Landau levels for an electromagnetic wave

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In this paper we show that the frequencies of propagating electromagnetic waves (photons) in a rotating dielectric medium obey Landau quantization. We show that the degeneracy of right and left helicities of photons is broken on the lowest Landau level. In homogeneous space this level is shown to be helical; i.e., left and right helical photons counterpropagate. This leads to a helical vortical effect for photons, which can be understood as an inverse of the optical torque.

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I. INTRODUCTION

Photons are spin-1 massless particles and are described by helicity, a scalar product of spin and propagation direction, which can only take $+1$ or -1 values. It is a natural spin-momentum locking of a photon. Therefore, when a photon's momentum direction is adiabatically changed and returned to the original direction, the spin acquires a phase—the Berry phase [1]. This phase was experimentally observed in a system of coiled optical fiber [2,3]. Berry-curvature-modified semiclassical equations of photons' wave-packet motion were also developed to predict and measure qualitatively similar effects due to the topology of light [4,5].

Due to the natural Berry phase of an electromagnetic wave, there are a number of edge modes that share a resemblance with the fermionic Hall phases. Chiral electromagnetic edge modes occurring at the boundaries and domain walls of gyrotropic systems such as magnetized plasma [6], optical isomers [7], metals with anomalous Hall effect [8], and others are analogous to the quantum Hall effect [9,10] for photons [11] and were recently observed [12,13] in photonic crystals. Surface electromagnetic waves occurring at the dielectric media interface at which the dielectric constant changes sign (see Refs. [14,15]) were recently shown to be a topological insulator analog [16–18] for electromagnetic waves [19].

In this paper we describe another topological property of electromagnetic waves revealed under medium rotation. The polarization rotation of an electromagnetic wave propagating in a rotating medium was studied in many papers, both theoretically [20–23] and experimentally [24]. Moreover, it is understood that the motion of the medium is an effective vector potential seen by photons propagating in the medium (for example, see Refs. [14,25]). In Refs. [26,27] the Landau level quantization of photons inside the rotating medium was shown. Further on, the Aharonov-Bohm effect for photons was theoretically proposed in Ref. [28]. In this paper we study the propagation of photons inside a uniformly rotating dielectric medium and show, in accord with Ref. [26], there are solutions similar to the solution of the Schrödinger equation for an electron in a magnetic field, namely, the Landau wave functions and corresponding Landau levels [29]. Importantly, compared to Refs. [26,27] we find an additional solution, namely, a gapless helical zeroth Landau level. In this case, photons with opposite helicities counterpropagate along the axis of rotation and do not mix with each other. Because of this, there is a finite-temperature nonzero helicity current in a gas of photons, the so-called helical vortical effect (or chiral vortical effect). The helical Landau level is analogous to the chiral zeroth Landau level of three-dimensional Dirac fermions in a magnetic field (for example, see Ref. [30]). So the helical vortical effect for photons is analogous to the chiral magnetic effect for Dirac fermions (see Ref. [31] for a review). We note that the helical Landau level found in this paper is called the zero mode, whose existence is due to the nontrivial topology (Berry phase) of electromagnetic waves described by Maxwell equations [1,2]. Note, a solution physically similar to the helical mode solution was shown [4,32] to occur in a whispering-gallery setup. Finally, the helical vortical effect can be understood as an inverse to the optical torque that circularly polarized light exerts on the dielectric medium it propagates through [33].

In a recent experimental work creation of synthetic Landau levels for photons in optical resonators was reported [34]. It is possible that the Landau level for photons and the helical vortical effect can be observed in experiments with slow light [35].

II. LANDAU LEVELS FOR ELECTROMAGNETIC WAVES

Maxwell equations describing the propagation of an electromagnetic wave in the dielectric medium described by constant ϵ and μ in the absence of currents and charges are

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}, \tag{1}$$

$$\nabla \cdot \mathbf{D} = 0, \quad \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}. \tag{2}$$

Assume that the dielectric medium is moving with a speed $\mathbf{v}$. Relations between the components of the electromagnetic field in the moving dielectric are (see Sec. 76 of Ref. [14])

$$\mathbf{D} + \frac{1}{c}[\mathbf{v}\times\mathbf{H}] = \epsilon\left(\mathbf{E} + \frac{1}{c}[\mathbf{v}\times\mathbf{B}]\right), \tag{3}$$

$$\mathbf{B} + \frac{1}{c}[\mathbf{E}\times\mathbf{v}] = \mu\left(\mathbf{H} + \frac{1}{c}[\mathbf{D}\times\mathbf{v}]\right). \tag{4}$$

In the limit $|\mathbf{v}|/c \ll 1$, to the lowest order in $|\mathbf{v}|/c$, we write

$$\mathbf{D} = \epsilon\mathbf{E} + \frac{\epsilon\mu - 1}{c}[\mathbf{v}\times\mathbf{H}], \tag{5}$$

$$\mathbf{B} = \mu\mathbf{H} - \frac{\epsilon\mu - 1}{c}[\mathbf{v}\times\mathbf{E}]. \tag{6}$$

For the sake of generality we assume an infinite system with homogeneous $\epsilon > 0$ and $\mu > 0$. We then introduce Riemann-Silberstein fields,

$$\mathbf{F}^{(\pm)} = \sqrt{\epsilon}\mathbf{E} \pm i\sqrt{\mu}\mathbf{H}, \tag{7}$$

where the $\pm$ sign corresponds to photon helicity. For time-independent velocity $\mathbf{v}$, we rewrite the Maxwell equations,

$$\left(\boldsymbol{\nabla} - \frac{\epsilon\mu - 1}{c^2}\mathbf{v}\partial_t\right)\times\mathbf{F}^{(\pm)} = \pm i\frac{\sqrt{\epsilon\mu}}{c}\partial_t\mathbf{F}^{(\pm)}, \tag{8}$$

$$\boldsymbol{\nabla}\cdot\left[\mathbf{F}^{(\pm)} \pm \frac{\epsilon\mu - 1}{i\sqrt{\epsilon\mu}c}[\mathbf{v}\times\mathbf{F}^{\pm}]\right] = 0, \tag{9}$$

which bear a similarity with the Dirac equation (for a review of such an approach see Ref. [36]). Here velocity $\mathbf{v}$ plays the role of a vector potential of an effective magnetic field. We assume velocity to have a cylindrical symmetry, described by

$$\mathbf{v} = v(-y\mathbf{e}_x + x\mathbf{e}_y), \tag{10}$$

where v is the angular velocity. Compare vector field (10) describing rotation with the symmetric gauge of the magnetic field. As an example, one can keep in mind a dielectric medium of cylindric form, which is rotating about its axis. However, as mentioned above, we are going to study a rotating system infinite in all directions. Even though it is not experimentally possible, we wish to study this case in order to understand the nature of solutions. For finite systems it is straightforward to set boundary conditions by integrating Eqs. (8) and (9). We search for solutions in the form $\propto e^{-i\omega t}e^{ip_z z}$. For the sake of simplicity, we introduce $\Omega = \frac{\sqrt{\epsilon\mu}}{c}\omega$ and $V = \frac{\epsilon\mu - 1}{c^2}v\omega$. In the following we choose $V > 0$, and as mentioned above we assume the system to be infinite in all directions. Components of Eq. (8) are written as

$$\Pi_y F_z^{(\pm)} - ip_z F_y^{(\pm)} = \pm\Omega F_x^{(\pm)}, \tag{11}$$

$$ip_z F_x^{(\pm)} - \Pi_x F_z^{(\pm)} = \pm\Omega F_y^{(\pm)}, \tag{12}$$

$$\Pi_x F_y^{(\pm)} - \Pi_y F_x^{(\pm)} = \pm\Omega F_z^{(\pm)}, \tag{13}$$

where $\Pi_y \equiv (-i\nabla_y + Vx)$ and $\Pi_x \equiv (-i\nabla_x - Vy)$ are the updated momentum operators. After straightforward transformations, assuming all components of $\mathbf{F}^{(\pm)}$ are nonzero, we obtain for the $F_z^{(\pm)}$ component an equation

$$\left(\Pi_y^2 + \Pi_x^2\right)F_z^{(\pm)} = \left(\Omega^2 - p_z^2\right)F_z^{(\pm)}. \tag{14}$$

The equation has exactly the form of the Schrödinger equation describing an electron in a uniform magnetic field, chosen to be described in a symmetric gauge [29]. Hence we obtain the Landau solutions to the equation. We label the eigenvalues and energies by index n, and write

$$F_{z,n}^{(\pm)} = e^{-V|\zeta|^2/2}\left(\partial_{\bar{\zeta}} - \frac{V}{2}\zeta\right)^n f(\bar{\zeta}), \tag{15}$$

$$\Omega_n^2 = 4V\left(n + \frac{1}{2}\right) + p_z^2, \tag{16}$$

where $\zeta = x + iy$, $\bar{\zeta} = x - iy$, and $f(\bar{\zeta})$ is an arbitrary function of $\bar{\zeta}$. The function can be presented through basis states as

$$f(\bar{\zeta}) = \sum_m f_m(\bar{\zeta}) = \sum_m \sqrt{N_m}\bar{\zeta}^m, \tag{17}$$

where N_m is a renormalization constant. Each f_m corresponds to the mth orbit of the state on the nth level. For example, for $n = 0$ each f_m corresponds to an orbit with a radius $|\zeta|_m = \sqrt{m/V}$.

The other two components of $\mathbf{F}^{\pm}$ are expressed through $F_z^{(\pm)}$ as

$$F_x^{(\pm)} = \frac{\pm\Omega}{\Omega^2 - p_z^2}\left(i\Pi_y \pm \frac{p_z}{\Omega}\Pi_x\right)F_z^{(\pm)}, \tag{18}$$

$$F_y^{(\pm)} = \frac{\pm\Omega}{i\left(\Omega^2 - p_z^2\right)}\left(\Pi_x \pm i\frac{p_z}{\Omega}\Pi_y\right)F_z^{(\pm)}. \tag{19}$$

The solutions found from Eq. (8) are consistent with Eq. (9). This can be seen by taking the divergence operation of expression (8).

The obtained spectrum, keeping in mind that $\omega > 0$ and $\epsilon\mu - 1 > 0$, is rewritten in a more transparent form:

$$\omega_n = 2\frac{\epsilon\mu - 1}{\epsilon\mu}v\left(n + \frac{1}{2}\right) + \sqrt{\left(2\frac{\epsilon\mu - 1}{\epsilon\mu}v\right)^2\left(n + \frac{1}{2}\right)^2 + \frac{c^2 p_z^2}{\epsilon\mu}}. \tag{20}$$

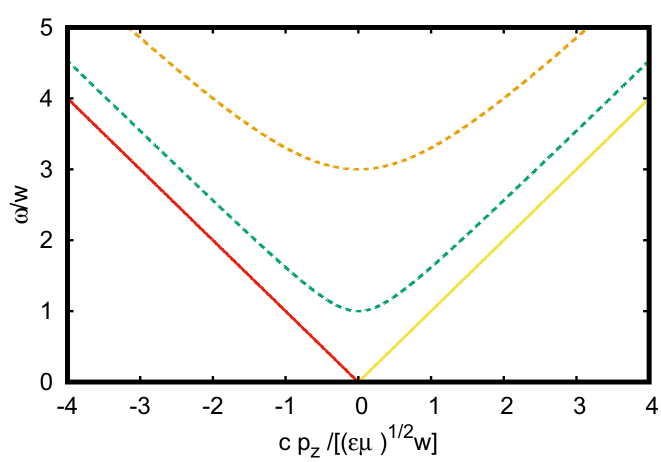

FIG. 1. The spectrum of the Landau levels for electromagnetic waves. Levels ω_n described by Eq. (20) are plotted as a dashed green line for $n = 0$ and a dashed orange line for $n = 1$. Two branches of the lowest Landau level described by Eq. (24) are plotted in red and blue. This level is gapless and helical; i.e., red corresponds to + helicity while blue corresponds to − velocity for a $v > 0$ choice of angular velocity of rotation. We have introduced a characteristic energy scale $w = 2\frac{\epsilon\mu - 1}{\epsilon\mu}v$.

Note that the spectrum does not depend on the helicity index ($\pm$). Hence the obtained solutions are degenerate in helicity. See Fig. 1 for a description of the $n = 0$ and $n = 1$ Landau levels.

III. HELICAL MODE

In the previous section we assumed that all components of $\mathbf{F}^{(\pm)}$ are nonzero. We observe that there is an ambiguity of expressions (18) and (19) if one sets $F_z^{(\pm)} = 0$ and $\Omega^2 = p_z^2$ in them. Hence, in the following we examine the $F_z^{(\pm)} = 0$ case. In the following we again consider infinite geometry and assume propagation in z direction, such that $\mathbf{F}^{(\pm)} \propto e^{-i\omega t} e^{ip_z z}$. In this case we obtain, for Eq. (8),

$$i p_z F_y^{(\pm)} = \mp \Omega F_x^{(\pm)}, \tag{21}$$

$$i p_z F_x^{(\pm)} = \pm \Omega F_y^{(\pm)}, \tag{22}$$

$$\Pi_x F_y^{(\pm)} - \Pi_y F_x^{(\pm)} = 0, \tag{23}$$

from which one gets $F_x^{(\pm)} = is F_y^{(\pm)}$, with solution index $s = \pm$. For $s = +$ we get $(\Pi_x - i\Pi_y) F_y^{(\pm)} = -i(2\partial_\zeta + V\bar{\zeta}) F_y^{(\pm)} = 0$, which can be met by $F_y^{(+)} = f(\bar{\zeta}) \exp(-\frac{V}{2}|\zeta|^2)$ solution, with arbitrary function $f(\bar{\zeta})$, given a requirement for the solution to decay as $|\zeta| \to \infty$. For $s = -$ we get $(\Pi_x + i\Pi_y) F_y^{(\pm)} = -i(2\partial_{\bar{\zeta}} - V\zeta) F_y^{(\pm)} = 0$, which cannot be met by solutions decaying at large $|\zeta|$. Hence, only the $s = +$ solution exists for the $v > 0$ choice of the angular velocity of dielectric media rotation. The spectrum of the $s = +$ solution for $\pm$ helicity is

$$\omega_\pm = \mp \sqrt{\frac{c^2}{\epsilon\mu}}\, p_z, \tag{24}$$

and, together with the $\omega > 0$ condition, we obtain a requirement that $+$ helical photons must have $p_z < 0$, and that $-$ helical photons must have $p_z > 0$. Hence, such a solution is purely helical; i.e., opposite helicities counterpropagate. See Fig. 1 for the plot of the helical Landau level. If the rotation is switched to an opposite, $v \to -v$, the propagation structures of different helicities switch places. Note that the spectrum (24) satisfies the $\Omega^2 = p_z^2$ assumption which we started this section with. Hence solutions described by $F_z^{(\pm)} \neq 0$ obtained in Eq. (15) and the helical $F_z^{(\pm)} = 0$ solution are different.

IV. HELICAL VORTICAL EFFECT FOR PHOTONS

Here we calculate the equilibrium helicity current in the direction of the rotation axis. For that we assume a gas of photons propagating inside a hypothetical infinite and rotating dielectric medium. The current of a given photon helicity is

$$j_z^{[\pm]} = \sum_n \int_{-\infty}^{+\infty} \frac{dp_z}{4\pi^2} \left(\frac{\epsilon\mu - 1}{c^2} |v| \omega_n \right) \frac{d\omega_n}{dp_z} g(\omega_n), \tag{25}$$

where a $(\frac{\epsilon\mu-1}{c^2}|v|\omega_n) > 0$ factor is due to the Landau level degeneracy [29], $\frac{d\omega_n}{dp_z}$ is the photon velocity along the z direction, and $g(\omega_n) = (e^{\omega_n/T} - 1)^{-1}$ is the Bose-Einstein distribution function at nonzero temperature T. Summation is over all Landau levels obtained in previous sections and given by Eqs. (20) and (24). Only the lowest Landau level given by Eq. (24) is helical; hence it is the only level that contributes to the helicity current. Calculations show that the helicity current is

$$j_z^{[\mathrm{H}]} = j_z^{[-]} - j_z^{[+]} = \left(\frac{\epsilon\mu - 1}{c^2} v \right) \frac{T^2}{12}. \tag{26}$$

This current means that, in a hypothetical infinite and rotating dielectric medium, opposite helicities of photons will counterpropagate. The net photon current is zero; i.e., $j_z^{[-]} + j_z^{[+]} = 0$ as expected in equilibrium. This effect can be understood as an inverse to the optical torque that circularly polarized light exerts on the dielectric medium it propagates through [33].

V. DISCUSSION AND CONCLUSIONS

For an inhomogeneous case when ϵ and μ are functions of coordinates, Eqs. (8) and (9) will change (see, for a review, Ref. [36]). For example, imagine a rotating cylinder, for which $\epsilon = 1$ and $\mu = 1$ outside of the cylinder, and $\epsilon \neq 1$ inside. Helicities will then be mixed due to the inhomogeneous ϵ and μ functions. The helicity degeneracy of solutions of Maxwell equations described by Eqs. (15) and (20) will not change in finite geometry. However, the helical mode solutions (24) will change due to helicity mixing. Also, it is important to note that there is a natural limit on the radius of a rotating cylinder given by condition $|\mathbf{v}|/c \ll 1$. Therefore, if it will be possible to excite such a helical mode in finite geometry, there will be spatial separation of copropagating opposite-helicity waves. For example, in a particular direction of cylinder rotation, the intensity of $+$ helicity of the $p_z > 0$ wave will peak at the surface of the cylinder, while intensity of $-$ helicity will peak closer to the axis of the cylinder.

We note a striking similarity of the helical mode obtained in the present paper to the chiral lowest Landau level of a three-dimensional Dirac fermion (for example, see Ref. [30]). The similarity is due to the nontrivial Berry curvature of photons and Dirac fermions. In a rotating gas of photons at finite temperature, opposite helicities counterpropagate along the axis of rotation and result in a finite helicity current [see Eq. (26)]. It is the helical vortical effect, an analog of the chiral magnetic effect for Dirac fermions (see Ref. [31] for a review). The calculation of the helical vortical effect (chiral vortical effect) for photons appeared in this paper shortly after Refs. [37,38]. We believe the results of Refs. [37,38] and of the present paper, all being obtained by different methods, complement each other. The present paper utilizes the zero-mode description of the helical vortical effect.

It is tempting to search for photon Landau levels in pulsars; however, we note the dielectric function in the pulsar atmosphere is $\epsilon \sim 1$, and the levels are hard to resolve due to the $\epsilon\mu - 1 \ll 1$ factor. It is possible that Landau levels for photons proposed in this paper can be observed in experiments with slow light [35]. It requires further thorough investigation.

To conclude, in this paper we have described the occurrence of the Landau quantization of the frequency of photons propagating in a rotating dielectric medium. Solutions described in Sec. II are in accord with previous studies [21,23,26]. In Sec. III we have found an additional solution, which we called the helical mode solution [see Eq. (24)]. This solution describes photons with counterpropagating helicities. The solution leads to a helical vortical effect in a gas of photons at finite temperature. We note that the helical vortical effect can be understood as an inverse of the optical torque [33].

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