Multipartite analysis of average-subsystem entropies

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So-called average subsystem entropies are defined by first taking partial traces over some pure state to define density matrices, then calculating the subsystem entropies, and finally averaging over the pure states to define the average subsystem entropies. These quantities are standard tools in quantum information theory, most typically applied in bipartite systems. We shall first present some extensions to the usual bipartite analysis (including a calculation of the average tangle and a bound on the average concurrence), follow this with some useful results for tripartite systems, and finally extend the discussion to arbitrary multipartite systems. A particularly nice feature of tripartite and multipartite analyses is that this framework allows one to introduce an "environment" to which small subsystems can couple.

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I. INTRODUCTION

Entanglement in bipartite systems is completely characterized in quantum theory, but there exist very many open questions when one is seeking to quantify the entanglement among three or more subsystems; the tripartite or multipartite situations. In this article we shall focus mainly on the study of entanglement entropy (and mutual entropy), as defined among multiple subsystems when we impose the condition that the total global system is in a pure state.

An extremely useful technique in this analysis is the "average subsystem approach" championed by Page [1], whereby we shall average over all possible pure states, using this to define the average-subsystem entropies. The explicit results and bounds we have found will allow us to understand the degree of entanglement and the amount of entropy induced in making the partition into multiple subsystems.

We consider a global system partitioned into two or more subsystems, and assume the total Hilbert space factorizes as follows: $\mathcal{H} = \bigotimes_{i=1}^{N} \mathcal{H}_i$. This does not mean that the states are always separable, they can be entangled among each other, or entangled with only some of the other subsystems (provided that the Schmidt rank is greater than unity).[1] For the classification of entanglement of states in quantum information theory it is usual to use the term *k separable* for states of an N-partite system that satisfy

$$|\Psi\rangle = \bigotimes_{i=1}^{k} |\phi_i\rangle, \tag{1.1}$$

where $k \leqslant N$. The nonseparable states will be called entangled.

In order to fully quantify entanglement among different subsystems, in a $N = 2$ bipartite system it is enough (when the total system is in a pure state) to consider the von Neumann (entanglement) entropy between the two subsystems, but in the case of $N > 2$ multipartite systems this quantity does not provide us with a fully general measure of entanglement, and there is no universally agreed upon standard quantity for measuring multipartite entanglement [2–6].

Two of the quantities that have been used in the literature are, for example, the concurrence or tangle, which serves to partially quantify the entanglement in multipartite systems [7,8]. But neither of these is really a fully satisfactory quantifiable and universal measure of entanglement. For instance, using the notion of partial trace to define $\rho_A = \mathrm{tr}_B\{|\psi\rangle\langle\psi|\}$, concurrence is defined as [8]

$$\mathcal{C}(\rho_A) = \sqrt{2\big(1 - \mathrm{tr}\{\rho_A^2\}\big)}, \tag{1.2}$$

and "tangle" is defined as the square of concurrence

$$\tau = \mathcal{C}^2. \tag{1.3}$$

(Concurrence and tangle are thus relatively easy to calculate.) Is is interesting to note that concurrence and tangle are closely related to Tsallis and Renyi entropies:

$$S_{\mathrm{Tsallis}} = \frac{1 - \mathrm{tr}(\rho^q)}{q-1}, \quad S_{\mathrm{Renyi}} = \frac{\ln \mathrm{tr}(\rho^q)}{1-q}, \tag{1.4}$$

and that the von Neumann entropy is a specific case of these entropies

$$S_{\text{von Neumann}} = \lim_{q\to 1} S_{\mathrm{Tsallis}} = \lim_{q\to 1} S_{\mathrm{Renyi}}. \tag{1.5}$$

It is easy to see from Eqs. (1.2) and (1.3) that tangle can be expressed in terms of

$$\tau = \mathcal{C}^2 = 2\big[1 - \mathrm{tr}\big(\rho_A^2\big)\big] = 2S_{\mathrm{Tsallis}}(q=2). \tag{1.6}$$

The case $q = 2$ of Tsallis entropy is often called "quadratic entropy".

Other entanglement measures have been mooted, such as negativity, discord, and entanglement of formation, but they

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[1]Warning: for mathematicians, separable is a technical term that effectively means the Hilbert space is either finite dimensional or at worst has denumerable dimensionality. For physicists, separable is a statement that a particular element of the Hilbert space can be written in factorizable form. This conflict in usage is unfortunately standard.

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are either trivial or more difficult to work with in the average subsystem framework. For instance, the concept of negativity is rather subtle. One has [8]

$$\mathcal{N}(\rho) = \tfrac{1}{2}(||\rho^{T_A}|| - 1), \tag{1.7}$$

involving a partial transpose with respect to the bipartite subsystem decomposition. [If the density matrix is $\rho_{ij,kl}$ then $(\rho^{T_A})_{ij,kl} = \rho_{kj,il}$. Also note that $||X|| = \mathrm{tr}\{\sqrt{X^\dagger X}\}$.] Calculating the negativity can often be relatively difficult, though for pure states it simplifies to [9,10]

$$\mathcal{N}(\rho) = \tfrac{1}{2}([\mathrm{tr}(\sqrt{\rho_A})]^2 - 1). \tag{1.8}$$

Averaging over pure states

$$\langle \mathcal{N}(\rho)\rangle = \tfrac{1}{2}(\langle[\mathrm{tr}(\sqrt{\rho_A})]^2\rangle - 1). \tag{1.9}$$

This is not trivial, but at least tractable [9–11].

Another interesting quantity is the quantum discord, although this does not directly characterize the entanglement itself, but instead measures the extent to which the correlation are quantum as opposed to classical [12]. However, for pure states the discord reduces to the entanglement entropy, and so in the average subsystem framework we gain no extra usable information from considering the quantum discord.

Finally, another interesting measure of entanglement is given by the "entanglement of formation", which quantifies the minimum cost of preparing a state in terms of EPR pairs. It is given by

$$E_F(\rho) := \inf\left\{\sum_i p_i E(|\psi_i\rangle\langle\psi_i|) : \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|\right\}, \tag{1.10}$$

where $E(|\psi\rangle\langle\psi|) = S(\mathrm{tr}_B\{|\psi\rangle\langle\psi|\})$. This is also related to the concept of "entanglement cost". However, in the average subsystem framework the state ρ is by assumption a pure state $\rho = |\psi\rangle\langle\psi|$, and the entanglement of formation therefore trivializes to the usual von Neumann entropy $E_F(\rho) \to S(\rho_A) = S(\mathrm{tr}_B\{|\psi\rangle\langle\psi|\})$. So in the average subsystem framework we gain no extra usable information from considering the entanglement of formation.

It is also interesting to calculate the mutual information among different pairs of subsystems, or pairs of collections of subsystems, but note that this quantity is not really an entanglement measure because it merely considers the correlations between systems (that is, the decrease of uncertainty in one subsystem when we measure the other one). What is however clear is that if the mutual information between any two subsystems is zero, then there is no entanglement between them.

II. AVERAGE SUBSYSTEM ENTROPIES

In bipartite systems, the average subsystem entropies associated with the Hilbert space factorization $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ are defined by a simple three-step process [1] as follows:

(1) Take partial traces of some pure state to define two density matrices:

$$\rho_A = \mathrm{tr}_B\{|\psi\rangle\langle\psi|\}, \quad \rho_B = \mathrm{tr}_A\{|\psi\rangle\langle\psi|\}. \tag{2.1}$$

(2) Calculate the two subsystem entropies (which are equal to each other):

$$S_A = -\mathrm{tr}\{\rho_A \ln \rho_A\} = -\mathrm{tr}\{\rho_B \ln \rho_B\} = S_B. \tag{2.2}$$

(3) Average uniformly over the pure states $|\psi\rangle$ to define average entropies:

$$\langle S_A\rangle = \langle S_B\rangle. \tag{2.3}$$

By extension, in the context of a tripartite system the obvious generalization is to consider $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and modify the three-step process as follows:

(1) Take partial traces of some pure state to define *six* density matrices:

$$\rho_A = \mathrm{tr}_{BC}\{|\psi\rangle\langle\psi|\}, \quad \rho_B = \mathrm{tr}_{AC}\{|\psi\rangle\langle\psi|\},$$
$$\rho_C = \mathrm{tr}_{AB}\{|\psi\rangle\langle\psi|\}, \tag{2.4}$$

$$\rho_{AB} = \mathrm{tr}_C\{|\psi\rangle\langle\psi|\}, \quad \rho_{BC} = \mathrm{tr}_A\{|\psi\rangle\langle\psi|\},$$
$$\rho_{CA} = \mathrm{tr}_B\{|\psi\rangle\langle\psi|\}. \tag{2.5}$$

(2) Calculate *six* subsystem entropies (three of which are pairwise equal):

$$S_A = -\mathrm{tr}\{\rho_A \ln \rho_A\} = -\mathrm{tr}\{\rho_{BC} \ln \rho_{BC}\} = S_{BC}, \tag{2.6}$$

$$S_B = -\mathrm{tr}\{\rho_B \ln \rho_B\} = -\mathrm{tr}\{\rho_{CA} \ln \rho_{CA}\} = S_{CA}, \tag{2.7}$$

$$S_C = -\mathrm{tr}\{\rho_C \ln \rho_C\} = -\mathrm{tr}\{\rho_{AB} \ln \rho_{AB}\} = S_{AB}. \tag{2.8}$$

(3) Average uniformly over the pure states $|\psi\rangle$ to define average entropies:

$$\langle S_A\rangle = \langle S_{BC}\rangle, \quad \langle S_B\rangle = \langle S_{CA}\rangle, \quad \langle S_C\rangle = \langle S_{AB}\rangle. \tag{2.9}$$

In multipartite systems the basic ideas are the same, but the algebra can quickly get messy; there are up to $2^N - 2$ nontrivial ways of grouping the subsystems (the empty Hilbert space, and the full Hilbert space, will be deemed "trivial"); more on this multipartite construction below.

III. BIPARTITE ENTANGLEMENT

Returning to bipartite systems, in Ref. [1] Page established a number of interesting results regarding these average subsystem entropies. In particular, even *before* any averaging is enforced,

$$S_A = S_B \leqslant \ln \min\{n_A, n_B\}, \tag{3.1}$$

where $n_A = \dim(\mathcal{H}_A)$, and $n_B = \dim(\mathcal{H}_B)$. Page then considered the effect of taking a uniform average over all pure states on $\mathcal{H}_{AB}$.

The central result of Ref. [1] is that the average subsystem entropy is then extremely close to its maximum possible value. (The "average subsystem" is very close to being "maximally mixed".) When combined with the exact result derived by Sen in Ref. [13], wherein Sen provided a formal analytic proof of a conjecture by Page, and the discussion below, this can be strengthened to a strict lower bound on the subsystem entropy, and in the "thermodynamic limit" can be strengthened to an equality. Below we shall first present some extensions to the usual bipartite analysis, follow this with some useful results for tripartite systems, and finally extend the discussion to arbitrary multipartite systems.

A. Exact results

Page conjectured [1] and Sen proved [13] (under certain mild technical assumptions, and with minor change of their notation) that when measured in natural units (nats) [14,15], the average dimensionless entropy of a subsystem is exactly given by

$$S_{n_A,n_B} = \langle S_A\rangle = \langle S_B\rangle = H_{mM} - H_M - \frac{m-1}{2M}. \quad (3.2)$$

Here

$$m = \min\{n_A,n_B\}, \quad M = \max\{n_A,n_B\}, \quad (3.3)$$

where H_n is the nth harmonic number $H_n = \sum_{i=1}^n \frac{1}{n}$; see, for instance, Ref. [16].

Writing $n = n_A n_B$ for the total dimensionality, we note that $m \leqslant \sqrt{n}$ and $M \geqslant \sqrt{n}$, so that the average subsystem entropy can also be written as

$$S_{n_A,n_B} = \langle S_A\rangle = \langle S_B\rangle = H_n - H_{n/m} - \frac{m(m-1)}{2n}, \quad (3.4)$$

with $m = \min\{n_A,n_B\} \leqslant \sqrt{n}$. For example,

$$S_{1,n_B} = 0, \quad S_{2,n_B} = H_{2n_B-1} - H_{n_B},$$
$$S_{3,n_B} = H_{3n_B} - H_{n_B} - \frac{1}{n_B}, \quad (3.5)$$

and for $n_B \geqslant 4$ we have

$$S_{4,n_B} = H_{4n_B} - H_{n_B} - \frac{3}{2n_B}. \quad (3.6)$$

That the harmonic numbers show up here should in retrospect perhaps not be so surprising. Even in a classical statistical context the harmonic numbers arise in many situations where one is extremizing the Shannon entropy, $S = -\sum_{i=1}^n p_i \ln p_i$, subject to external constraints. For example, the harmonic numbers also show up in finite-space models for Zipf's law, where $p_i = 1/(i H_n)$ and $\sum_{i=1}^n p_i = 1$ [17]. These harmonic numbers are well studied in mathematics. Note, for instance, the standard mathematical asymptotic result [16]

$$H_n = \gamma + \ln n + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \frac{1}{256n^6} + O(n^{-8}). \quad (3.7)$$

There is also an explicit bound [16]

$$H_n = \gamma + \ln n + \epsilon_n, \quad \text{where } \epsilon_n \in \left(\frac{1}{2(n+1)}, \frac{1}{2n}\right). \quad (3.8)$$

Here, in view of the nonoverlapping nature of the bounds on ϵ_n, we know that ϵ_n is monotone decreasing. Stronger bounds are known; for instance, the Franel bound

$$H_n = \gamma + \ln n + \frac{1}{2n} - \frac{\hat{\epsilon}_n}{8n^2}, \quad \text{where } \hat{\epsilon}_n \in (0,1), \quad (3.9)$$

or the result

$$H_n = \gamma + \ln n + \frac{1}{2n} - \frac{1}{12n^2} + \frac{\hat{\epsilon}_n}{120n^4}, \quad \text{where } \hat{\epsilon}_n \in (0,1). \quad (3.10)$$

(Yet even stronger bounds on the harmonic numbers are known, but would be overkill for current purposes.)

Sometimes it is sufficient to consider the less stringent result [16]

$$H_n = \ln n + \tilde{\epsilon}_n, \quad \text{where } \tilde{\epsilon}_n \in \left(\frac{1}{n}, 1\right). \quad (3.11)$$

The purely mathematical explicit bound in Eq. (3.8), when combined with the exact result derived by Sen in Ref. [13], can be strengthened to a strict bound on the bipartite entropy

$$S_{n_A,n_B} = \langle S_A\rangle = \langle S_B\rangle \in \left(\ln m - \tfrac{1}{2},\ \ln m\right). \quad (3.12)$$

That is, the average subsystem entropy is always within $\frac{1}{2}$ nat (less than $\frac{1}{2\ln 2} < \frac{3}{4}$ of a bit) of its maximum possible value. Let us now formalize this statement.

Theorem:

$$S_{n_A,n_B} = \ln m + \Delta_{m,M},$$
$$\Delta_{m,M} \in \left(-\frac{m}{2M}, -\frac{m-1}{2M}\right) \subseteq \left(-\frac{1}{2},0\right). \quad (3.13)$$

(Page says something somewhat similar in Ref. [1] but only as an estimate; this is now a rigorous bound. There is no obvious way of *cleanly* improving this bound.)

Proof. To establish this bound, starting from the bound in Eq. (3.8), we write

$$S_{n_A,n_B} = \ln m + \Delta_{m,M},$$
$$\text{with } \Delta_{m,M} = \epsilon_{mM} - \epsilon_M - \frac{m-1}{2M}. \quad (3.14)$$

Now since ϵ_n is monotonically decreasing we certainly have

$$\epsilon_{mM} - \epsilon_M - \frac{m-1}{2M} \leqslant -\frac{m-1}{2M}. \quad (3.15)$$

But we also have an absolute lower bound

$$\begin{aligned}\epsilon_{mM} - \epsilon_M - \frac{m-1}{2M} &> \frac{1}{2(mM+1)} - \frac{1}{2M} - \frac{m-1}{2M}\\ &= \frac{1}{2(mM+1)} - \frac{m}{2M}\\ &> -\frac{m}{2M} > -\frac{1}{2}. \end{aligned} \quad (3.16)$$

That is,

$$S_{n_A,n_B} = \ln m + \Delta_{m,M},$$
$$\text{with } \Delta_{m,M} \in \left(-\frac{m}{2M}, -\frac{m-1}{2M}\right) \subset \left(-\frac{1}{2},0\right). \quad (3.17)$$

This is the result we were seeking. ■

B. Symmetric average subsystem information

In terms of the (symmetric) average subsystem information (as defined by Page in Ref. [1]), we now have

$$\begin{aligned} I_{n_A,n_B} &= S_{\max;n_A,n_B} - S_{n_A,n_B}\\ &= \ln m - S_{n_A,n_B} = -\Delta_{m,M}. \end{aligned} \quad (3.18)$$

We then have the rigorous bounds

$$I_{n_A,n_B} \in \left(\frac{m-1}{2M}, \frac{m}{2M}\right) \subseteq \left(0, \frac{1}{2}\right). \quad (3.19)$$

That is, the (symmetric) average subsystem information is always less than $\frac{1}{2}$ nat, which is less than $\frac{1}{2\ln 2} < \frac{3}{4}$ of a bit; thus

this (symmetric) definition of average subsystem information leads to a very tight bound.

C. Asymmetric average subsystem information

In contrast, in the follow-up reference [18], Page defines (redefines) the average subsystem information in an *asymmetrical* manner:

$$\begin{aligned}\tilde{I}_{n_A,n_B} &= S_{\max;n_A} - S_{n_A,n_B} = \ln n_A - S_{n_A,n_B}\\ &= \ln(n_A/m) + I_{n_A,n_B}\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}\tilde{I}_{n_B,n_A} &= S_{\max;n_B} - S_{n_A,n_B} = \ln n_B - S_{n_A,n_B}\\ &= \ln(n_B/m) + I_{n_A,n_B}.\end{aligned} \tag{3.21}$$

Then for the difference we always have

$$\tilde{I}_{n_A,n_B} - \tilde{I}_{n_B,n_A} = \ln(n_A/n_B), \tag{3.22}$$

while for the average

$$\bar{I}_{n_A,n_B} = \tfrac{1}{2}\left(\tilde{I}_{n_A,n_B} + \tilde{I}_{n_B,n_A}\right) = \ln(M) + I_{n_A,n_B}. \tag{3.23}$$

We note that the average $\bar{I}_{n_A,n_B} \approx \ln M$ is symmetric, and, since $I_{n_A,n_B} \in (0,\frac{1}{2})$, it is utterly dominated by the dimensionality of the larger Hilbert space. In view of the very tight bound on I_{n_A,n_B}, this means that (to within $\frac{1}{2}$ nat) for all practical purposes we have

$$\tilde{I}_{n_A,n_B} \approx \ln\left(\frac{n_A}{m}\right) = \ln\left(\frac{n_A}{\min\{n_A,n_B\}}\right) \tag{3.24}$$

and

$$\tilde{I}_{n_B,n_A} \approx \ln\left(\frac{n_B}{m}\right) = \ln\left(\frac{n_B}{\min\{n_A,n_B\}}\right). \tag{3.25}$$

That is, the modified average subsystem information, $\tilde{I}_{n_A,n_B} \neq \tilde{I}_{n_B,n_A}$, really says nothing much about the subsystem beyond specifying the dimensionalities of the two Hilbert subspaces. (Specifically, Page's asymmetric subsystem information, $\tilde{I}_{n_A,n_B} \neq \tilde{I}_{n_B,n_A}$, contains at most $\frac{1}{2}$ a nat of "useful" information.)

D. Mutual information and other measures of entanglement

It should be emphasized that mutual information is certainly not the same as what Page calls the subsystem information. (See discussion above, and Refs. [1,18], for details.) In general (using industry standard terminology) one has

$$I_{A:B} = S_A + S_B - S_{AB}. \tag{3.26}$$

For the bipartite system considered by Page, where the total system is in a pure state, one has $S_A = S_B$ and $S_{AB} = 0$, so yielding the particularly simple result

$$I_{A:B} = 2S_A = 2S_B. \tag{3.27}$$

More specifically, after applying the "average subsystem" argument

$$\langle I_{A:B}\rangle = 2\langle S_A\rangle = 2\langle S_B\rangle \approx 2\ \ln\min\{n_A,n_B\}. \tag{3.28}$$

While at first glance this seems uninteresting, when combined with Page's asymmetric subsystem information this leads to

$$\begin{aligned}\langle\tilde{I}_{A,B}\rangle + \langle\tilde{I}_{B,A}\rangle + \langle I_{A:B}\rangle \approx{}& \ln\left(\frac{n_A}{\min\{n_A,n_B\}}\right)\\ &+ \ln\left(\frac{n_B}{\min\{n_A,n_B\}}\right)\\ &+ 2\ \ln\min\{n_A,n_B\},\end{aligned} \tag{3.29}$$

from which we obtain the approximate sum rule

$$\langle\tilde{I}_{A,B}\rangle + \langle\tilde{I}_{B,A}\rangle + \langle\hat{I}_{A:B}\rangle \approx \ln(n_A n_B) \approx \ln n_{AB}. \tag{3.30}$$

Here this approximation is now valid to within $\frac{3}{2}$ nat.

In counterpoint, while in bipartite systems the entanglement is well determined in terms of the von Neumann entropy, it is possible also to bound the concurrence, and to actually calculate the average tangle. Let $n = n_A n_B$, while $m = \min\{n_A,n_B\}$ and $M = \max\{n_A,n_B\}$, so $n = mM$. Then it is known that [1,19]

$$\begin{aligned}\langle\mathrm{tr}(\rho_A^2)\rangle &= \frac{n_A+n_B}{n+1} = \frac{n_A+n_B}{n_A n_B+1}\\ &= \frac{m+M}{n+1} = \frac{m+M}{mM+1}.\end{aligned} \tag{3.31}$$

Thus the averaged tangle is given by

$$\langle\tau\rangle = 2\left(1 - \frac{m+M}{mM+1}\right) = \frac{2(m-1)(M-1)}{mM+1}. \tag{3.32}$$

Now note that for the concurrence $\langle\mathcal{C}\rangle^2 \leqslant \langle\mathcal{C}^2\rangle = \langle\tau\rangle$, so we certainly have the bound

$$\langle\mathcal{C}\rangle \leqslant \sqrt{\frac{2(m-1)(M-1)}{mM+1}}. \tag{3.33}$$

E. Thermodynamic limit—bipartite

The usual classical thermodynamic limit is the infinite volume limit, and the closest one can get to this notion in a quantum von Neumann context is to let one Hilbert space factor become arbitrarily large, while the other remains fixed. (Specifically let $m = \min\{n_A,n_B\}$ be held fixed, while $M = \max\{n_A,n_B\} \to \infty$.) In that limit the smaller average subsystem achieves maximum entropy (maximal mixing). We can state this more formally as follows.

Theorem:

$$\lim_{M\to\infty} S_{n_A,n_B} = \lim_{M\to\infty}\langle S_A\rangle = \lim_{M\to\infty}\langle S_B\rangle = \ln m. \tag{3.34}$$

Proof. We note the standard mathematical result

$$\lim_{M\to\infty} H_{mM} - H_M = \ln m. \tag{3.35}$$

But then

$$\lim_{M\to\infty} S_{n_A,n_B} = \lim_{M\to\infty}\left\{H_{mM} - H_M - \frac{m-1}{2M}\right\} = \ln m, \tag{3.36}$$

as claimed. ■

Thus, in the thermodynamic limit of the average subsystem approach, the finite-dimensional subsystem is always

maximally entangled with the infinite-dimensional subsystem. It is also possible to calculate the averaged tangle in this thermodynamic $M \to \infty$ limit, finding that

$$\lim_{M\to\infty} \langle \tau \rangle = \frac{2(m-1)}{m} = 2\left(1 - \frac{1}{m}\right). \tag{3.37}$$

Specifically, the "tangle deficit," the deviation from maximal tangle, is

$$\Delta\tau = \lim_{M\to\infty} \langle \tau \rangle - \langle \tau \rangle = 2\left(1 - \frac{1}{m}\right) - \frac{2(m-1)(M-1)}{mM+1}$$
$$= \frac{2(m^2-1)}{m(mM+1)} \leqslant \frac{2}{M}. \tag{3.38}$$

So the "average tangle" is always within $\frac{2}{M}$ nat of maximal mixing. Similarly, in the thermodynamic limit the average concurrence is bounded by

$$\lim_{M\to\infty} \langle \mathcal{C} \rangle \leqslant \sqrt{2\left(1 - \frac{1}{m}\right)}. \tag{3.39}$$

F. Wrap up

While the mathematical validity of the bipartite analysis is unassailable, in certain circumstances the physical relevance of the input assumptions can be questionable. In particular, the fact that the total system is always taken to be a pure state is not always entirely physically appropriate, which is one reason why we now turn to a tripartite analysis.

We emphasize this point: consideration of a global pure state can be very useful for some physical systems, but in some cases we cannot simply divide those systems into *only* two isolated subsystems. It might then be necessary to consider some overall encompassing environment, or to consider more than two subsystems into which the (total) Hilbert space is to be factorized, thereby making the system separable in that more subtle sense.

IV. TRIPARTITE ENTANGLEMENT

Let us now consider a tripartite system, to be modeled by the Hilbert space $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$. Let us first see how far we can get without making any averaging assumptions. Take the entire universe to be in a pure state, so at all times $S_{ABC} = 0$ and the subsystem entropies satisfy

$$S_A = S_{BC}, \quad S_B = S_{AC}, \quad S_C = S_{AB}. \tag{4.1}$$

For the average entanglement entropy, we now have

$$\langle S_A \rangle = \langle S_{BC} \rangle \approx \ln \min\{n_A, n_B n_C\}, \tag{4.2}$$

$$\langle S_B \rangle = \langle S_{CA} \rangle \approx \ln \min\{n_B, n_C n_A\}, \tag{4.3}$$

$$\langle S_C \rangle = \langle S_{AB} \rangle \approx \ln \min\{n_C, n_A n_B\}, \tag{4.4}$$

with all three of these approximations holding to within $\frac{1}{2}$ a nat. This now allows us to write

$$\langle S_A + S_B + S_C \rangle \approx \min\{\ln n_A, \ln n_B + \ln n_C\} + \min\{\ln n_B, \ln n_C + \ln n_A\} + \min\{\ln n_C, \ln n_A + \ln n_B\}. \tag{4.5}$$

For convenience, temporarily permute $\{A,B,C\}$ so that $n_A \leqslant n_B \leqslant n_C$. Then we have

$$\langle S_A + S_B + S_C \rangle \approx \ln n_A + \ln n_B + \min\{\ln n_C, \ln n_A + \ln n_B\}, \tag{4.6}$$

which implies

$$\langle S_A + S_B + S_C \rangle \approx \ln n_A + \ln n_B + \min\{\ln n_C, \ln n - \ln n_C\}, \tag{4.7}$$

whence we deduce

$$\langle S_A + S_B + S_C \rangle \approx \ln n + \min\{0, \ln n - 2 \ln n_C\}. \tag{4.8}$$

Undoing the permutation we see

$$\langle S_A + S_B + S_C \rangle \approx \ln n + \min\{0, \ln n - 2 \max\{\ln n_A, \ln n_B, \ln n_C\}\}. \tag{4.9}$$

This approximation for the sum of subsystem entropies is related to the existence of bounds on variable-length compound jumps [20].

A. "Rest of the universe"—the environment

Now suppose one subsystem is much larger than the other two. Specifically let subsystem C denote the environment (the "rest of the universe"), while subsystems A and B are free to interact with each other (and, for now, are free to interact with the environment). Specifically let us assume that $n_C \geqslant n_A n_B$. This implies both $n_C \geqslant n_A$ and $n_C \geqslant n_B$, which furthermore implies both $n_B n_C \geqslant n_A$ and $n_C n_A \geqslant n_B$. So in this situation

$$\langle S_A \rangle = \langle S_{BC} \rangle \approx \ln n_A, \quad \langle S_B \rangle = \langle S_{CA} \rangle \approx \ln n_B,$$
$$\langle S_C \rangle = \langle S_{AB} \rangle \approx \ln\{n_A n_B\}. \tag{4.10}$$

That is, $\langle S_C \rangle$ is *not* the total entropy of the rest of the universe, it is merely the extent to which the rest of the universe is entangled with the AB subsystem.

B. Mutual information

For the tripartite ABC system we are advocating here the situation is more interesting than for the bipartite AB system. For the tripartite system $S_{AB} = S_C$ (and $S_A \neq S_B$ in general) so

$$I_{A:B} = S_A + S_B - S_{AB} = S_A + S_B - S_C. \tag{4.11}$$

Now averaging over the pure states in ABC, we have

$$\langle I_{A:B} \rangle = S_{n_A, n_B n_C} + S_{n_B, n_A n_C} - S_{n_C, n_A n_B}. \tag{4.12}$$

So in the situation where C is a suitably large environment, $n_C \geqslant n_A n_B$ as discussed above, and using the harmonic numbers H_n as introduced above, we have the exact result

$$\langle I_{A:B} \rangle = \left[H_{n_A n_B n_C} - H_{n_A n_C} - \frac{n_A - 1}{2 n_B n_C}\right] + \left[H_{n_A n_B n_C} - H_{n_B n_C} - \frac{n_B - 1}{2 n_B n_C}\right] - \left[H_{n_A n_B n_C} - H_{n_C} - \frac{n_A n_B - 1}{2 n_C}\right]. \tag{4.13}$$

Then after a little simplification

$$\langle I_{A:B}\rangle = H_{n_A n_B n_C} + H_{n_C} - H_{n_A n_C} - H_{n_B n_C} + \frac{(n_A-1)(n_B-1)(n_A n_B + n_A + n_B)}{2 n_A n_B n_C}. \quad (4.14)$$

It is now relatively easy to see that

$$\langle I_{A:B}\rangle \leqslant \tfrac{1}{2}. \quad (4.15)$$

So in the tripartite ABC system the average mutual information between the two "small" subsystems A and B, never exceeds $\frac{1}{2}$ nat. More formally we have the following.

Theorem: Provided $n_A n_B \leqslant n_C$ we have

$$\langle I_{A:B}\rangle \leqslant \frac{n_A n_B}{2 n_C} \leqslant \frac{1}{2}. \quad (4.16)$$

Proof. We start from

$$\langle I_{A:B}\rangle = H_{n_A n_B n_C} + H_{n_C} - H_{n_A n_C} - H_{n_B n_C} + \frac{(n_A-1)(n_B-1)(n_A n_B + n_A + n_B)}{2 n_A n_B n_C} \quad (4.17)$$

and again use

$$H_n = \gamma + \ln n + \epsilon_n, \quad \text{where } \epsilon_n \in \left(\frac{1}{2(n+1)}, \frac{1}{2n}\right). \quad (4.18)$$

Then the ln's and γ's cancel and

$$\langle I_{A:B}\rangle = \epsilon_{n_A n_B n_C} + \epsilon_{n_C} - \epsilon_{n_A n_C} - \epsilon_{n_B n_C} + \frac{(n_A-1)(n_B-1)(n_A n_B + n_A + n_B)}{2 n_A n_B n_C}. \quad (4.19)$$

But then

$$\langle I_{A:B}\rangle \leqslant \frac{1}{2 n_A n_B n_C} + \frac{1}{2 n_C} + \frac{(n_A-1)(n_B-1)(n_A n_B + n_A + n_B)}{2 n_A n_B n_C}. \quad (4.20)$$

That is

$$\langle I_{A:B}\rangle \leqslant \frac{n_A^2 n_B^2 - n_A^2 - n_B^2 + n_A + n_B + 1}{2 n_A n_B n_C}. \quad (4.21)$$

We can rewrite this as

$$\langle I_{A:B}\rangle \leqslant \frac{n_A^2 n_B^2}{2 n_A n_B n_C} - \frac{n_A^2 + n_B^2 - n_A - n_B - 1}{2 n_A n_B n_C}, \quad (4.22)$$

that is

$$\langle I_{A:B}\rangle \leqslant \frac{n_A n_B}{2 n_C} - \frac{n_A^2 + n_B^2 - n_A - n_B - 1}{2 n_A n_B n_C}, \quad (4.23)$$

whence

$$\langle I_{A:B}\rangle \leqslant \frac{n_A n_B}{2 n_C} - \frac{n_A(n_A-1) + n_B(n_B-1) - 1}{2 n_A n_B n_C}. \quad (4.24)$$

Now consider the following two cases

(i) If *either* $n_A > 1$ *or* $n_B > 1$, then we have $n_A(n_A-1) + n_B(n_B-1) - 1 > 0$, and so certainly

$$\langle I_{A:B}\rangle \leqslant \frac{n_A n_B}{2 n_C} \leqslant \frac{1}{2}. \quad (4.25)$$

(ii) If *both* $n_A = 1$ *and* $n_B = 1$, then we go back to the exact result of Eq. (4.17) and note that $\langle I_{A:B}\rangle \to H_{n_C} + H_{n_C} - H_{n_C} - H_{n_C} + 0 = 0$.

In either case we certainly have

$$\langle I_{A:B}\rangle \leqslant \frac{n_A n_B}{2 n_C} \leqslant \frac{1}{2}. \quad (4.26)$$

So the average mutual information between the two small subsystems A and B in the tripartite pure-state ABC system never exceeds $\frac{1}{2}$ nat (as long as the subsystem C is dominant in the sense that $n_A n_B \leqslant n_C$). ■

Note that the mutual information is a measure of the uncertainty remaining in one subsystem when the other one is measured, and can be defined for both classical and quantum systems. Entanglement however is a purely quantum concept, and so it cannot be completely characterized by the mutual information. It is important to remember that in pure bipartite systems the entanglement is directly characterized by the von Neumann entropy, but not so in tripartite (multipartite) systems [21].

Indeed, it is not entirely clear how to formulate a specific and practical measure of the entanglement in multipartite systems that are high-dimensional (greater than dimension two). None of the quantities explained above seem fully adequate to this case, but what can certainly be said is this: the measure of global correlations (that is, the mutual entropy) is very small, so it is expected that the entanglement between systems will be also very small. We will check this specifically in the case of thermodynamic limit.

What, in counterpoint, can we say about $\langle I_{A:C}\rangle$ and $\langle I_{B:C}\rangle$, the mutual information between A or B with the environment C? We note that as long as ABC is a pure state we have (even before averaging)

$$I_{A:C} = S_A + S_C - S_{AC} = S_A + S_C - S_B,$$
$$I_{B:C} = S_B + S_C - S_{BC} = S_B + S_C - S_A. \quad (4.27)$$

But we already know that after averaging

$$\langle S_A\rangle + \langle S_B\rangle \approx \langle S_C\rangle \quad \text{(to within 1 nat)}. \quad (4.28)$$

So we see

$$\langle I_{A:C}\rangle \approx 2\langle S_A\rangle, \quad \langle I_{B:C}\rangle \approx 2\langle S_B\rangle \quad \text{(to within 1 nat)}. \quad (4.29)$$

So *these* particular mutual information scenarios do not yield any extra useful insight.

We can also lump two of the subsystems together, and calculate the mutual bipartite information, obtaining as in the previous case that (even before averaging)

$$I_{A:(BC)} = 2S_A, \quad I_{B:(CA)} = 2S_B,$$
$$I_{C:(AB)} = 2S_C = 2S_{AB}, \quad (4.30)$$

thereby verifying that (overall) we have a completely entangled system.

C. Thermodynamic limit—tripartite

For the bipartite AB system, the whole point (usually) is to keep the total dimensionality n_{AB} fixed, while letting the A and B subsystems trade dimensionality with each other. For the tripartite ABC system, however, the environment C

(the rest of the universe) is used to initially entangle the AB subsystem with the rest of the universe, but then largely "comes along for the ride" (as long as $n_C \geqslant n_A n_B$). So there is no real loss of generality in taking the limit $n_C \to \infty$. This does not mean we are making any restrictive assumptions concerning the actual thermodynamic entropy of the rest of the universe; it is a much milder statement that the rest of the universe could in principle have an arbitrarily high-dimensional Hilbert space. Under these conditions we have (at all times) the following limits:

$$\lim_{n_C\to\infty}\langle S_A\rangle = \ln n_A, \quad \lim_{n_C\to\infty}\langle S_B\rangle = \ln n_B, \tag{4.31}$$

which is the maximum entropy compatible with dimensionality, and

$$\lim_{n_C\to\infty}\langle S_C\rangle = \ln(n_A n_B), \tag{4.32}$$

which is the maximum entropy compatible with the total system being a pure state. In this limit we therefore have the *equality*

$$\lim_{n_C\to\infty}(\langle S_A\rangle + \langle S_B\rangle) = \lim_{n_C\to\infty}\langle S_C\rangle, \tag{4.33}$$

an equality which (in this limit) reproduces the classical thermodynamic arguments. An immediate consequence of this result is

$$\lim_{n_C\to\infty}\langle I_{A:B}\rangle = 0. \tag{4.34}$$

That is, for an infinite dimensional environment C, the mutual information between the subsystems A and B in a pure-state ABC system is zero. The fact that things simplify so nicely for an infinite dimensional environment should perhaps not be all that surprising in view of the fact that even in purely classical thermodynamics an infinite volume limit (infinite degrees of freedom) is necessary for the existence of phase transitions. In counterpoint, an infinite dimensional environment is also necessary if for some reason one wishes to drive the Shannon entropy to infinity [22]. On the other hand, it is well known that when the mutual entropy is zero, the subsystems are completely independent, that is, we can confirm that the entanglement between these two subsystems is zero.

V. MULTIPARTITE ENTANGLEMENT

We now seek to further generalize these considerations to explore a generic multipartite context. Many results carry over (with minor increase in algebraic complexity) from the bipartite and tripartite results.

A. Framework

For multipartite decompositions the basic idea is to write

$$\mathcal{H} = \bigotimes_{i=1}^{N}\mathcal{H}_i, \quad n_i = \dim(\mathcal{H}_i), \quad n = \dim(\mathcal{H}), \tag{5.1}$$

and then define partial traces

$$\rho_i = \mathrm{tr}_{(\text{all subspaces except } \mathcal{H}_i)}\,\rho = \mathrm{tr}_{\{\mathcal{H}/\mathcal{H}_i\}}\rho, \tag{5.2}$$

or even more generally

$$\begin{aligned}\rho_{ijk\ldots} &= \mathrm{tr}_{(\text{all subspaces except } \mathcal{H}_i,\,\mathcal{H}_j,\,\mathcal{H}_k,\,\ldots)}\,\rho \\ &= \mathrm{tr}_{\{\mathcal{H}/(\mathcal{H}_i\otimes\mathcal{H}_j\otimes\mathcal{H}_k\otimes\ldots)\}}\rho.\end{aligned} \tag{5.3}$$

The Page-Sen result [1,13] translates (at the most elementary level) to the statement that for each individual i we have

$$\langle S(\rho_i)\rangle = S_{n_i,n/n_i} = H_{m_i M_i} - H_{M_i} - \frac{m_i - 1}{2M_i}. \tag{5.4}$$

Here in the obvious manner

$$m_i = \min\{n_i, n/n_i\}, \quad M_i = \max\{n_i, n/n_i\}, \tag{5.5}$$

and where H_n is the nth harmonic number [16]. Note we always have $m_i M_i = n$, so that we can deduce

$$\langle S(\rho_i)\rangle = S_{n_i,n/n_i} = H_n - H_{n/m_i} - \frac{m_i(m_i-1)}{2n}, \tag{5.6}$$

with $m_i = \min\{n_i, n/n_i\}$.

More generally, we also have

$$\begin{aligned}\langle S(\rho_{ijk\ldots})\rangle &= S_{n_{ijk\ldots},n/n_{ijk\ldots}} \\ &= H_{m_{ijk\ldots}M_{ijk\ldots}} - H_{M_{ijk\ldots}} - \frac{m_{ijk\ldots}-1}{2M_{ijk\ldots}},\end{aligned} \tag{5.7}$$

where now we define

$$m_{ijk\ldots} = \min\left\{n_{ijk\ldots}, \frac{n}{n_{ijk\ldots}}\right\}, \quad M_{ijk\ldots} = \max\left\{n_{ijk\ldots}, \frac{n}{n_{ijk\ldots}}\right\}. \tag{5.8}$$

Again $M_{ijk\ldots}m_{ijk\ldots} = n$ so that

$$\begin{aligned}\langle S(\rho_{ijk\ldots})\rangle &= S_{n_{ijk\ldots},n/n_{ijk\ldots}} \\ &= H_n - H_{n/m_{ijk\ldots}} - \frac{m_{ijk\ldots}(m_{ijk\ldots}-1)}{2n},\end{aligned} \tag{5.9}$$

where $m_{ijk\ldots}$ is as defined above.

B. Bounds

One obvious comment, based on the bipartite analysis, is that for any collection $ijk\ldots$ of subsystems with collective dimensionality $n_{ijk\ldots}$ and $m_{ijk\ldots} = \min\{n_{ijk\ldots}, n/n_{ijk\ldots}\}$ we have the rigorous bound

$$\begin{aligned}\langle S(\rho_{ijk\ldots})\rangle &= S_{n_{ijk\ldots},n/n_{ijk\ldots}} = \ln(m_{ijk\ldots}) + \Delta, \\ \Delta &\in \left(-\frac{1}{2}, 0\right).\end{aligned} \tag{5.10}$$

So we see that any collection of these average subsystems is close to being maximally mixed, in fact within $\frac{1}{2}$ a nat of maximal mixing.

C. Mutual information

The mutual information is in principle easy to deal with, just algebraically messy.

Let us define three disjoint subsystems ABC as follows:

$$\begin{aligned}A &: \mathcal{H}_{ijk\ldots} = (\mathcal{H}_i \otimes \mathcal{H}_j \otimes \mathcal{H}_k \otimes \ldots), \\ B &: \mathcal{H}_{pqr\ldots} = (\mathcal{H}_p \otimes \mathcal{H}_q \otimes \mathcal{H}_r \otimes \ldots),\end{aligned} \tag{5.11}$$

where none of the indices overlap, and then define C by setting

$$C : \mathcal{H}_{(ijk\ldots pqr\ldots)^*} = \mathcal{H}/(\mathcal{H}_{ijk\ldots} \otimes \mathcal{H}_{pqr\ldots}). \tag{5.12}$$

Then we can immediately apply the tripartite analysis to this situation. In particular, as long as A and B are "small" and C is "large" for the mutual information we certainly have

$$\langle I_{A:B}\rangle = \langle I_{(ijk\ldots):(pqr\ldots)}\rangle \leqslant \frac{n_{ijk\ldots}n_{pqr\ldots}}{2n_{(ijk\ldots pqr\ldots)^*}} = \frac{n_{ijk\ldots}^2 n_{pqr\ldots}^2}{2n} \leqslant \frac{1}{2}. \tag{5.13}$$

So the average mutual information between any two small collections of subsystems, A and B, in the multipartite pure-state system never exceeds $\frac{1}{2}$ nat as long as the subsystem collection C is dominant in the sense that $n_{ijk\ldots}n_{jkl\ldots} \leqslant n_{(ijk\ldots pqr\ldots)^*}$. More prosaically by small collections we mean that, when considering the collections A, B, and C, the product of the two total dimensions of the small collections is less than or equal to the dimension of whatever is left over: $n_A n_B \leqslant n_C = n_{\text{total}}/(n_A n_B)$. That is, for small collections we require $n_A^2 n_B^2 \leqslant n_{\text{total}}$. As promised the only slightly tricky thing is keeping track of all the indices.

Note that this result is only applicable to pairs[2] of small collections of subsystems; pairs of large collections (and also any small collection when compared with a large collection) will have some degree of entanglement, often close to maximal. Overall, the system is not separable at all.

D. Thermodynamic limit—multipartite

In the thermodynamic limit, letting $n \to \infty$ while keeping $n_{ijk\ldots}$ fixed, we see

$$\lim_{n\to\infty} \langle S(\rho_{ijk\ldots})\rangle = \lim_{n\to\infty} S_{n_{ijk\ldots},n/n_{ijk\ldots}} = \ln(n_{ijk\ldots}). \tag{5.14}$$

So we see that, in the thermodynamic limit, any collection of these average subsystems is exactly maximally mixed, and maximally entangled with the complementary collection of average subsystems (not just close to maximal). Indeed in the thermodynamic limit the mutual information between any collection of (nonoverlapping) average subsystems vanishes:

$$\lim_{n\to\infty} \langle I_{A:B}\rangle = \lim_{n\to\infty} \langle I_{(ijk\ldots):(pqr\ldots)}\rangle = 0. \tag{5.15}$$

As in the tripartite case, we can affirm that the entanglement between pairs of small collections of subsystems is zero in the thermodynamic limit. That is, we can always choose a pair of small collections of subsystems of the overall entangled system that will be completely unentangled.

VI. DISCUSSION

In this article we have studied so-called average subsystem entropies [1,10,13,18,19] in bipartite, tripartite, and multipartite scenarios. This analysis takes the "universe" to be in a random pure state, splits the universe into subsystems which are *not* pure (and so have nonzero subsystem entropies), and averages over the pure states. This model for subsystem entropies has been found to be a useful one in many different contexts. The most common applications found in the literature are for bipartite systems [1,10,13,18,19], where subsystem entropies are typically within 1 nat of maximal mixing, but we have argued herein that it is often more useful to look at tripartite or even multipartite decompositions of the universe. (Indeed multipartite decompositions have attracted and continue to attract considerable attention [3,4,6,12,21,24,25].)

Tripartite analyses are particularly useful in that they allow one to introduce a notion of environment for the other two subsystems to interact with. In the tri-partite context the situation is cleanest when two subsystems are small compared to the third (specifically, $n_A n_B \leqslant n_C$). In this situation the subsystems A and B (and even AB) are close to maximally entangled with C, while the entanglement between A and B (as measured by average mutual information) is utterly minimal (less than 1 nat). The thermodynamic limit ($n_C \to \infty$, while $n_A n_B$ is held fixed) is particularly well behaved, with the subsystems A and B (and even AB) maximally entangled with C, while the entanglement between A and B (as measured by average mutual information) is zero.

In the multipartite context it is convenient to lump the individual subsystems into "collections" A, B, and C. As long as two collections are small compared to the third, then the tripartite analysis sketched above will still hold, including the existence of the thermodynamic limit.

While we were originally inspired to consider these ideas based on an analysis of the entropy budget in the Hawking radiation process [26–28], and also in a cosmological context [29,30], wherein it can be seen that the quantum consideration of the rest of the universe can be crucial, absolutely nothing in the current article specifically depends on the physics of black holes or any cosmological model (neither general relativity black holes nor analog black holes [31–40]; neither experimental [41–52] nor theoretical [53–61]), and nothing herein depends on any specific aspect of the Hawking evaporation process—such as the distinction between event and apparent horizons [62–65]. The use of bipartite, tripartite, and multipartite decompositions of Hilbert spaces, and the use of the average subsystem approach, are general tools of quantum information theory, and we have tried to carefully separate out the general features from the specific applications [1–8,10,12,13]. Making this clean conceptual separation has allowed us to completely sidestep the highly contentious issues associated with the "information puzzle" [24,66–84]. Of course there are implications for the information puzzle—see specifically Refs. [26,27]—but we shall not touch on these issues in the present article.

[2]In Ref. [23] (and references therein) an interesting alternative approach to multipartite entangled systems was developed, analyzing bipartite measures for pure states averaged over all possible bipartitions of the system. This was done in order to characterize the entanglement of the system by a single measure, with a view to then studying the robustness of entanglement. This approach seems orthogonal to the average subsystem approach discussed herein.

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