Time evolution of linear and generalized Heisenberg algebra nonlinear Pöschl-Teller coherent states

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We analyze the time evolution of two kinds of coherent states for a particle in a Pöschl-Teller potential. We find a pair of canonically conjugate operators and compare the behavior of their time evolution for both coherent states. The nonlinear ones are more localized. The trajectory in the phase space of the mean values of these two operators is a kind of generalization of the Rose algebraic curves. The new pair of canonically conjugate variables leads to a fourth-order Schrödinger equation which has the same energy spectrum as the Pöschl-Teller system.

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I. INTRODUCTION

The idea of coherent states was first proposed in a very important paper of Schrödinger [1] and it is linked to the question of maintaining maximum localizability during the time evolution of quantum systems. This idea first succeeded in the case of the harmonic oscillator in which the coherent state evolution maintains the minimum uncertainty. In the 1960s, the papers of Klauder [2,3] and Glauber in quantum optics [4] showed the applicability of coherent states in physics. For other systems which are not the harmonic oscillator some generalizations of the coherent states [5–9] have been introduced and were called nonlinear coherent states; the question of the localizability in the evolution of these systems is currently an interesting subject of investigation [10–16].

In this paper, we study the Pöschl-Teller potential (PT) [17], which has been used in molecular systems [18,19] and, in a certain number of articles, to model infinite quantum wells (see, for example, [20] and references therein). For that system we calculate the evolution of the mean values of a new pair of canonically conjugate operators both for generalized Heisenberg algebra (GHA) nonlinear coherent states [7] and also for the linear ones. This pair of operators is defined in terms of ladder operators which obey for the PT system the same algebra as the usual annihilation and creation operators for the harmonic oscillator. We find that during time evolution the mean values of the conjugate operators present a fluctuation which oscillates between a minimum and a maximum value, exhibiting then a localized behavior. This time evolution is more localized for the mean values constructed with the GHA nonlinear than with the linear coherent states. These two operators determine a phase space in which the trajectory of their mean values follows a kind of generalization of the Rose algebraic curves. We use the new pair of canonically conjugate operators to construct a fourth-order Hamiltonian whose energy spectrum is the same as the Pöschl-Teller one, that is, these two systems are isospectral.

In Sec. II we make a brief review of the GHA and introduce generalized creation and annihilation operators, in terms of which we construct a new pair of canonically conjugate variables and define linear and nonlinear coherent states. In Sec. III we apply these results to a particle in a Pöschl-Teller potential and find the realization of the GHA and of its associated harmonic oscillator algebra. In Sec. IV we construct linear and nonlinear coherent states for the case $\nu = 0$ of the Pöschl-Teller potential. We present the behavior of the time evolution of the uncertainty relation and show that as time goes by the uncertainty has a maximum value and it is always smaller or equal for the nonlinear than for the linear case. In Sec. V we show that in terms of the new pair of canonical variables the system obeys a fourth-order Hamiltonian whose solutions are the harmonic oscillator ones. In Sec. VI we present our final comments.

II. GENERALIZED HEISENBERG ALGEBRA AND ITS ASSOCIATED CANONICALLY CONJUGATE OPERATORS

A. Brief review of the generalized Heisenberg algebra

The generalized Heisenberg algebra (GHA), constructed a few years ago [21–23], is a family of Heisenberg-type algebras depending on a characteristic function of the dimensionless Hamiltonian H, $f(H)$, and it is described by the generators $H, A, A^\dagger$ satisfying

$$HA^\dagger = A^\dagger f(H), \tag{1}$$

$$AH = f(H)A, \tag{2}$$

$$[A, A^\dagger] = f(H) - H, \tag{3}$$

where $A = (A^\dagger)^\dagger, H = H^\dagger$ is the Hamiltonian of the physical system under consideration, and $f(H)$ is a function of H, called the characteristic function of the algebra. By its choice, we obtain a large class of Heisenberg-type algebras. In particular, if H is the Hamiltonian of the harmonic oscillator, A and $A^\dagger$ are the usual a and $a^\dagger$ annihilation and creation operators. Note that $H \equiv \frac{\mathcal{H}}{b}$, where $\mathcal{H}$ is the dimensional Hamiltonian of the system and b is a constant with dimension of energy.

The Fock space representation of the GHA is given through a general vector $|m\rangle$ which is required to be an eigenvector of the Hamiltonian, $H|m\rangle = \epsilon_m|m\rangle$, where $\epsilon_m = f^{(m)}(\epsilon_0)$, and $f^{(m)}(\epsilon_0)$ is the definition of the mth iterate of ϵ_0 under f. ϵ_0 is the lowest eigenvalue of H with respect to the vacuum

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state $|0\rangle$. Besides, $H = H(N)$ where $N|m\rangle = m|m\rangle$. Using relations (1)–(2), we have

$$A^\dagger|m\rangle = \mathcal{N}_m|m+1\rangle, \tag{4}$$

$$A|m\rangle = \mathcal{N}_{m-1}|m-1\rangle, \tag{5}$$

where

$$\mathcal{N}_m^2 = \epsilon_{m+1} - \epsilon_0. \tag{6}$$

The Casimir operator of the algebra is

$$\mathfrak{C} = A^\dagger A - H, \quad \mathfrak{C}|m\rangle = \epsilon_0|m\rangle \forall m. \tag{7}$$

In [22] it was shown that choosing for the characteristic function of the GHA the linear function $f(x) = x + 1$ the algebra presented in Eqs. (1) and (2) becomes the harmonic oscillator algebra. For $f(x) = qx + 1$ we obtain in Eqs. (1) and (2) the q-oscillator algebra. Moreover, it was shown in [24] that there is a class of quantum systems described by these generalized Heisenberg algebras. This class is characterized by those quantum systems having energy eigenvalues written as

$$\epsilon_{n+1} = f(\epsilon_n), \tag{8}$$

where ϵ_{n+1} and ϵ_n are successive energy levels and $f(x)$ is a different function for each physical system. This function $f(x)$ is exactly the characteristic function that appears in the construction of the algebra in Eqs. (1) and (2). In the algebraic description of this class of quantum systems, $A^\dagger$ and A are creation and annihilation operators.

Note that using Eq. (8) $\mathcal{N}_m$ is the eigenvalue of the operator C_H defined according to

$$C_H|m\rangle \equiv \sqrt{f(H) - \epsilon_0}|m\rangle = \sqrt{\epsilon_{m+1} - \epsilon_0}|m\rangle. \tag{9}$$

B. GHA and generalized harmonic oscillator creation and annihilation operators

Let us now discuss the meaning of a GHA in a Fock space. To start with, we note that the GHAs, given by Eqs. (1)–(6), can be constructed once we know the energy spectrum of the corresponding systems, which completely determines the characteristic function. This endues that isospectral systems have the same GHA. These isospectral systems will be distinguished by the physical realization of the operators A and $A^\dagger$ (see, for example, [25]). In fact, we can think of GHA as a metasystem in the sense that the same algebraic structure allows more than one physical realization.

We will now define an operator D, that together with its adjoint operator $D^\dagger$ and the operator N, presents the same algebraic structure as the harmonic oscillator operators a, $a^\dagger$, and N.

We consider a general GHA system whose Hamiltonian is H. Using the operator C_H given in (9), we can define the operator D as

$$D \equiv \sqrt{N+1}\,\frac{1}{C_H}A. \tag{10}$$

$D^\dagger$ is the Hermitian conjugate of D.

We know how each of the operators in definition (10) acts on the Fock space, so it can easily be seen that D acts on a Fock state $|m\rangle$ as the usual annihilation operator a, $D|n\rangle = \sqrt{n}\,|n-1\rangle$. It is then simple to prove that the operators $D, D^\dagger, N$ obey

$$[D, D^\dagger] = 1, \tag{11}$$

$$[N, D] = -D, \tag{12}$$

$$[N, D^\dagger] = D^\dagger. \tag{13}$$

C. Algebraic canonically conjugate operators ξ and ρ

The algebra of the operators N, D, and $D^\dagger$, given by Eqs. (11)–(13), allows one to define canonically conjugate positionlike and momentumlike (ξ, ρ) operators for any system described by a GHA, as follows:

$$\xi = \frac{L}{\sqrt{2}}(D + D^\dagger), \tag{14}$$

$$\rho = i\frac{\hbar}{\sqrt{2}L}(D^\dagger - D), \tag{15}$$

where L is some constant with dimension of length specific to the physical system described by the Hamiltonian H; conversely, we can now write operators D and $D^\dagger$ in terms of (ξ, ρ) as

$$D = \frac{\xi}{\sqrt{2}L} + \frac{iL}{\sqrt{2}\hbar}\rho, \tag{16}$$

$$D^\dagger = \frac{\xi}{\sqrt{2}L} - \frac{iL}{\sqrt{2}\hbar}\rho. \tag{17}$$

From the commutation relation of D and $D^\dagger$, we have the interesting result that

$$[\xi, \rho] = i\hbar, \tag{18}$$

ensuing that the pair of variables (ξ, ρ) is canonically conjugate.

D. Nonlinear and linear coherent states

The fact that, as we have shown above, there are two different algebras associated with any system, that is, the GHA algebra generated by H, A, and $A^\dagger$ [(1)–(3)] and the harmonic oscillator algebra generated by N, D, and $D^\dagger$ [(11)–(13)], allows us to build both linear and GHA nonlinear coherent states from the Fock space, as we have now two different pairs of annihilation and creation operators, $(A, A^\dagger)$ and $(D, D^\dagger)$, acting on it.

Linear coherent states are the coherent states for the harmonic oscillator; they were first introduced mathematically [1–4] and only later it was realized that they could be generated in the laboratory [26]. Their well-known expression is

$$|z\rangle_L = e^{\frac{-|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n\rangle, \tag{19}$$

where $\epsilon_n = n + 1/2$ is the dimensionless harmonic oscillator nth energy level, $n! = (\epsilon_n - \epsilon_0)! \equiv \prod_{k=1}^{n}(\epsilon_k - \epsilon_0)$ and z is a complex number.

It has also been shown [27] that GHA nonlinear coherent states corresponding to any other system can be constructed and that their general form is

$$|z\rangle_{\rm NL} = \mathcal{N}(z)\sum_{n=0}^{\infty}\frac{z^n}{\mathcal{N}_{n-1}!}|n\rangle, \tag{20}$$

where $\mathcal{N}_{n-1} = \sqrt{\epsilon_n - \epsilon_0}$ is given in (6). $|z\rangle_{\rm NL}$ is a solution of the equation $A|z\rangle_{\rm NL} = z|z\rangle_{\rm NL}$.

The time evolution of the states is obtained by the application of the unitary operator,

$$U(t) = \exp(-i\mathcal{H}t/\hbar), \tag{21}$$

where $\mathcal{H}$ is the Hamiltonian associated with the GHA in question. Note that for the linear coherent state, its time evolution is not the same as the harmonic oscillator one.

In the next section we will explicitly discuss these operators $D, D^\dagger, \xi$, and ρ and construct linear and nonlinear coherent states in the case of a particle in a Pöschl-Teller potential.

III. THE CASE OF THE PÖSCHL-TELLER POTENTIAL

Let us consider the system consisting of a particle in the Pöschl-Teller potential,

$$V(x) = \frac{\hbar^2\pi^2}{2mL^2}\frac{\nu(\nu+1)}{\sin^2(\pi x/L)}, \tag{22}$$

where $\nu \geqslant 0$ and $0 < x < L$; this potential is infinite for x outside this range.

The corresponding Schrödinger equation is

$$\mathcal{H}\Psi_n(x) = \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\Psi_n(x) \tag{23}$$

$$= b\,\epsilon_n\Psi_n(x) = b\,(n+\nu+1)^2\,\Psi_n(x), \tag{24}$$

where $n = 0,1,2,3,\ldots$, $b = \hbar^2\pi^2/2mL^2$, and m is the mass of the particle; its solutions are

$$\Psi_n(x) = c_n(\nu)\,\sin^{\nu+1}(\pi x/L)\,C_n^{\nu+1}(\cos(\pi x/L)), \tag{25}$$

where

$$c_n(\nu) = \Gamma(\nu+1)\frac{2^{\nu+1/2}}{\sqrt{L}}\sqrt{\frac{n!(n+\nu+1)}{\Gamma(n+2\nu+2)}}$$

is a normalization constant and $C_n^{\nu+1}$ is a Gegenbauer polynomial [28].

The dimensionless energy spectrum is then given by

$$\epsilon_n = (n+\nu+1)^2; \tag{26}$$

we can easily see that $\epsilon_{n+1} = (n+2+\nu)^2 = (\sqrt{\epsilon_n}+1)^2$. The characteristic function for this physical system is then $f(x) = (\sqrt{x}+1)^2$. In order to obtain the corresponding GHA, generated by the dimensionless Hamiltonian $H \equiv \frac{\mathcal{H}}{b}$ and the creation and annihilation operators $A^\dagger$, A, we substitute the above characteristic function in Eqs. (1)–(3):

$$[H,A^\dagger] = 2\,A^\dagger\sqrt{H} + A^\dagger, \tag{27}$$

$$[H,A] = -2\sqrt{H}\,A - A, \tag{28}$$

$$[A,A^\dagger] = 2\sqrt{H} + 1. \tag{29}$$

The square root of the generator H is well defined since this is a Hermitian and positive definite operator.

Note that this characteristic function is the same for the free particle in an infinite square-well potential [25]. As a matter of fact this characteristic function is the same for all systems having an energy spectrum of the form $\epsilon_n = (n+a)^2$, where a is a positive real number. Consequently, this family of systems has the same GHA.

The Fock space representation of the algebra generated by H, A, and $A^\dagger$, as in Eqs. (27)–(29), is obtained considering eigenstates $|n\rangle$ of H. The action of these algebra generators on $|n\rangle$ is given by

$$H\,|n\rangle = (n+\nu+1)^2\,|n\rangle, \quad n = 0,1,2,\ldots, \tag{30}$$

$$A^\dagger\,|n\rangle = \sqrt{(n+1)(n+2\nu+3)}\;|n+1\rangle, \tag{31}$$

$$A\,|n\rangle = \sqrt{n(n+2\nu+2)}\;|n-1\rangle, \tag{32}$$

where $\mathcal{N}_n^2 = \epsilon_{n+1} - \epsilon_0 = (n+1)(n+2\nu+3)$ and, of course, we have also

$$\mathcal{H}|n\rangle = b(n+\nu+1)^2\,|n\rangle \quad n = 0,1,2,\ldots. \tag{33}$$

Note that $A|0\rangle = 0$. This is equivalent to the ν-discrete representations of the $su(1,1)$ algebra ([8,29,30]). This equivalence happens for systems whose energy spectrum is of the form $an^2 + bn + c$, a, b, and c positive numbers, but not for any system.

We depart from the operators H, A, and $A^\dagger$ above, that obey the GHA given in (27)–(29) and act on the Fock space according to (30)–(32). In the Hilbert space of functions (25) their physical realization in terms of the position and differential operators is

$$\mathcal{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x), \tag{34}$$

$$A = g(N)\left(-\frac{L}{\pi}\frac{d}{dx}\sin\frac{\pi x}{L} + (N+\nu+1)\cos\frac{\pi x}{L}\right), \tag{35}$$

$$A^\dagger = \left(\frac{L}{\pi}\sin\frac{\pi x}{L}\frac{d}{dx} + \cos\frac{\pi x}{L}(N+\nu+1)\right)g(N), \tag{36}$$

where

$$g(N) \equiv \sqrt{\frac{(N+\nu+2)(N+2\nu+3)}{(N+\nu+1)(N+2\nu+2)}}. \tag{37}$$

A different physical realization of the ladder operators that satisfy the dynamical $su(2)$ can be found in [8] for a modified Pöschl-Teller potential.

In our case, the operators D and $D^\dagger$ defined in Eq. (10) are given by

$$D = \frac{1}{\sqrt{N+2\nu+3}}A, \tag{38}$$

$$D^\dagger = A^\dagger\frac{1}{\sqrt{N+2\nu+3}}. \tag{39}$$

Note that these two new operators are constructed with the generators of the GHA, namely, A, $A^\dagger$, and $\sqrt{H} = N+\nu+1$.

Therefore the canonical variables ξ and ρ can be written in terms of the Pöschl-Teller operators x and d/dx as

$$\xi = \frac{L}{\sqrt{2}}\left[\frac{1}{\sqrt{N+2\nu+3}}A + A^{\dagger}\frac{1}{\sqrt{N+2\nu+3}}\right], \quad (40)$$

$$\rho = \frac{i\hbar}{\sqrt{2}L}\left[\frac{1}{\sqrt{N+2\nu+3}}A - A^{\dagger}\frac{1}{\sqrt{N+2\nu+3}}\right], \quad (41)$$

where operators A and $A^{\dagger}$ are given above in relations (35) and (36). A GHA algebraic structure depends only on the energy spectrum but since one may have different physical systems with the same energy spectrum, the so-called isospectral systems [31,32], what we have to do in order to fully connect a GHA to a given physical system is to realize the GHA operators A, $A^{\dagger}$ in its Hilbert space; this is achieved once one realizes the operators A and $A^{\dagger}$ in terms of x, d/dx, and H, as was done for the Pöschl-Teller in Eqs. (35) and (36). In this way, the canonically conjugate operators ξ and ρ given in equations above are operators acting on the Hilbert space of a particle in a Pöschl-Teller potential.

From (30)–(32) and (38) and (39), we obtain the action of the operators D and $D^{\dagger}$ on the Fock space:

$$D|n\rangle = \frac{1}{\sqrt{N+2\nu+3}}A|n\rangle = \sqrt{n}\,|n-1\rangle, \quad (42)$$

$$D^{\dagger}|n\rangle = A^{\dagger}\frac{1}{\sqrt{N+2\nu+3}}|n\rangle = \sqrt{n+1}\,|n+1\rangle. \quad (43)$$

Then, from definitions (14) and (15) we have the variables ξ and ρ.

From (42) and(43), in terms of these operators the Pöschl-Teller Hamiltonian is written,

$$\mathcal{H} = b(D^{\dagger}D + \nu + 1)^2. \quad (44)$$

IV. COHERENT STATES FOR THE CASE $\nu = 0$

Considering that the characteristic function and consequently the GHA are the same for all systems with the energy spectrum $\epsilon_n = (n+\nu+1)^2$, $n = 0,1,2,3\ldots$, and remembering that, as mentioned in the beginning of Sec. II, the m-energy level is the m iterate of ϵ_0 according to $\epsilon_m = f^{(m)}(\epsilon_0)$, this implies that for each value of ν corresponds a different value of the vacuum energy ϵ_0. In fact, the Casimir of the Pöschl-Teller system is, according to (7) and (26), given by $\epsilon_0(\nu) = (\nu+1)^2$; this means that this system and the infinite square-well potential are different representations of the same algebra. The results obtained for a definite value of ν are then qualitatively equivalent to any other value. Therefore, for the sake of simplicity, from now on we choose $\nu = 0$, which happens to be the free particle in an infinite square-well potential.

Besides up to now we have considered the Fock space starting from $n = 0$. But in the case of the free particle in an infinite square-well potential, it is usual to start it from $n = 1$. To follow this standard procedure from now on we replace in all the equations [(20) and (30)–(43)] the operator N by $N-1$ and the quantum number n by $n-1$. Notice that the new function $g(N)$ is obtained from Eq. (37) replacing N by $N-1$.

A. Constructing GHA nonlinear coherent states

We now construct GHA nonlinear coherent states $|z\rangle_{\rm NL} = \sum_{n\geqslant 1} c_n|n\rangle$ written in terms of the eigenvectors $|n\rangle$, as the coherent states that obey the equation,

$$A|z\rangle_{\rm NL} = z|z\rangle_{\rm NL}, \quad (45)$$

where $n \geqslant 1$; $|1\rangle$ is now the ground state, i.e., $A|1\rangle = 0$. The action of the annihilation operator A on the Fock states $|n\rangle$ is given by Eq. (32), leading to [7]

$$|z\rangle_{\rm NL} = \mathcal{C}_{\rm NL}(|z|)\sum_{n=1}^{\infty}\frac{z^{n-1}}{\prod_{l=1}^{n}(l^2-1)}|n\rangle, \quad (46)$$

where z is a complex number. We can see that, for this case, $\mathcal{N}_{n-1}$ appearing in the general form, Eq. (20), is $\mathcal{N}_{n-1} = \sqrt{\epsilon_n - \epsilon_0} = \sqrt{n^2-1}$.

The states above satisfy the conditions of normalizability, continuity in the label, and completeness [6,7]. From normalizability, that is, $\langle z_{\rm NL}|z_{\rm NL}\rangle = 1$, we find the normalization factor $\mathcal{C}_{\rm NL}(z)$ to be

$$\mathcal{C}_{\rm NL}^2(|z|) = \left[\sum_{n=1}^{\infty}\frac{|z|^{2n-2}}{\prod_{l=1}^{n}(l^2-1)}\right]^{-1} = \frac{|z|^2}{2I_2(2|z|)}, \quad (47)$$

where $I_n(x)$ are the modified Bessel functions of the second kind [33], and z belongs to the whole complex plane. It is important to mention that the particular nonlinear coherent states (46) are the $\nu = 0$ Barut-Girardello coherent states ([8,29,30]). In fact, we can show that this generally happens when the energy spectrum is a polynomial of the form $an^2 + bn + c$, where a, b, and c are positive numbers. Since this is not the case for the spectrum of the q oscillators, their corresponding GHA nonlinear coherent states [7] are more general than Barut-Girardelo ones.

B. Constructing linear coherent states

Now, we have a second approach. The existence of a second annihilation operator, namely D, acting on the Fock space as

$$D|n\rangle = \sqrt{n-1}\,|n-1\rangle, \quad (n \geqslant 1), \quad (48)$$

with $D|1\rangle = 0$, allows us to *also* have for the system in question coherent states similar to the ones given by (19). These linear coherent states obey the relation,

$$D|z\rangle_L = z|z\rangle_L, \quad (49)$$

and can be written as ($n \geqslant 1$)

$$|z\rangle_L = e^{\frac{-|z|^2}{2}}\sum_{n=1}^{\infty}\frac{z^{n-1}}{\sqrt{(n-1)!}}\,|n\rangle, \quad (50)$$

where $\langle x|n\rangle = \Psi_n(x)$ is the wave function of the PT system and z belongs to the whole complex plane.

It is worthwhile to remark that both coherent states, nonlinear and linear ones, given by Eqs. (46) and (50) can be constructed for the same physical system. Both sets of coherent states satisfy the conditions of normalizability, continuity in the label, and resolution of identity.

Having at our disposal two different sets of coherent states for the same physical system, in the next section we calculate the mean values of the pair of canonically conjugate variables (ξ,ρ) and compare their behavior.

C. Initial uncertainty relation of ξ and ρ for the linear coherent states

1. Mean values of ξ and ρ

Using Eqs. (14), (15), (42), and (43), the mean values of ξ and ρ on the states $|z\rangle_L$, Eq. (50), are

$$\langle\xi\rangle_z = \langle z|\xi|z\rangle_L = \frac{L}{\sqrt{2}}\langle z|D+D^\dagger|z\rangle_L = \sqrt{2}\,L\,\mathrm{Re}(z), \quad (51)$$

$$\langle\rho\rangle_z = \langle z|\rho|z\rangle_L = i\frac{\hbar}{L\sqrt{2}}\langle z|D^\dagger - D|z\rangle_L = \frac{\sqrt{2}\,\hbar}{L}\mathrm{Im}(z), \quad (52)$$

where Re(z) and Im(z) mean the real and imaginary parts of the complex number z.

2. Mean values of ξ^2 and ρ^2

The averages of ξ^2 and ρ^2 are

$$\langle\xi^2\rangle_z = \langle z|\xi^2|z\rangle_L = \frac{L^2}{2}\langle z|D^2+(D^\dagger)^2+DD^\dagger+D^\dagger D|z\rangle_L$$
$$= \frac{L^2}{2}(z^2+\bar{z}^2+2|z|^2+1), \quad (53)$$

$$\langle\rho^2\rangle_z = \langle z|\rho^2|z\rangle_L = -\frac{\hbar^2}{2L^2}\langle z|D^2+(D^\dagger)^2-DD^\dagger - D^\dagger D|z\rangle_L$$
$$= -\frac{\hbar^2}{2L^2}(z^2+\bar{z}^2-2|z|^2-1). \quad (54)$$

Note that the mean values (51)–(54) are exactly the same as the mean values of the operators x, p, x^2, and p^2 for the coherent states of the harmonic oscillator. In particular, as the coherent state is defined for z belonging to the entire complex plane, and the mean values of ξ and ρ, (51) and (52), are related to the real and imaginary part of z, this implies that the respective spectra of ξ and ρ are the whole real line.

3. Variances and uncertainty relation

The variances of ξ and ρ are

$$\Delta\xi = \sqrt{\langle\xi^2\rangle_z - \langle\xi\rangle_z^2} = \frac{L}{\sqrt{2}}, \quad (55)$$

$$\Delta\rho = \sqrt{\langle\rho^2\rangle_z - \langle\rho\rangle_z^2} = \frac{\hbar}{L\sqrt{2}}, \quad (56)$$

what leads to the minimum Heisenberg uncertainty relation:

$$\Delta\xi\,\Delta\rho = \frac{\hbar}{2}. \quad (57)$$

This means that we can construct a wave packet (a linear coherent state) obeying the minimum Heisenberg uncertainty relation. Obviously, unlike the harmonic oscillator case, this uncertainty will not remain the same under time evolution.

D. Time evolution of the mean values of ξ and ρ for the linear coherent states

The time evolution of the states is obtained by the application of the unitary operator,

$$U(t) = \exp(-i\mathcal{H}t/\hbar), \quad (58)$$

where $\mathcal{H}$ is given by Eq. (44).

We have for the time evolution of $|z\rangle_L$,

$$U(t)\,|z\rangle_L = e^{-i\mathcal{H}t/\hbar}\,|z\rangle_L \quad (59)$$
$$= e^{-|z|^2/2}\sum_{n\geqslant 1}\frac{z^{n-1}}{\sqrt{(n-1)!}}\,e^{-i\mathcal{H}t/\hbar}\,|n\rangle$$
$$= e^{-|z|^2/2}\sum_{n\geqslant 1}\frac{z^{n-1}}{\sqrt{(n-1)!}}\,e^{-ibn^2t/\hbar}\,|n\rangle. \quad (60)$$

1. Time evolution of the uncertainty relation

The uncertainty relation for the canonical variables ξ,ρ as functions of t is given by

$$\Delta\xi\,\Delta\rho = \sqrt{(\langle\xi(t)^2\rangle - \langle\xi(t)\rangle^2)(\langle\rho(t)^2\rangle - \langle\rho(t)\rangle^2)}. \quad (61)$$

From now on we write $z = re^{i\varphi}$ and the time evolution of the mean values of the operators ξ and ρ become

$$\langle\xi(t)\rangle = (\langle z|U^\dagger(t))\,\xi\,(U(t)|z\rangle_L)$$
$$= \frac{L}{\sqrt{2}}(\langle r,\varphi|U^\dagger(t))\,(D+D^\dagger)\,(U(t)|r,\varphi\rangle_L)$$
$$= \sqrt{2}\,Le^{-r^2}\sum_{n\geqslant 1}\frac{r^{2n-1}\cos[bt(2n+1)/\hbar - \varphi]}{(n-1)!}, \quad (62)$$

and

$$\langle\rho(t)\rangle = (\langle z|U^\dagger(t))\,\rho\,(U(t)|z\rangle_L)$$
$$= i\frac{\hbar}{L\sqrt{2}}(\langle z|U^\dagger(t))\,(D^\dagger - D)\,(U(t)|z\rangle_L)$$
$$= -\frac{\hbar\sqrt{2}}{L}e^{-r^2}\sum_{n\geqslant 1}\frac{r^{2n-1}\sin[bt(2n+1)/\hbar - \varphi]}{(n-1)!}, \quad (63)$$

where $|z\rangle_L \equiv |r,\varphi\rangle_L$.

The time evolution of the mean values of the squares of operators ξ and ρ is

$$\langle\xi(t)^2\rangle = (\langle r,\varphi|U^\dagger(t))\,\xi^2\,(U(t)|r,\varphi\rangle_L)$$
$$= \frac{L^2}{2}(\langle r,\varphi|U^\dagger(t))\,(D^2+(D^\dagger)^2+D^\dagger D+DD^\dagger)\,(U(t)|r,\varphi\rangle_L) \quad (64)$$

$$= L^2\left(e^{-r^2}\sum_{n\geqslant 1}\frac{r^{2n}}{(n-1)!}\cos[4b(n+1)t/\hbar-2\varphi]+r^2+\frac{1}{2}\right), \tag{65}$$

$$\langle\rho(t)^2\rangle = (\langle r,\varphi|U^\dagger(t))\,\rho^2\,(U(t)|r,\varphi\rangle_L)$$
$$= -\frac{\hbar^2}{2L^2}(\langle r,\varphi|U^\dagger(t))\,(D^2+(D^\dagger)^2-D^\dagger D-DD^\dagger)\,(U(t)|r,\varphi\rangle_L)$$
$$= -\frac{\hbar^2}{L^2}\left(e^{-r^2}\sum_{n\geqslant 1}\frac{r^{2n}}{(n-1)!}\cos[4b(n+1)t/\hbar-2\varphi]-r^2-\frac{1}{2}\right), \tag{66}$$

and allows one to calculate the time evolution of the uncertainty, (61), for the linear coherent states.

E. Time evolution of the uncertainty relation for the GHA nonlinear coherent states

If we now replace the linear coherent state $|r,\varphi\rangle_L$ by the nonlinear one,

$$|r,\varphi\rangle_{\mathrm{NL}} = \frac{1}{\sqrt{2I_2(2|z|)}}\sum_{n=1}^{\infty}\frac{r^{n-1}e^{i(n-1)\varphi}}{\prod_{l=2}^{n}(l^2-1)}|n\rangle, \tag{67}$$

whose time evolution is given by

$$U(t)\,|r,\varphi\rangle_{\mathrm{NL}} = e^{-i\mathcal{H}t/\hbar}\,|r,\varphi\rangle_{\mathrm{NL}} \tag{68}$$

$$= \frac{1}{\sqrt{2I_2(2|z|)}}\sum_{n\geqslant 1}\frac{r^{n-1}}{\prod_{l=2}^{n}(l^2-1)}\,e^{-i(bn^2t/\hbar-(n-1)\varphi)}\,|n\rangle, \tag{69}$$

and repeat the calculations shown in Sec. IV D, we obtain

$$\langle\xi(t)\rangle_{\mathrm{NL}} = \frac{\sqrt{2}\,\lambda\,r^2}{I_2(2r)}\sum_{n=1}^{\infty}\frac{r^{2n-1}\cos((2n+1)bt/\hbar-\varphi)}{\sqrt{(n+2)}\Gamma(n)\Gamma(n+2)}, \tag{70}$$

$$\langle\rho(t)\rangle_{\mathrm{NL}} = -\frac{\sqrt{2}\,\hbar\,r^2}{\lambda I_2(2r)}\sum_{n=1}^{\infty}\frac{r^{2n-1}\sin((2n+1)bt/\hbar-\varphi)}{\sqrt{(n+2)}\Gamma(n)\Gamma(n+2)}, \tag{71}$$

$$\langle\xi(t)^2\rangle_{\mathrm{NL}} = \frac{\lambda^2r^2}{I_2(2r)}\left(\sum_{n=1}^{\infty}\frac{\sqrt{n+1}\,r^{2n}\cos(4(n+1)bt/\hbar-2\varphi)}{\sqrt{(n+2)(n^2+4n+3)}\,\Gamma(n)\Gamma(n+2)}+\sum_{n=1}^{\infty}\frac{(n-1)r^{2(n-1)}}{\Gamma(n)\Gamma(n+2)}+\frac{I_2(2r)}{2r^2}\right), \tag{72}$$

$$\langle\rho(t)^2\rangle_{\mathrm{NL}} = \frac{\hbar^2r^2}{\lambda^2I_2(2r)}\left(\sum_{n=1}^{\infty}\frac{\sqrt{n+1}\,r^{2n}\sin(4(n+1)bt/\hbar-2\varphi)}{\sqrt{(n+2)(n^2+4n+3)}\,\Gamma(n)\Gamma(n+2)}-\sum_{n=1}^{\infty}\frac{(n-1)r^{2(n-1)}}{\Gamma(n)\Gamma(n+2)}+\frac{I_2(2r)}{2r^2}\right). \tag{73}$$

The uncertainty relation for the nonlinear case is given by Eq. (61) but now using the nonlinear averages given by Eqs. (70)–(73).

F. Phase-space trajectories for mean values of ξ and ρ

If we analyze the expressions for the time evolution of the mean values of our canonical variables ξ and ρ, namely expressions (62) and (63) or (70) and (71), respectively, for the linear and GHA nonlinear coherent states, we can see that they are a kind of generalization of the Rose algebraic curves in the Cartesian parametric form, which are written as

$$x(t) = \frac{\cos(k-1)t+\cos(k+1)t}{2}, \tag{74}$$

$$y(t) = \frac{-\sin(k-1)t+\sin(k+1)t}{2}, \tag{75}$$

where k is an integer. By generalization we mean that the number of terms is not equal to two but the relevant number of terms increases with r; moreover, the coefficients are different for each term and, finally, we have the φ phase. It is worthwhile to note that looking at expressions (62) and (63) or (70) and (71) we immediately see that if r is very small, only the first term in the sum is relevant; in this case, the curve is simply an ellipsis.

In fact, in Fig. 1 it is shown for the linear case that for a very small r the curve is an ellipsis and for larger values of r we have generalized Rose curves. The curves are similar for the nonlinear case.

G. Uncertainty relations for linear and nonlinear coherent states

We now analyze the time evolution of the uncertainty relation for both cases, linear and nonlinear ones. In Figs. 2–4 we compare the time evolution of both linear and nonlinear uncertainty relations, for $r = 0.5$, $r = 1$, and $r = 1.5$; $\varphi = 0$

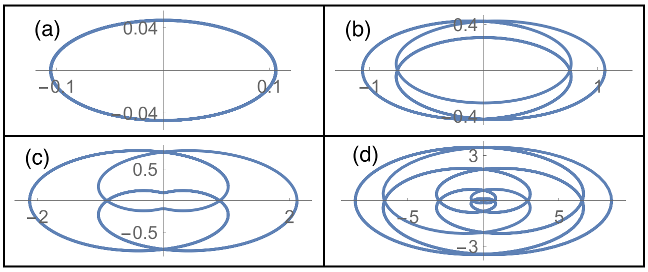

FIG. 1. ξ/λ and $(\rho\lambda/\hbar)$ phase-space cycles. In all graphs, $\hbar = 1, \lambda = 1.5, \varphi = 0$. (a) $r = 0.05$; (b) $r = 0.5$; (c) $r = 1$; (d) $r = 4$. ξ/λ is the horizontal axis and $\rho\lambda/\hbar$ is the vertical axis.

for all three cases. In these numerical simulations we have taken $b/\hbar = 1$. Note that this choice implies that the product mL^2 has a specific well-determined value. Looking at the three figures we see that the uncertainty oscillates between 0.5 and a maximum value which increases with r both for linear and GHA nonlinear coherent states. We can also see that the uncertainty is always smaller or equal for the nonlinear than for the linear case. In fact, they are equal only in a few points and for the maximum there is approximately a factor two. The original idea behind the concept of coherent states [1] is to construct a wave packet whose behavior is as close as possible to the classical trajectory. In this sense the best coherent state in our case is the nonlinear one.

In Fig. 5 we can see numerically how the maximum uncertainty increases with r for the nonlinear case. For high values of r this behavior is approximately linear.

Figures 2–5 show that the time evolution of the coherent state along the trajectory in the ξ-ρ phase space, like in Fig. 1, presents a fluctuation in the variance, which oscillates between a minimum (that is always equal to $1/2$) and a maximum value, exhibiting then a localized behavior. In Figs. 2–5 we are plotting $\Delta\xi\,\Delta\rho/\hbar$, which is dimensionless.

V. FOURTH-ORDER HAMILTONIAN

Let us take the variable ξ as the "position" and ρ as the "momentum." As these variables are canonically conjugate, Eq. (18), and following the usual operator equivalence of quantum mechanics, the momentum ρ can be written as the differential operator given by

$$\rho = -i\hbar\frac{d}{d\xi}. \tag{76}$$

In these variables, the operators D and $D^\dagger$ now read

$$D = \frac{1}{\sqrt{2}L}\xi + \frac{L}{\sqrt{2}}\frac{d}{d\xi}, \tag{77}$$

$$D^\dagger = \frac{1}{\sqrt{2}L}\xi - \frac{L}{\sqrt{2}}\frac{d}{d\xi}. \tag{78}$$

So, in the canonically conjugate variables$(\xi, -i\hbar\frac{d}{d\xi})$, the Hamiltonian (44) is

$$\mathcal{H} = \frac{\pi^2\hbar^2}{2mL^2}\left[-\frac{L^2}{2}\frac{d^2}{d\xi^2} + \frac{1}{2L^2}\xi^2 + \nu + \frac{1}{2}\right]^2, \tag{79}$$

which is just

$$\mathcal{H} = \frac{\pi^2\hbar^2}{2mL^2}\left[H_{\rm HO} + \nu + \frac{1}{2}\right]^2, \tag{80}$$

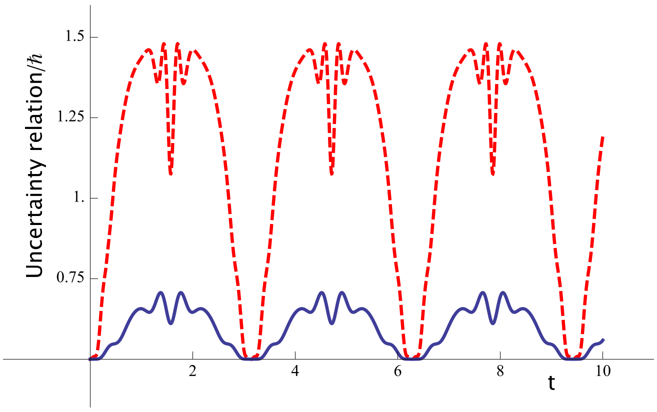

FIG. 3. Time evolution of the uncertainty relation for $r = 1$, $\varphi = 0$, and $b/\hbar = 1$. Continuous (blue) line refers to nonlinear coherent state and dashed (red) line refers to linear coherent state.

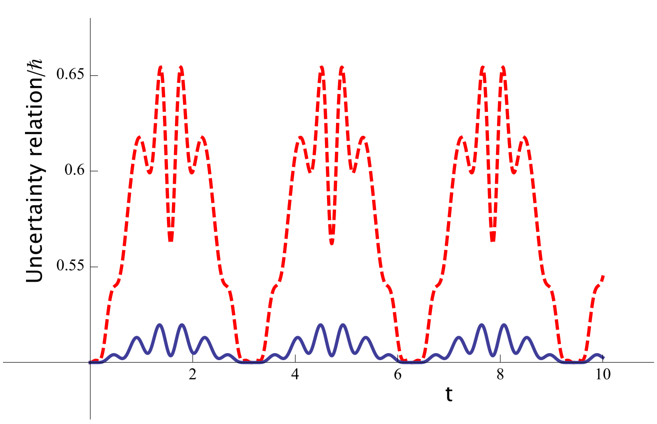

FIG. 2. Time evolution of the uncertainty relation for $r = 0.5$, $\varphi = 0$, and $b/\hbar = 1$. Continuous (blue) line refers to nonlinear coherent state and dashed (red) line refers to linear coherent state.

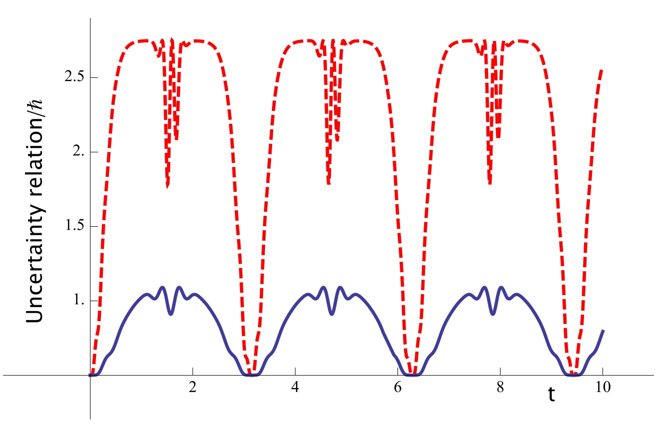

FIG. 4. Time evolution of the uncertainty relation for $r = 1.5$, $\varphi = 0$, and $b/\hbar = 1$. Continuous (blue) line refers to nonlinear coherent state and dashed (red) line refers to linear coherent state.

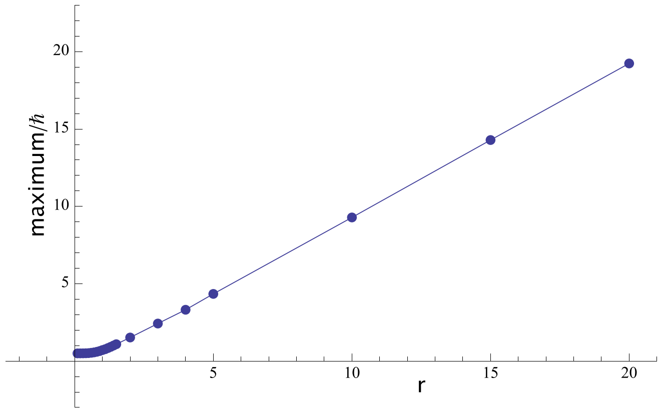

FIG. 5. Behavior of the nonlinear coherent state maximum of uncertainty for increasing values of r. The continuous line is a guide to the eye.

where $H_{\rm HO}$ is the dimensionless harmonic oscillator Hamiltonian. In terms of the dimensional one,

$$\mathcal{H}_{\rm HO} = \left[-\frac{\hbar^2}{2m}\frac{d^2}{d\xi^2} + \frac{m\omega^2}{2}\xi^2\right], \tag{81}$$

where

$$\omega \equiv \frac{\hbar}{mL^2}, \tag{82}$$

we have

$$\mathcal{H} = \frac{\pi^2 mL^2}{2\hbar^2}\left[\mathcal{H}_{\rm HO} + \frac{\hbar^2}{mL^2}\left(\nu + \frac{1}{2}\right)\right]^2. \tag{83}$$

The time-independent Schrödinger equation in the variable ξ, $\mathcal{H}\Psi_n = \mathcal{E}_n\Psi_n$, where $\mathcal{E}_n$ are the energies of levels n ($n \geqslant 0$), is then

$$\frac{\pi^2 mL^2}{2\hbar^2}\left[\mathcal{H}_{\rm HO} + \hbar\omega\left(\nu + \frac{1}{2}\right)\right]^2\Psi_n = \mathcal{E}_n\Psi_n, \tag{84}$$

which is a fourth-order differential equation. But this is an equation of the type $(\mathcal{L}.\mathcal{L})\Psi$, where the linear differential operator $\mathcal{L} = H_{\rm HO} + \nu + 1/2$. So, the solutions of the fourth-order differential equation (84) are linear combinations of the two solutions of $\mathcal{L}\Psi_n = \epsilon_n\Psi_n$, where $\epsilon_n = n + \nu + 1$ are the energy levels of the harmonic oscillator. As one of these solutions is nonphysical, we have just

$$\Psi_n(y) = (\pi L^2)^{-1/4}\frac{1}{2^n n!}H_n(y)e^{-y^2/2}, \tag{85}$$

where $y = \frac{\xi}{L}$ and H_n are the Hermite polynomials; the energy levels of the fourth-order Hamiltonian associated with the Pöschl-Teller potential are then

$$\mathcal{E}_n = \left(\frac{\pi^2\hbar^2}{2mL^2}\right)(n + \nu + 1)^2. \tag{86}$$

This is an interesting result because given a quantum system described by a GHA, we can find variables of the (ξ,ρ) type (14,15) that obey the commutation relations of the position and momentum of the harmonic oscillator. This means that we have two different systems with the same energy spectrum, in this case, namely $\epsilon_n = (n + \nu + 1)^2$: The first is the system described by the second-order Hamiltonian (23) whose solutions are of the form (25) and the other is the system described by the fourth-order Hamiltonian (79) with solutions given by (85).

VI. FINAL COMMENTS

In this paper we use the Pöschl-Teller system in order to discuss the following important theoretical issues. First, for that system we realize physically ladder operators which have the same behavior as the creation and annihilation operators in the harmonic oscillator case. These ladder operators satisfy an algebra called generalized Heisenberg algebra (GHA). Second, using the GHA annihilation operator we have a GHA nonlinear coherent state; next, constructing another annihilation operator, we are able to define, for the same Pöschl-Teller system, a linear coherent state. We analyze the time evolution of those coherent states and find that both present an oscillating and nonspreading behavior and that the nonlinear one is always more localized. Third, by means of some operator transformations, we arrive at a new pair of canonically conjugate operators for the Pöschl-Teller system. Finally, using this pair of canonically conjugate operators we construct a fourth-order Hamiltonian which has the same energy spectrum (isospectral) as the Pöschl-Teller system.

We show that given a physical system and its associated GHA, whose generators are the ladder operators A and $A^\dagger$ and its characteristic function $f(H)$, we can always introduce the operator D (10), and its Hermitian conjugate $D^\dagger$, in terms of that particular algebra. D, $D^\dagger$, and N present the same algebraic structure of the harmonic oscillators operators a, $a^\dagger$, and N. Departing from D and $D^\dagger$ we define another pair of canonically conjugate position and momentumlike operators (ξ, ρ) (14) and (15).

As a GHA algebraic structure depends only on the energy spectrum, in order to fully connect it to a given physical system we must realize the GHA generators A, $A^\dagger$ in terms of x, d/dx, and H; this is done for the Pöschl-Teller system in Eqs. (35) and (36). In this way, the canonically conjugate operators ξ and ρ given in (40) and (41) are operators acting on the Hilbert space of a particle in a Pöschl-Teller potential.

We show that for a system described by a given GHA two different coherent states can be constructed, namely linear (19) and nonlinear ones (20). We analyze the physical consequences of the new canonically conjugate operators for the case of a particle in a Pöschl-Teller potential. To this end we calculate the time evolution of their mean values for both linear and GHA nonlinear coherent states and find that they evolve in the phase space $(\langle\xi\rangle, \langle\rho\rangle)$ according to the kinds of generalizations of the Rose algebraic curves (see Fig. 1). We also study the time evolution of the uncertainty relation and we see that it oscillates between $\hbar/2$ and a maximum value which increases with the radial label r of the coherent states both for linear and GHA nonlinear coherent states. An interesting result is that the uncertainty is either equal or smaller for the GHA nonlinear coherent states than for the linear ones. Therefore GHA nonlinear coherent states take to wave packets with a closer behavior to the classical trajectory than the wave packet constructed with linear ones. Anyhow, in both cases our wave packets are always localized.

The existence of operators D and $D^\dagger$ for the Pöschl-Teller allowed the definition of appropriate linear coherent states for that system. The fact of having two types of coherent states enlarges the possibility of constructing them experimentally for that system. Thus, the behavior of any relevant quantum variables can be compared for both types. In fact, linear and nonlinear ([9,34–36]) coherent states are currently constructed in the laboratory for optical systems. Concerning molecular systems, in [37] the authors have described two molecular systems by numerically modeling them by Morse and Pöschl-Teller potentials. In particular, the Pöschl-Teller potential could describe, approximately, the homonuclear diatomic molecule H_2. Therefore, using the first 10 bound states of the molecule of H_2, their nonlinear coherent states of the Morse and Pöschl-Teller systems are approximations for the coherent states of those molecules. This can be a first step in the attempt to obtain nonlinear coherent states for a molecular system in the laboratory. We hope that this will be addressed in the near future.

It is important to comment that whenever the operator D and its adjoint $D^\dagger$ can be defined for a GHA system there will be canonically conjugate variables ξ and ρ for which the mean values $\langle n|\xi^2|n\rangle$ and $\langle n|\rho^2|n\rangle$ increase with n. This is an interesting feature and these operators deserve further investigation, namely on their physical meaning.

Finally, using the fact that the operators ξ and ρ satisfy the relation (18) and assuming the expression $\rho = -i\hbar d/d\xi$ (the traditional form of canonically conjugate operators), we can construct a new physical system having a fourth-order Hamiltonian in $d/d\xi$ (83). This Hamiltonian, in spite of acting in a different Hilbert space of the Pöschl-Teller Hilbert space [see Eqs. (84)–(86)], has the same energy spectrum as the Pöschl-Teller Hamiltonian (22) and (23). These two systems are then isospectral.

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