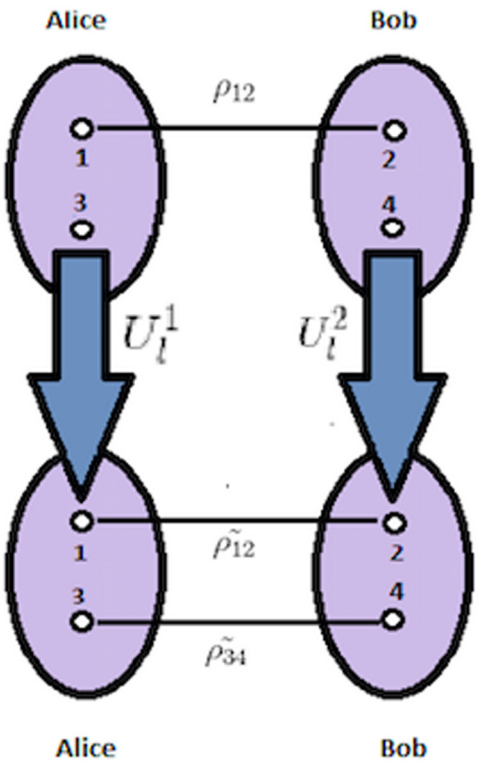

FIG. 1. Application of local cloning operations U_l^1 and U_l^2 on qubits 1 and 2 of the input state ρ_{12} and the blank states $|\Sigma_3\rangle\langle\Sigma_3|$ and $|\Sigma_4\rangle\langle\Sigma_4|$ on Alice's and Bob's labs, respectively, to get the output states $\tilde{\rho}_{12}$ and $\tilde{\rho}_{34}$.

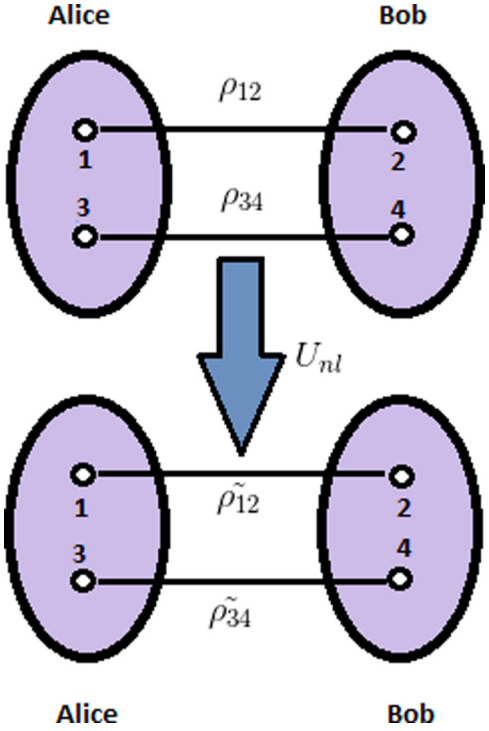

FIG. 2. Application of nonlocal cloning operation U_{nl} on the input state ρ_{12} and blank state ρ_{34} to get the output states $\tilde{\rho}_{12}$ and $\tilde{\rho}_{34}$.

I is the 4×4 identity matrix in the computational basis. An interesting property of this state is that it will always have the same representation in any basis of our choice, i.e., $\rho = \frac{1}{4}(|x_1\rangle\langle x_1| + |x_2\rangle\langle x_2| + |x_3\rangle\langle x_3| + |x_4\rangle\langle x_4|)$ in an arbitrary basis $\{|x_1\rangle, |x_2\rangle, |x_3\rangle, |x_4\rangle\}$. So, in such states we cannot introduce coherence by merely changing the basis.

Therefore, in this work, one of our aims is to introduce coherence in the genuinely incoherent state by the process of cloning and broadcasting, which is otherwise impossible with the change of basis. In the nonlocal cloning machine (see Fig. 2), we have two inputs: the state to be cloned (ρ_{12}) and the blank state (ρ_{34}), where the copy is made. The outputs of the cloning machine ($\tilde{\rho_{12}}$ and $\tilde{\rho_{34}}$) are independent of the nature of the blank state (ρ_{34}). Hence, we propose that we use the most incoherent two-qubit state as the blank state (ρ_{34}) so that in the postcloning scenario, we would be able to introduce coherence in it. A similar argument can be extended for the local cloning situation (described by Fig. 1), where we can use single-qubit state $\rho = \frac{I}{2}$ as the blank state in qubits 3 and 4. After introducing coherence, we can further convert the coherence in $\tilde{\rho_{34}}$ to entanglement [28]. We can even transform the coherence in $\tilde{\rho_{34}}$ to discord [29].

In this paper, our primary objective is to study the broadcast of coherence as a correlation present in a pair of qubits to other pairs via local and nonlocal cloning, with the pairs being spread out across two labs. Here, we have used the l_1 norm as a measure of coherence. Given a state ρ, with its matrix elements as ρ_{ij} the amount of coherence present in the state ρ in the basis $\{|i\rangle\}$ is given by the quantity $C(\rho) = \sum_{i,j=1}^{d} |\langle i|\rho|j\rangle|$, where $i \neq j$. It is interesting to note that coherence is a basis-dependent quantity, as the representation of coherence will be different in different basis. Since the l_1 norm is a function of the off-diagonal elements of the given density matrix representation, clearly, it is evident that the value of the coherence will be zero in the eigenbasis of the density matrix, where there are no off-diagonal elements. It is intuitive that for all other basis the physical findings followed by implications will be similar. So, without any loss of generality, here we have computed the coherence in the two-qubit computational basis $\{|00\rangle, |01\rangle\, |10\rangle\, |11\rangle\}$.

II. OPTIMAL AND NONOPTIMAL BROADCASTING OF COHERENCE

Let us begin with a situation where we have two distant parties, Alice and Bob, and they share a two-qubit mixed state ρ_{12} which can be canonically expressed as

$$\rho_{12} = \frac{1}{4}\Bigg[I_{2\times2} \otimes I_{2\times2} + \sum_{i=1}^{3}(x_i\sigma_i \otimes I) + \sum_{i=1}^{3}(y_i I \otimes \sigma_i) + \sum_{i,j=1}^{3}(t_{ij}\sigma_i \otimes \sigma_j)\Bigg] = \{\vec{X}, \vec{Y}, T_{3\times3}\}, \quad (1)$$

where $\vec{X} = \{x_1, x_2, x_3\}$ and $\vec{Y} = \{y_1, y_2, y_3\}$ are Bloch vectors with $0 \leqslant \|\vec{x}\| \leqslant 1$ and $0 \leqslant \|\vec{y}\| \leqslant 1$. Here, t_{ij} $(i, j = \{1,2,3\})$ represents the elements of the correlation matrix $(T = [t_{ij}]_{3\times3})$, whereas $\sigma_i = (\sigma_1,\sigma_2,\sigma_3)$ are the Pauli matrices and I is the identity matrix. We consider the two-qubit

state ρ_{12} shared between Alice and Bob as the input state to the two types of cloners: a universal symmetric optimal Buzek-Hillery (B-H) nonlocal cloner (U_{bh}^{nl} or simply U_{nl}) and a local cloner (U_{bh}^{l} or simply U_l) (see Appendix A).

We separately investigate both *optimal* and *nonoptimal broadcasting*. In order to broadcast coherence between the desired pairs (1,2) and (3,4) (see Figs. 1 and 2), we have to maximize the amount of coherence between the nonlocal output pairs (1,2) and (3,4), regardless of the local output states between (1,3) and (2,4). However, for optimal broadcasting, we should ideally have no coherence between the local pair of qubits (1,3) and (2,4) in order to have the maximum amount of coherence in a nonlocal pair of qubits.

Definition: Optimal broadcasting (local). As shown in Fig. 1, we start with two parties, Alice and Bob, sharing a coherent input state ρ_{12}, with qubits 1 and 2 of ρ_{12} on Alice's side and Bob's side, respectively. Also, qubits 3 and 4 serve as blank states ($|\Sigma_3\rangle\langle\Sigma_3| = \frac{I}{2}$ and $|\Sigma_4\rangle\langle\Sigma_4| = \frac{I}{2}$) on Alice's and Bob's side, respectively. A coherent input state ρ_{12} is said to be broadcast optimally after the application of local unitary part $U_l^1 \otimes U_l^2$ of the cloning operation, on the qubits 1 and 2, respectively, if for some values of the input state parameters the nonlocal output states, given by

$$\begin{aligned}\tilde{\rho}_{12} &= \mathrm{Tr}_{34}[\tilde{\rho}_{1234}] \\ &= \mathrm{Tr}_{34}\big[U_l^1 \otimes U_l^2(\rho_{12} \otimes |\Sigma_3\rangle\langle\Sigma_3| \otimes |\Sigma_4\rangle\langle\Sigma_4|)\big], \\ \tilde{\rho}_{34} &= \mathrm{Tr}_{12}[\tilde{\rho}_{1234}] \\ &= \mathrm{Tr}_{12}\big[U_l^1 \otimes U_l^2(\rho_{12} \otimes |\Sigma_3\rangle\langle\Sigma_3| \otimes |\Sigma_4\rangle\langle\Sigma_4|)\big],\end{aligned}$$

are coherent, i.e., $C(\tilde{\rho}_{12}) \neq 0, C(\tilde{\rho}_{34}) \neq 0$, while the local output states, given by

$$\begin{aligned}\tilde{\rho}_{13} &= \mathrm{Tr}_{24}[\tilde{\rho}_{1234}] \\ &= \mathrm{Tr}_{24}\big[U_l^1 \otimes U_l^2(\rho_{12} \otimes |\Sigma_3\rangle\langle\Sigma_3| \otimes |\Sigma_4\rangle\langle\Sigma_4|)\big], \\ \tilde{\rho}_{24} &= \mathrm{Tr}_{13}[\tilde{\rho}_{1234}] \\ &= \mathrm{Tr}_{13}\big[U_l^1 \otimes U_l^2(\rho_{12} \otimes |\Sigma_3\rangle\langle\Sigma_3| \otimes |\Sigma_4\rangle\langle\Sigma_4|)\big],\end{aligned}$$

are incoherent, i.e., $C(\tilde{\rho}_{13}) = 0, C(\tilde{\rho}_{24}) = 0$.

Definition: Optimal broadcasting (nonlocal). As shown in Fig. 2, we start with two parties, Alice and Bob, sharing a coherent input state ρ_{12}, with qubits 1 and 2 on Alice's side and Bob's side, respectively, and a blank state $\rho_{34} = \frac{I}{4}$, with qubits 3 and 4 jointly shared by both of them. A coherent input state ρ_{12} is said can be broadcasted optimally after the application of a nonlocal unitary part U_{nl} of the cloning operation together on qubits 1 and 2, if for some values of the input state parameters, the nonlocal output states, given by

$$\begin{aligned}\tilde{\rho}_{12} &= \mathrm{Tr}_{34}[\tilde{\rho}_{1234}] = \mathrm{Tr}_{34}[U_{nl}(\rho_{12} \otimes \rho_{34})], \\ \tilde{\rho}_{34} &= \mathrm{Tr}_{12}[\tilde{\rho}_{1234}] = \mathrm{Tr}_{12}[U_{nl}(\rho_{12} \otimes \rho_{34})],\end{aligned}$$

are coherent, i.e., $C(\tilde{\rho}_{12}) \neq 0$ and $C(\tilde{\rho}_{34}) \neq 0$, while the local output states, given by

$$\begin{aligned}\tilde{\rho}_{13} &= \mathrm{Tr}_{24}[\tilde{\rho}_{1234}] = \mathrm{Tr}_{24}[U_{nl}(\rho_{12} \otimes \rho_{34})], \\ \tilde{\rho}_{24} &= \mathrm{Tr}_{13}[\tilde{\rho}_{1234}] = \mathrm{Tr}_{13}[U_{nl}(\rho_{12} \otimes \rho_{34})],\end{aligned}$$

are incoherent, i.e., $C(\tilde{\rho}_{13}) = 0$ and $C(\tilde{\rho}_{24}) = 0$.

Interestingly, we find that it is impossible to broadcast coherence optimally via both nonlocal and local cloning transformations. We show this result for the computational basis $\{|00\rangle, |01\rangle\, |10\rangle\, |11\rangle\}$ with the general mixed state [given in Eq. (1)] as the input resource without any loss of generality.

Theorem 1. Given a two-qubit general mixed state ρ_{12} and B-H local cloning transformations (see Appendix A), it is impossible to optimally broadcast the coherence within ρ_{12} into two coherent states: $\tilde{\rho}_{12} = \mathrm{Tr}_{34}[U_l^1 \otimes U_l^2(\rho_{12} \otimes |\Sigma_3\rangle\langle\Sigma_3| \otimes |\Sigma_4\rangle\langle\Sigma_4|)]$ and $\tilde{\rho}_{34} = \mathrm{Tr}_{12}[U_l^1 \otimes U_l^2(\rho_{12} \otimes |\Sigma_3\rangle\langle\Sigma_3| \otimes |\Sigma_4\rangle\langle\Sigma_4|)]$, where $|\Sigma_3\rangle\langle\Sigma_3| = |\Sigma_4\rangle\langle\Sigma_4| = \frac{I}{2}$.

Proof. Here, we consider the input state as the most general two-qubit mixed state ρ_{12}. The local unitary part $U_l^1 \otimes U_l^2$ of the B-H cloning transformation (see Appendix A) is applied on qubits of $\rho_{12} = \{x, y, T\}$. We have local output states as $\tilde{\rho}_{13} = \{\mu x, \mu x, T_l\}$ and $\tilde{\rho}_{24} = \{\mu y, \mu y, T_l\}$, where $T_l = \mathrm{diag}(2\lambda, 2\lambda, 1-4\lambda)$ is its 3×3 diagonal correlation matrix and the nonlocal output states $\tilde{\rho}_{12} = \tilde{\rho}_{34} = \{\mu x, \mu y, \mu^2 T\}$. Here, $\mu = 1 - 2\lambda$ and λ are the cloning machine parameters (see Appendix A), x, y represent the Bloch vectors, and T represents the 3×3 correlation matrix of the input state. The coherence calculated by using the l_1 norm of the local output states in the computational basis is given by $C(\tilde{\rho}_{13}) = 2\sqrt{x_1^2 + x_2^2} + 2\lambda(1 - 2\sqrt{x_1^2 + x_2^2}) > 0$ on Alice's side and $C(\tilde{\rho}_{24}) = 2\sqrt{y_1^2 + y_2^2} + 2\lambda(1 - 2\sqrt{y_1^2 + y_2^2}) > 0$ on Bob's side. Each of these quantities are positive, as the norm $||\vec{v}|| = \sqrt{v_1^2 + v_2^2}$ for a $2d$ vector $\vec{v} = \{v_1, v_2\}$ is a non-negative quantity and in the minimum case of $x_1 = x_2 = 0$ and $y_1 = y_2 = 0$, still, $C(\tilde{\rho}_{13}) = C(\tilde{\rho}_{24}) = 2\lambda > 0$. This is true for all values of λ lying between 0 and 1/2.

Now for $\lambda = \frac{1}{6}$, the above B-H local cloner reduces to a state-independent B-H optimal local cloner. Subsequently, the coherence of the local output states changes to $C(\tilde{\rho}_{13}) = 1/3 + (4/3)\sqrt{x_1^2 + x_2^2}) > 0$ and $C(\tilde{\rho}_{24}) = 1/3 + (4/3)\sqrt{y_1^2 + y_2^2}) > 0$. Since the coherence of local output states is nonvanishing, it is clear enough that it is impossible to optimally broadcast quantum coherence via local cloning. ■

Theorem 2. Given the two-qubit general mixed state ρ_{12} and B-H nonlocal cloning transformations (see Appendix A), it is impossible to optimally broadcast the coherence within ρ_{12} into two coherent states: $\tilde{\rho}_{12} = \mathrm{Tr}_{34}[U_{bh}^{nl}(\rho_{12} \otimes \rho_{34})]$ and $\tilde{\rho}_{34} = \mathrm{Tr}_{12}[U_{bh}^{nl}(\rho_{12} \otimes \rho_{34})]$, where $\rho_{34} = \frac{I}{4}$.

Proof. Here, we start with most general version of the two-qubit mixed state $\rho_{12} = \{x, y, T\}$. When the nonlocal unitary part U_{bh}^{nl} or U_{nl} (see Fig. 2) of the B-H cloning transformation is applied on qubits of ρ_{12}, then we have local output states as $\tilde{\rho}_{13} = \{\mu x, \mu x, T_{nl}\}$ and $\tilde{\rho}_{24} = \{\mu y, \mu y, T_{nl}\}$, where $T_{nl} = \mathrm{diag}(2\lambda, 2\lambda, 1-8\lambda)$ is its 3×3 diagonal correlation matrix and the nonlocal output states $\tilde{\rho}_{12} = \tilde{\rho}_{34} = \{\mu x, \mu y, \mu T\}$. Here, $\mu = 1 - 4\lambda$, with λ being the cloning machine parameter (see Appendix A), x, y represent the Bloch vectors, and T represents the correlation matrix of the input state. The coherence is calculated by using l_1 norm measure in the computational basis. Hence, for the local output states, we have $C(\tilde{\rho}_{13}) = 2\sqrt{x_1^2 + x_2^2} + 2\lambda(1 - 4\sqrt{x_1^2 + x_2^2}) > 0$ on Alice's side and $C(\tilde{\rho}_{24}) = 2\sqrt{y_1^2 + y_2^2} + 2\lambda(1 - 4\sqrt{y_1^2 + y_2^2}) > 0$ on Bob's side. Each of these quantities is positive as the norm $||\vec{v}|| = \sqrt{v_1^2 + v_2^2}$ for a $2d$ vector $\vec{v} = \{v_1, v_2\}$ is a non-negative

quantity, and in the minimum case of $x_1 = x_2 = 0$ and $y_1 = y_2 = 0$, still, $C(\tilde{\rho}_{13}) = C(\tilde{\rho}_{24}) = 2\lambda > 0$. This is true for all values of λ lying between 0 and 1/4.

Now, for $\lambda = \frac{1}{10}$, the above B-H nonlocal cloning machine reduces to a state-independent B-H optimal nonlocal cloner. Subsequently, the coherence of local output states changes to, $C(\tilde{\rho}_{13}) = 1/5 + (6/5)\sqrt{x_1^2 + x_2^2} > 0$ and $C(\tilde{\rho}_{24}) = 1/5 + (6/5)\sqrt{y_1^2 + y_2^2} > 0$. Therefore, we see that it is impossible to broadcast quantum coherence optimally, as we have nonvanishing quantum coherence for the local output states. ■

At this point, we note that it is impossible to optimally broadcast coherence between the desired qubits. However, we cannot rule out the entire possibility of broadcasting. If we relax the condition $C(\tilde{\rho}_{13}) = 0, C(\tilde{\rho}_{24}) = 0$, then we show that it will be possible to broadcast coherence nonoptimally.

Definition : Non-optimal broadcasting (local). A coherent state ρ_{12} can be broadcasted nonoptimally after the application of local unitary part $U_l^1 \otimes U_l^2$ (see Fig. 1) of the cloning operation, if for some values of the input state parameters, the coherence of nonlocal output states is greater than the coherence of local output states. Mathematically, for nonoptimal broadcasting via local cloning, we should have the following conditions:

$C(\tilde{\rho}_{12}) > C(\tilde{\rho}_{13})$ and $C(\tilde{\rho}_{12}) > C(\tilde{\rho}_{24})$

$C(\tilde{\rho}_{34}) > C(\tilde{\rho}_{13})$ and $C(\tilde{\rho}_{34}) > C(\tilde{\rho}_{24})$

Definition: Nonoptimal broadcasting (nonlocal). A coherent state ρ_{12} can be broadcasted nonoptimally after the application of nonlocal unitary part U_{nl} (see Fig. 2) of the cloning operation, if for some values of the input state parameters, the coherence of nonlocal output states is greater than coherence of local output states. Mathematically, for nonoptimal broadcasting via nonlocal cloning, we should have the following conditions:

$C(\tilde{\rho}_{12}) > C(\tilde{\rho}_{13})$ and $C(\tilde{\rho}_{12}) > C(\tilde{\rho}_{24})$

$C(\tilde{\rho}_{34}) > C(\tilde{\rho}_{13})$ and $C(\tilde{\rho}_{34}) > C(\tilde{\rho}_{24})$

Under this relaxation, the quantum coherence of the nonlocal output states $\tilde{\rho}_{12}$ and $\tilde{\rho}_{34}$ can either increase or decrease from the initial resource state ρ_{12}, which implies the following two scenarios after broadcasting:

$C(\tilde{\rho}_{34}) < C(\rho_{12})$ and $C(\tilde{\rho}_{12}) < C(\rho_{12})$

$C(\tilde{\rho}_{34}) > C(\rho_{12})$ and $C(\tilde{\rho}_{12}) > C(\rho_{12})$

Interestingly, in this work we show that it is indeed impossible to increase the quantum coherence by both nonlocal and local cloning operation. We present this result in form of the following theorems by taking the two-qubit most general state [described in Eq. (1)] as the input state. We have measured the coherence in the following proofs in computational basis without any loss of generality.

Theorem 3. Given a two-qubit general mixed state ρ_{12} and B-H local cloning transformation (see Appendix A), we can always create two states $\tilde{\rho}_{12}$ and $\tilde{\rho}_{34}$, such that (a) $C(\tilde{\rho}_{12}) < C(\rho_{12})$ and (b) $C(\tilde{\rho}_{34}) < C(\rho_{12})$ in the computational basis $\{|0\rangle, |1\rangle\}$, where $\mu = 1 - 2\lambda$, with λ being the machine parameter.

Proof. We start with the most general representation of a two-qubit mixed state $\rho_{12} = \{x, y, T\}$. When the B-H local cloning transformation $U_l^1 \otimes U_l^2$ is applied to ρ_{12}, we get the nonlocal output states as $\tilde{\rho}_{12} = \tilde{\rho}_{34} = \{\mu x, \mu y, \mu^2 T\}$. Here, $\mu = 1 - 2\lambda$, with λ being the cloning machine parameter, and x, y represent the Bloch vectors and $T = [t_{ij}]_{3\times 3}$ represents the correlation matrix of the input state. Using the l_1 norm as the measure of coherence in the computational basis, we have $C(\rho_{12}) = 1/2[a_1 + a_2 + a_3]$ and $C(\tilde{\rho}_{12}) = C(\tilde{\rho}_{34}) = 1/2[b_1 + b_2 + b_3]$.

Here,

$$
\begin{aligned}
a_1 &= \sqrt{(t_{13} + t_{21})^2 + (t_{11} - t_{22})^2} \\
&\quad + \sqrt{(t_{12} - t_{21})^2 + (t_{11} + t_{22})^2} \\
a_2 &= \sqrt{(t_{13} - x_1)^2 + (t_{23} - x_2)^2} \\
&\quad + \sqrt{(t_{13} + x_1)^2 + (t_{23} + x_2)^2} \\
a_3 &= \sqrt{(t_{31} - y_1)^2 + (t_{32} - y_2)^2} \\
&\quad + \sqrt{(t_{31} + y_1)^2 + (t_{32} + y_2)^2} \\
b_1 &= \mu^2\sqrt{(t_{13} + t_{21})^2 + (t_{11} - t_{22})^2} \\
&\quad + \mu^2\sqrt{(t_{12} - t_{21})^2 + (t_{11} + t_{22})^2} \\
b_2 &= \mu\sqrt{(\mu t_{13} - x_1)^2 + (\mu t_{23} - x_2)^2} \\
&\quad + \mu\sqrt{(\mu t_{13} + x_1)^2 + (\mu t_{23} + x_2)^2} \\
b_3 &= \mu\sqrt{(\mu t_{31} - y_1)^2 + (\mu t_{32} - y_2)^2} \\
&\quad + \mu\sqrt{(\mu t_{31} + y_1)^2 + (\mu t_{32} + y_2)^2}. \qquad (2)
\end{aligned}
$$

If we can show that the conditions $a_1 > b_1$, $a_2 > b_2$, and $a_3 > b_3$ hold true at the same time, then we can always conclude that $C(\rho_{12}) > C(\tilde{\rho}_{12})$ and $C(\rho_{12}) > C(\tilde{\rho}_{34})$.

At first, if we observe the terms a_1 and b_1 carefully, we find that $b_1 = \mu^2 a_1 < a_1$ because $\mu = 1 - 2\lambda < 1$ and $a_1 \geqslant 0$.

Now, we need to compare a_2 and b_2. If we observe the terms in a_2 and b_2 and ignore the constant 1/2, they can be thought of as distance between two points in a Cartesian plane. Let us denote the distance between two points A and B in a two-dimensional space (like the XY Cartesian plane) by $\|AB\|$. Let us consider $\tilde{b_2} = \sqrt{(\mu t_{13} - x_1)^2 + (\mu t_{23} - x_2)^2} + \sqrt{(\mu t_{13} + x_1)^2 + (\mu t_{23} + x_2)^2}$ and as $\mu < 1$, so, we have, $b_2 < \tilde{b_2}$. Now, assume four points in Cartesian plane, namely, $A(t_{13}, t_{23})$, $B(x_1, x_2)$, $C(-x_1, -x_2)$, and $D(\mu t_{13}, \mu t_{23})$ without O as the origin. Now, as $\mu < 1$, hence, it is always true that $\|OD\| = \sqrt{(\mu t_{13})^2 + (\mu t_{23})^2} = \mu\sqrt{(t_{13})^2 + (t_{23})^2} = \mu\|OA\|$; that is, $\|OD\| < \|OA\|$. So, D always lies on $\overline{OA}$, inside ΔABC. Now, from Fig. 3, we have, $\tilde{b_2} = \|DC\| + \|DB\|$ and $a_2 = \|AC\| + \|AB\|$. Now, we have proved $\tilde{b_2} < a_2$ in Appendix B by plane geometry.

Hence, we have $b_2 < \tilde{b_2} < a_2$. Similarly, we can show $b_3 < a_3$. Hence, we have proved that $b_1 + b_2 + b_3 < a_1 + a_2 + a_3$. ■

Theorem 4. Given a two-qubit general mixed state ρ_{12} and B-H nonlocal cloning transformation (see Appendix A), we can always create two states $\tilde{\rho}_{12}$ and $\tilde{\rho}_{34}$ such that (a) $C(\tilde{\rho}_{12}) = \mu C(\rho_{12}) < C(\rho_{12})$ and (b) $C(\tilde{\rho}_{34}) = \mu C(\rho_{12}) < C(\rho_{12})$ in the computational basis, where $\mu = 1 - 4\lambda$, with λ being the machine parameter.

Proof. First of all we apply the Buzek-Hillary nonlocal cloning transformation to the two-qubit general mixed quantum states $\rho_{12} = \{x, y, T\}$. The nonlocal output states are given by $\tilde{\rho}_{12} = \tilde{\rho}_{34} = \{\mu x, \mu y, \mu T\}$. Here, $\mu = 1 - 4\lambda$, with λ

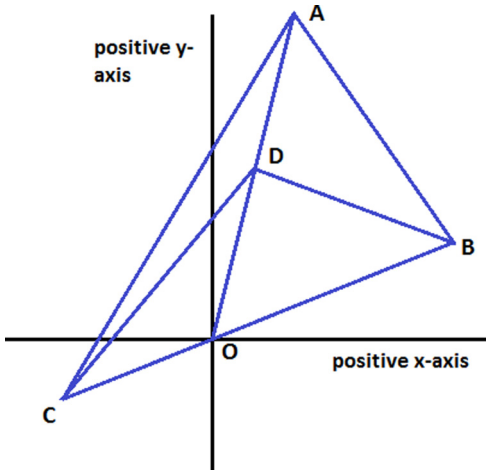

FIG. 3. ΔABC in the xy plane, with point D in its interior.

being the cloning machine parameter, x, y represent the Bloch vectors, and $T = [t_{ij}]_{3\times 3}$ represents the correlation matrix of the input state. The coherence present in these states, calculated in the computational basis by using the l_1 norm, are given by $C(\rho_{12}) = \frac{1}{2}X$ and $C(\tilde{\rho}_{12}) = C(\tilde{\rho}_{34}) = \frac{\mu}{2}X$, where,

$$\begin{aligned} X = [&\sqrt{(t_{13}+t_{21})^2+(t_{11}-t_{22})^2} \\ &+\sqrt{(t_{12}-t_{21})^2+(t_{11}+t_{22})^2} \\ &+\sqrt{(t_{13}-x_1)^2+(t_{23}-x_2)^2} \\ &+\sqrt{(t_{13}+x_1)^2+(t_{23}+x_2)^2} \\ &+\sqrt{(t_{31}-y_1)^2+(t_{32}-y_2)^2} \\ &+\sqrt{(t_{31}+y_1)^2+(t_{32}+y_2)^2}]. \end{aligned} \tag{3}$$

Now, we know, $\mu = 1-4\lambda < 1$ and $\lambda < 1/4$. So, clearly, $C(\tilde{\rho}_{12}) = \mu C(\rho_{12}) < C(\rho_{12})$ and $C(\tilde{\rho}_{34}) = \mu C(\rho_{12}) < C(\rho_{12})$. Hence, we have created two correlations across the labs of Alice and Bob with lesser coherence compared to that of the input state via nonlocal cloning. ■

III. NONOPTIMAL BROADCASTING FOR PARTICULAR MIXED STATES

In this section, we consider the standard examples of a two-qubit mixed state as a resource state. These include states representing a mixture of the most coherent state (MCS) and the most incoherent state (MIS), and Bell diagonal states (BDS). We apply the standard B-H cloner nonlocally, to find the range where nonoptimal broadcasting is possible, in terms of input state parameters. In all the examples, we have chosen a computational basis for measuring coherence via the l_1 norm.

A. Example A: Mixture of MIS and MCS

Here, we consider a two-qubit state representing a convex mixture of the most coherent two-qubit state (ρ_{MCS}) and the most incoherent two-qubit state (ρ_{MIS}) in a computational basis. In a computational basis, we have $\rho_{\mathrm{MIS}} = \frac{I}{4}$, where I is a 4×4 identity matrix and $\rho_{\mathrm{MCS}} = |\mathrm{MCS}\rangle\langle\mathrm{MCS}|$, where $|\mathrm{MCS}\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$. Now, assuming p as the mixing parameter, we have $\rho_{12} = p\rho_{\mathrm{MCS}} + (1-p)\rho_{\mathrm{MIS}}$, where $0 \leqslant p \leqslant 1$. Formally, we represent this class of states in terms of Bloch vectors as $\rho_{12} = \{\vec{x}^{\mathrm{MIX}}, \vec{y}^{\mathrm{MIX}}, T^{\mathrm{MIX}}\}$, where $\vec{x}^{\mathrm{MIX}} = \{p,0,0\}$ and $\vec{y}^{\mathrm{MIX}} = \{p,0,0\}$ are the Bloch vectors, and $T^{\mathrm{MIX}} = \begin{pmatrix} p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is a 3×3 correlation matrix, with p being the same mixing parameter.

1. Local cloning

Here, we use the general local cloning machine (see Appendix A) to obtain the local output states as $\tilde{\rho}_{13} = \{\mu\vec{x}^{\mathrm{MIX}}, \mu\vec{x}^{\mathrm{MIX}}, T_l\}$ and $\tilde{\rho}_{24} = \{\mu\vec{y}^{\mathrm{MIX}}, \mu\vec{y}^{\mathrm{MIX}}, T_l\}$ and the nonlocal output states as $\tilde{\rho}_{12} = \tilde{\rho}_{34} = \{\mu\vec{x}^{\mathrm{MIX}}, \mu\vec{x}^{\mathrm{MIX}}, \mu^2 T^{\mathrm{MIX}}\}$. Now we have $\mu = 1-2\lambda$, with λ being the cloning machine parameter and T^{MIX} is the same as the correlation matrix of ρ_{12}. Here, x^{MIX} and y^{MIX} are the input state Bloch vectors for the ρ_{12} as described above and $T_{nl} = \mathrm{diag}[2\lambda, 2\lambda, 1-4\lambda]$ is the 3×3 diagonal correlation matrix of the local output states.

Using the l_1 norm, the coherence for the local output states is given by $C(\tilde{\rho}_{13}) = C(\tilde{\rho}_{24}) = 2\lambda + 2p(1-2\lambda)$, while that of the nonlocal output states is given by $C(\tilde{\rho}_{12}) = C(\tilde{\rho}_{34}) = p(1-2\lambda)(3-2\lambda)$. The expression for the condition of broadcasting in the general local cloning case reduces to $p(1-2\lambda)^2 > 2\lambda$.

To obtain the state independent version, we substitute the value of λ as $\frac{1}{6}$ (see Appendix A). In this case, for the local output states we have $C(\tilde{\rho}_{13}) = C(\tilde{\rho}_{24}) = \frac{1+4p}{3}$, and for the nonlocal output states we have $C(\tilde{\rho}_{12}) = C(\tilde{\rho}_{34}) = \frac{16p}{9}$. Hence, the broadcasting in the state-independent local cloning case is possible when $p > \frac{3}{4}$.

2. Nonlocal cloning

Here, we use the general nonlocal cloning machine (see Appendix A) to obtain the local output states as $\tilde{\rho}_{13} = \{\mu\vec{x}^{\mathrm{MIX}}, \mu\vec{x}^{\mathrm{MIX}}, T_{nl}\}$ and $\tilde{\rho}_{24} = \{\mu\vec{y}^{\mathrm{MIX}}, \mu\vec{y}^{\mathrm{MIX}}, T_{nl}\}$, and the nonlocal output states as $\tilde{\rho}_{12} = \tilde{\rho}_{34} = \{\mu\vec{x}^{\mathrm{MIX}}, \mu\vec{x}^{\mathrm{MIX}}, \mu T^{\mathrm{MIX}}\}$. Now we have $\mu = 1-4\lambda$, with λ being the cloning machine parameter and T^{MIX} is the same as the correlation matrix of ρ_{12}. Here, x^{MIX} and y^{MIX} are the input state Bloch vectors for the ρ_{12} as described above, and $T_{nl} = \mathrm{diag}[2\lambda, 2\lambda, 1-8\lambda]$ is the 3×3 diagonal correlation matrix of the local output states.

Using the l_1 norm, the coherence for the local output states is given by $C(\tilde{\rho}_{13}) = C(\tilde{\rho}_{24}) = 2\lambda + 2p(1-4\lambda)$, while that of the nonlocal output states is given by $C(\tilde{\rho}_{12}) = C(\tilde{\rho}_{34}) = 3p(1-4\lambda)$. The broadcasting in the general nonlocal cloning case is possible when $p > 2\lambda + 4p\lambda$.

To obtain the state-independent version, we substitute the value of λ as $\frac{1}{10}$ (see Appendix A). In this case, for the local output states we have $C(\tilde{\rho}_{13}) = C(\tilde{\rho}_{24}) = \frac{1+6p}{5}$, and for the nonlocal output states we have $C(\tilde{\rho}_{12}) = C(\tilde{\rho}_{34}) = \frac{9p}{5}$. Hence, the condition for broadcasting in the state independent nonlocal cloning case reduces to $p > \frac{1}{3}$.

Hence, in this example it is evident that with respect to the computational basis, a nonlocal state-independent cloning machine performs much better than a state-independent local cloning machine when it comes to broadcasting of coherence.

B. Example B: Bell diagonal states (BDS)

A Bell diagonal state is a two-qubit state that is diagonal in the Bell basis. In other words, it is a mixture of the four Bell states. It can be written as $\rho_{12} = p_1|\Phi^+\rangle\langle\Phi^+| + p_2|\Psi^+\rangle\langle\Psi^+| + p_3|\Phi^-\rangle\langle\Phi^-| + p_4|\Psi^-\rangle\langle\Psi^-|$, where $p_1 + p_2 + p_3 + p_4 = 1$. A Bell diagonal state is separable if all the probabilities are less or equal to $1/2$. After transforming ρ_{12} from Bell basis to computational basis $\{|0\rangle, |1\rangle\}$, we convert the four-parameter description to a three-parameter description via β coordinate transformation. In the β coordinate system, the set of Bell diagonal states can be visualized as a tetrahedron where the four Bell states are the corners. The following change of the coordinate system makes the plotting of states easy, which is given by $\beta_0 = \frac{1}{2}(p_1 + p_2 + p_3 + p_4)$, $\beta_1 = \frac{1}{2}(p_1 - p_2 - p_3 + p_4)\beta_2 = \frac{1}{\sqrt{2}}(p_1 - p_4)$, $\beta_3 = \frac{1}{\sqrt{2}}(p_2 - p_3)$.

The coordinate β_0 will always be equal to $1/2$, and $(\beta_1, \beta_2, \beta_3)$ can be plotted in 3D easily. In the β-coordinate system, Bell diagonal states are mathematically described as $\rho_{12} = \{\vec{0},\vec{0},T^b\}$, where $\vec{0}$ is the null vector and $T^b = \text{diag}[\sqrt{2}(\beta_2 - \beta_3), -2\beta_1, \sqrt{2}(\beta_2 + \beta_3)]$ is the correlation matrix. The β-coordinate system has the advantage that two of the edges are parallel to axes of the coordinate system.

1. Local cloning

When we apply the general local unitary part $(U_l^1 \otimes U_l^2)$ (see Appendix A) of the cloning machine, the local output states are given by $\tilde{\rho}_{13} = \{\vec{0},\vec{0},T_l\}$, $\tilde{\rho}_{24} = \{\vec{0},\vec{0},T_l\}$ and the nonlocal output states are $\tilde{\rho}_{12} = \tilde{\rho}_{34} = \{\vec{0},\vec{0},\mu^2 T^b\}$, where $\mu = 1 - 2\lambda$, with λ being the machine parameter. T_l is a 3×3 diagonal matrix with the diagonal elements being 2λ, 2λ and $1 - 4\lambda$, whereas T^b is the input state correlation matrix for a Bell diagonal state, as described earlier.

In the case of a general local cloner, the coherence of local output states is given by $C(\tilde{\rho}_{13}) = C(\tilde{\rho}_{24}) = 2\lambda$, while on the other hand, the coherence of nonlocal output states is given by $C(\tilde{\rho}_{12}) = C(\tilde{\rho}_{34}) = \frac{|(2\beta_2 - \beta_3)(1-2\lambda)^2|}{\sqrt{2}}$. Now, in order to obtain the state independent version of the local cloner, we substitute $\lambda = 1/6$ (see Appendix A). Hence, after substituting $\lambda = 1/6$, we have $C(\tilde{\rho}_{13}) = C(\tilde{\rho}_{24}) = \frac{1}{3}$ and $C(\tilde{\rho}_{12}) = C(\tilde{\rho}_{34}) = \frac{4|2\beta_2 - \beta_3|}{9\sqrt{2}}$. The condition of broadcasting in this case is $4|2\beta_2 - \beta_3| > 3\sqrt{2}$.

In Table I we have listed the broadcasting range for local cloning in terms of β_1, β_2, and β_3. We fix the values of β_1 and β_3 and substitute their respective values in the expression $4|2\beta_2 - \beta_3| > 3\sqrt{2}$ in order to arrive at the desired range of values in terms of β_2. However, for certain values of β_1 and β_3, we observe discontinuities in the intervals corresponding to the range of β_2, which suggests the presence of a no-broadcast zone in between.

In Fig. 4, we have superimposed the region of broadcasting on the tetrahedral region hosting all Bell diagonal states in the β-coordinate system, with axes β_1, β_2, and β_3. The light-blue region represents the no-broadcast zone, while the two regions on the dark-blue–magenta spectrum, along the edges of the two corners of the tetrahedron, highlight the regions where broadcasting of coherence is possible. In these two regions along the edges, the change in hue signifies how the values of the nonlocal coherence, $C(\tilde{\rho}_{12})$ or $C(\tilde{\rho}_{34})$, change with respect to the values of the β parameters. As the value of $C(\tilde{\rho}_{12})$ or $C(\tilde{\rho}_{34})$ increases, the hue changes from blue to magenta. From the figure, it is evident that in terms of coverage of the tetrahedron's volume, the zone of broadcasting is very small when compared to the no-broadcast region, thereby implying that we can broadcast coherence only in a limited set of Bell diagonal states via local state-independent cloning.

TABLE I. Broadcasting ranges for local state-independent cloning in the Bell diagonal state in terms of β_1, β_3, and β_2 parameters.

β_1	β_3	β_2
0.15	–0.2	$0.430 < \beta_2 \leqslant 0.460$
0.15	–0.15	$0.455 < \beta_2 \leqslant 0.460$
0.2	0.2	$-0.495 \leqslant \beta_2 < -0.430$
0.2	0.15	$-0.495 \leqslant \beta_2 < -0.455$
0.2	0.1	$-0.495 \leqslant \beta_2 < -0.480$
0.2	–0.1	$0.480 < \beta_2 \leqslant 0.495$
0.2	–0.2	$0.430 < \beta_2 \leqslant 0.495$
0.3	0.05	$-0.566 \leqslant \beta_2 < -0.505$ and $0.555 < \beta_2 \leqslant 0.566$
0.3	0.1	$-0.566 \leqslant \beta_2 < -0.480$
0.3	–0.1	$0.48033 < \beta_2 \leqslant 0.566$
0.4	0.05	$-0.636 \leqslant \beta_2 < -0.505$ and $0.555 < \beta_2 \leqslant 0.636$

2. Nonlocal cloning

Next, we use a general nonlocal cloner (see Appendix A) for cloning a Bell diagonal state. The output states in this case are given by $\tilde{\rho}_{13} = \{\vec{0},\vec{0},T_{nl}\}$, $\tilde{\rho}_{24} = \{\vec{0},\vec{0},T_{nl}\}$, and $\tilde{\rho}_{12} = \tilde{\rho}_{34} = \{\vec{0},\vec{0},\mu T^b\}$, where $\mu = 1 - 4\lambda$, with λ being the machine parameter. Here, T_{nl} is a 3×3 diagonal matrix with the diagonal elements being 2λ, 2λ, and $1 - 8\lambda$, whereas T^b is the input state correlation matrix for the Bell diagonal state, as described earlier.

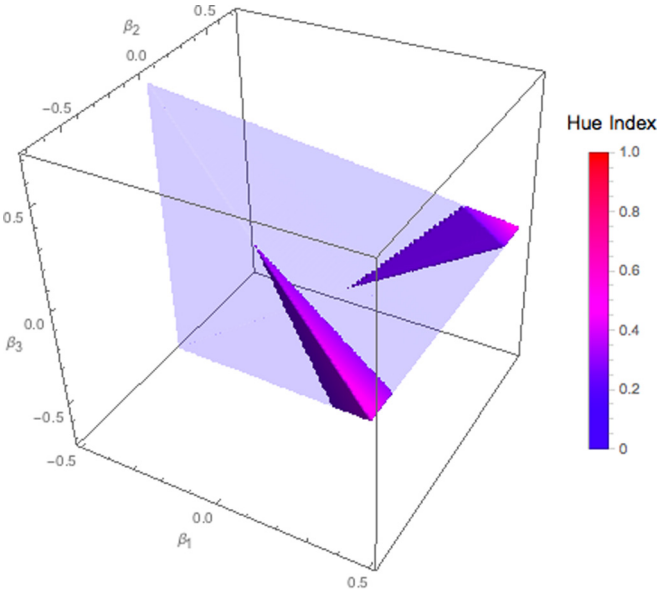

FIG. 4. Broadcasting of coherence in Bell diagonal states via local state-independent cloning. The hue index represents the trend in hue with respect to the value of nonlocal coherence $[C(\tilde{\rho}_{12})$ or $C(\tilde{\rho}_{34})]$ after normalized in a scale of 0 to 1.

TABLE II. Broadcasting ranges for state-independent nonlocal cloning in the case of Bell diagonal states in terms of β_1, β_3, and β_2 parameters.

β_1	β_3	β_2
–0.2	–0.2	$0.036 < \beta_2 \leqslant 0.212$
–0.2	–0.1	$0.136 < \beta_2 \leqslant 0.212$
–0.2	0.1	$-0.212 \leqslant \beta_2 < -0.136$
–0.2	0.2	$-0.212 \leqslant \beta_2 < -0.036$
–0.1	–0.2	$-0.283 \leqslant \beta_2 < -0.236$ and $0.036 < \beta_2 \leqslant 0.283$
–0.1	–0.1	$-0.283 \leqslant \beta_2 < -0.236$ and $0.136 < \beta_2 \leqslant 0.283$
–0.1	0.1	$-0.283 \leqslant \beta_2 < -0.136$ and $0.236 < \beta_2 \leqslant 0.283$
–0.1	0.2	$-0.283 \leqslant \beta_2 < -0.036$ and $0.236 < \beta_2 \leqslant 0.283$
0.1	–0.2	$-0.424 \leqslant \beta_2 < -0.236$ and $0.036 < \beta_2 \leqslant 0.424$
0.1	–0.1	$-0.424 \leqslant \beta_2 < -0.236$ and $0.136 < \beta_2 \leqslant 0.424$
0.1	0.1	$-0.424 \leqslant \beta_2 < -0.136$ and $0.236 < \beta_2 \leqslant 0.424$
0.1	0.2	$-0.424 \leqslant \beta_2 < -0.036$ and $0.236 < \beta_2 \leqslant 0.424$
0.2	–0.2	$-0.495 \leqslant \beta_2 < -0.236$ or $0.036 < \beta_2 \leqslant 0.495$
0.2	–0.1	$-0.495 \leqslant \beta_2 < -0.236$ or $0.136 < \beta_2 \leqslant 0.495$
0.2	0.1	$-0.495 \leqslant \beta_2 < -0.136$ or $0.236 < \beta_2 \leqslant 0.495$
0.2	0.2	$-0.495 \leqslant \beta_2 < -0.036$ or $0.236 < \beta_2 \leqslant 0.495$

The coherence of local output states is given by $C(\tilde{\rho}_{13}) = C(\tilde{\rho}_{24}) = 2\lambda$, while on the other hand, the coherence of nonlocal output states is given by, $C(\tilde{\rho}_{12}) = C(\tilde{\rho}_{34}) = \frac{1}{\sqrt{2}}[|(2\beta_2 - \beta_3)(-1+4\lambda)| + |\beta_3 - 4\beta_3\lambda|]$. To obtain the state independent version of the cloner, we substitute the value of λ by $1/10$ (see Appendix A). Now, after substituting $\lambda = 1/10$, we have $C(\tilde{\rho}_{13}) = C(\tilde{\rho}_{24}) = \frac{1}{5}$, $C(\tilde{\rho}_{12}) = C(\tilde{\rho}_{34}) = \frac{3}{5\sqrt{2}}[|(\beta_3 - 2\beta_2)| + |\beta_3|]$. In this case, the expression for the condition of broadcasting reduces to $3\sqrt{2}[|(\beta_3 - 2\beta_2)| + |\beta_3|] > 1$.

In Table II, we have shown some of the trends in broadcasting ranges obtained in terms of β_1, β_2, and β_3 after state independent nonlocal cloning transformations. We fix the values of β_1 and β_3 and substitute their respective values in the expression $3\sqrt{2}[|(\beta_3 - 2\beta_2)| + |\beta_3|] > 1$. Here, from $\beta_1 = -0.1$ onwards, discontinuities in the intervals corresponding to the ranges of the β_2 parameter are observed, which implies the presence of a no-broadcast zone in between.

In Fig. 5, the region of broadcasting is superimposed on the tetrahedral region hosting all Bell diagonal states in a β-coordinate system, with axes β_1, β_2, and β_3. The light-blue regions represent the no-broadcast zones, while the regions on the dark-blue–magenta spectrum, occupying a significant fraction of the tetrahedron's volume, highlight the regions where we can broadcast coherence. In the bluish-magenta region, the change in hue signifies how the values of the nonlocal coherence, $C(\tilde{\rho}_{12})$ or $C(\tilde{\rho}_{34})$, changes with respect to the values of the β parameters. As the value of $C(\tilde{\rho}_{12})$ or $C(\tilde{\rho}_{34})$ increases, the hue changes from blue to magenta. For instance, the concentrated presence of magenta-pink hue, especially in the regions along the tetrahedron's edge and in the regions surrounding the corners, suggests that nonlocal coherence is much higher in those regions, compared to the other regions in the broadcasting zone. Also, we find that there is a light-purple, cavitylike zone sandwiched in between the bluish-magenta regions, which points to a no-broadcast zone. This cavitylike zone accounts for discontinuities in the intervals corresponding to ranges of the β_2 parameter in Table II. Moreover, we observe in the figure that in terms of coverage of the tetrahedron's volume, the zone of broadcasting in the nonlocal cloning scenario is significantly larger when compared to that of the local cloning scenario (Fig. 4), thereby suggesting that nonlocal state-independent cloning performs much better when it comes to broadcasting of coherence in the Bell diagonal states.

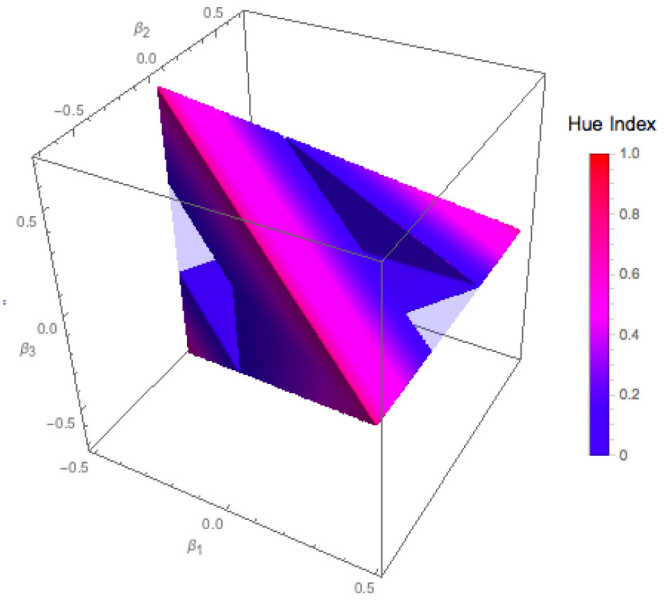

FIG. 5. Broadcasting ranges for nonlocal state-independent cloning in the case of Bell diagonal states. The hue index represents the trend in hue with respect to the value of the nonlocal coherence [$C(\tilde{\rho}_{12})$ or $C(\tilde{\rho}_{34})$] after normalized in a scale of 0–1.

IV. CONCLUSION

In short, in this work we have explored and studied the problem of broadcasting of coherence with the help of local and nonlocal cloning machines. We start with an incoherent pair of qubits along with a coherent pair of qubits as inputs to a quantum cloning machine, and after cloning, we induce coherence in the initially incoherent pair of qubits. First, we show that it is impossible to broadcast coherence optimally via cloning. However, we can broadcast coherence nonoptimally via cloning. Second, we show that in the case of nonoptimal broadcasting, the coherence induced in the output pairs of qubits is always lesser than the initial coherent pair of qubits, which was taken as input into the cloning machine. In this work, both these results have been proven with respect to the computational basis by taking the l_1 norm as a quantifier of coherence. We conjecture that the impossibility of optimal broadcasting and the possibility of nonoptimal broadcasting will hold true for any legitimate quantifier of coherence within the framework of resource theory of coherence. This is because for optimal broadcasting of coherence, by definition we must have zero coherence in the local output states with respect to any legitimate quantifier of coherence in the chosen basis. Hence, in such a scenario it is impossible for the local output states to be coherent with respect to one quantifier of coherence while being incoherent with another quantifier of coherence.

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APPENDIX A

The cloning of arbitrary quantum states is forbidden by a no-cloning theorem. However, there still exists the possibility of cloning arbitrary quantum states approximately with certain fidelity, or "imperfection." Quantum cloning transformations are (CPTP) a completely positive trace preserving map between two quantum systems, supported by an ancilla state. In this section, at first, we briefly describe the Buzek-Hillery (B-H) quantum cloning machine.

The unitary part of the B-H cloning machine (U_{bh}) is an M-dimensional quantum copying transformation acting on a state $|\Psi_i\rangle_{a_0}$ $(i = 1, \ldots, M)$, which is to be copied onto a blank state $|0\rangle_{a_1}$. Initially, the copy is prepared in state $|X\rangle_x$, which subsequently gets transformed into another set of state vectors $|X_{ii}\rangle_x$ and $|Y_{ij}\rangle_x$ as a result of application of the cloning machine. Here a_0, a_1, and x represent the input, blank, and machine qubits, respectively. In this case, these transformed state vectors belong to the orthonormal basis set in the M-dimensional space. The transformation scheme U_{bh} is given by

$$\begin{aligned} U_{bh}|\Psi_i\rangle_{a_0}|0\rangle_{a_1}|X\rangle_x \\ \rightarrow c|\Psi_i\rangle_{a_0}|\Psi_i\rangle_{a_1}|X_{ii}\rangle_x + d\sum_{j\neq i}^{M}\big(|\Psi_i\rangle_{a_0}|\Psi_j\rangle_{a_1} \\ +|\Psi_j\rangle_{a_0}|\Psi_i\rangle_{a_1}\big)|Y_{ij}\rangle_x, \end{aligned} \tag{A1}$$

where $i, j = \{1, \ldots, M\}$, and the coefficients c and d are real.

1. State independent cloning transformations

An optimal state independent version of the B-H cloner can be obtained from Eq. (A1) by imposing the unitarity and normalization conditions which give rise to the following set of constraints:

$$\begin{aligned} \langle X_{ii}|X_{ii}\rangle = \langle Y_{ij}|Y_{ij}\rangle = \langle X_{ii}|Y_{ji}\rangle = 1 \\ \langle X_{ii}|Y_{ij}\rangle = \langle Y_{ji}|Y_{ij}\rangle = \langle X_{ii}|X_{jj}\rangle = 0 \quad \text{and} \\ c^2 = \frac{2}{M+1}, \;\; d^2 = \frac{1}{2(M+1)}. \end{aligned} \tag{A2}$$

Here, $M = 2^m$ where m is the number of qubits in a given quantum register. In this transformation, the fidelity of the output copies does not depend on the input state because the shrinking or scaling property is independent of the input states.

In the case of the state-dependent cloner the unitarity constraints on the B-H cloning transformation in Eq. (A1) give rise to the following set of conditions:

$$c^2\langle X_{ii}|X_{ii}\rangle + d^2\sum_{j\neq i}^{M} 2\langle Y_{ij}|Y_{ij}\rangle = 1, \quad \langle Y_{ij}|Y_{kl}\rangle = 0, \tag{A3}$$

where $i \neq j$ and $ij \neq kl$ for $i,j,k,l = \{1, \ldots, M\}$, and the output states are orthonormal. We find that

$$c^2 = 1 - 2(M-1)d^2. \tag{A4}$$

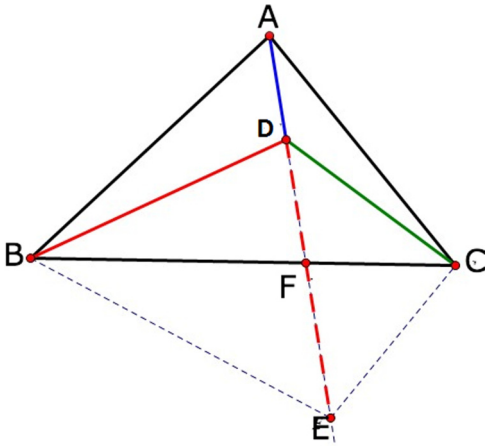

FIG. 6. ΔABC with point M in its interior.

If we substitute $\lambda = d^2$, c^2 becomes $1 - 2(M-1)\lambda$. We call λ as the machine parameter.

a. Local state-independent cloning machine

The optimal cloning machine described above with $M = 2$ becomes a local cloning machine (U^l_{bhsi}). By substituting the values of the coefficients $c = \sqrt{\frac{2}{3}}$ and $d = \sqrt{\frac{1}{6}}$ in Eq. (A1), we obtain the optimal state independent cloning machine which can be used for local copying purposes. In this case, the value of λ is $1/6$.

b. Nonlocal state-independent cloning machine

Similarly, the optimal cloning machine described above becomes a nonlocal cloning machine (U^{nl}_{bhsi}) for $M = 4$. Then the corresponding values of the coefficients c and d in Eq. (10) become $\sqrt{\frac{2}{5}}$ and $\sqrt{\frac{1}{10}}$, respectively. Now, by substituting the values of the coefficients $c = \sqrt{\frac{2}{5}}$ and $d = \sqrt{\frac{1}{10}}$ in Eq. (A1), we obtain the optimal state independent nonlocal cloning machine. In this case, the value of λ is $1/10$.

APPENDIX B

Lemma 5. Let M be a point in the interior of ΔABC. Then, $\overline{AC} + \overline{BC} > \overline{AD} + \overline{BD}$, $\overline{AB} + \overline{AC} > \overline{BD} + \overline{DC}$, and $\overline{BC} + \overline{BA} > \overline{DC} + \overline{DA}$.

Proof. As shown in Fig. 6, $\overline{AD}$ is extended to point E outside ΔABC, intersecting $\overline{BC}$ at F. By applying the triangle inequality on ΔACF and ΔBFD, we get $\overline{AC} + \overline{CF} > \overline{AF}$ and $\overline{DF} + \overline{BF} > \overline{BD}$, respectively. Also, as D lies on $\overline{AF}$, we have $\overline{AD} + \overline{DF} = \overline{AF}$.

Now, adding these two inequalities we have

$$\begin{aligned} &\overline{AC} + \overline{CF} + \overline{BF} + \overline{DF} > \overline{AF} + \overline{BD} \\ &\Rightarrow \overline{AC} + (\overline{CF} + \overline{BF}) + \overline{DF} > (\overline{AD} + \overline{DF}) + \overline{BD} \\ &\Rightarrow \overline{AC} + \overline{BC} + \overline{DF} > \overline{AD} + \overline{DF} + \overline{BD}. \end{aligned}$$

Cancelling out $\overline{DF}$, we finally get $\overline{AC} + \overline{BC} > \overline{AD} + \overline{BD}$. Similarly, the other two inequalities can be proved.

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