Unified framework to determine Gaussian states in continuous-variable systems

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Gaussian states are the backbone of quantum information protocols with continuous-variable systems whose power relies fundamentally on the entanglement between the different modes. In the case of global pure states, knowledge of the reduced states in a given bipartition of a multipartite quantum system bears information on the entanglement in such bipartition. For Gaussian states, the reduced states are also Gaussian, so their determination requires essentially the experimental determination of their covariance matrix. Here we develop strategies to determine the covariance matrix of an arbitrary n-mode bosonic Gaussian state through measurement of the total phase acquired when appropriate metaplectic evolutions, associated with quadratic Hamiltonians, are applied. Simply one-mode metaplectic evolutions, such rotations, squeezing, and shear transformations, in addition to a single two-mode rotation, allows us to determine all the covariance matrix elements of an n-mode bosonic system. All the single-mode metaplectic evolutions are applied conditionally to a state in which an ancilla qubit is entangled with the n-mode system. The ancillary system provides, after measurement, the value of the total phase of each evolution. The proposed method is experimentally suited to implement in the most currently used continuous-variable systems.

I. INTRODUCTION

Quantum information processing is a research area devoted to the study of information processing with quantum states. Its importance relies on the great advantages its protocols have in comparison with the currently known protocols of classical information processing. The theoretical realm of quantum information processing comprises quantum computation or simulation, and quantum communication protocols, with emphasis on quantum teleportation and quantum cryptography.

Whereas the first advances regarding the theoretical development and the experimental implementation of quantum information processing arose in systems with finite Hilbert spaces, more recently almost all the quantum information protocols have been extended to systems with infinite-dimensional Hilbert spaces, called continuous-variable (CV) systems [1–4]. For example, two schemes of quantum computation in discrete variables, i.e., quantum computation based on sequential applications of quantum gates and the one-way measurement-based quantum computation based on cluster states, were recently generalized to continuous-variable systems [5–7]. The same goes for the protocols of quantum teleportation, quantum cloning, quantum dense coding, and quantum cryptography [2,8–11], some of which have already been experimentally implemented [12–15].

It is worth noting that in the transition from discrete- to continuous-variable systems some advantages are gained, since several quantum information protocols are optimized using infinite-dimensional Hilbert spaces [2]. Moreover, the entanglement, the main resource in quantum processing protocols, can be efficiently produced using squeezed light and linear optics [16]. In addition, entanglement can also be detected more efficiently because the detectors for CVs in the optical domain are traditionally more efficient. Indeed, the generation and manipulation of highly entangled states is achievable in CV systems [17,18] and very often continuous-variable entanglement surpasses its discrete counterpart.

When dealing with entangled CV systems, Gaussian states (GSs) stand out as the paradigmatic ones [19–24]. These states constitute a powerful setting for quantum communication and quantum information protocols [10,25,26] and lie at the heart of CV optical and atomic technologies [27–29]. Considerable effort has been devoted to characterizing the informational properties and the entanglement structure of GSs [3,23,24,30–42]. Particularly noteworthy is the exceptional role of GSs in CV systems, since they are extremal with respect to various applications [43].

Highly multipartite entangled GSs (cluster states) can be produced, for example, by multimode frequency combs generated by a synchronously pumped optical parametric oscillator (SPOPO) [6,17,44–46]. Within this setup, a frequency comb with 60 entangled modes of the electromagnetic field was reported in [17]. Also, Gaussian states are easy to prepare and control in trapped ions, atomic ensembles, and optomechanical systems [3]. In particular, a trapped ionic system manipulated by laser light is now one of the most developed settings for the experimental investigation of quantum effects and processing of quantum information [28,47,48]. The trapping potential confines the system to a harmonic motion in the vibrational modes, whose ground state is a GS [28,49]. A scheme of quantum computation over the vibrational modes of a single trapped ion was recently suggested [50]. Highly entangled Gaussian states can also be generated, to

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a good approximation, with twin photons generated in the spontaneous parametric down-conversion (SPDC), since they can be performed as generalizations of two-mode squeezed states [29,51–53].

The complete determination of a generic (non-Gaussian) state relies on a fully tomographic process. However, GS characterization is achieved by specifying only its first and second canonical moments. The first moments can be freely adjusted by local phase-space displacements and play no role in determining entanglement properties of the state. Instead, the second moments determine the so-called covariance matrix (CM) and fully characterize the relevant informational properties of the GS, particularly its entanglement structure.

Each physical type of continuous-variable systems has its particular method for the determination of the covariance matrix of the system state. In the context of quadrature modes of the quantized electromagnetic field the traditional method is homodyne detection, which involves the interference of the input field to be probed with a local oscillator in a beam splitter. In this case, the value of the chosen measured quadrature is directly obtained from the difference of the photocurrent at the output ports of the beam splitter. The fluctuations around the mean values give the variances needed to infer the matrix elements of the CM of the input field quantum state. In CV systems where the quadrature measurement is not directly accessible, there exist two different strategies to determine the covariance matrix. These CV systems are generically massive oscillators and the first strategy involves the measurement of a qubit ancilla properly coupled to the oscillators, which directly gives phase-space values of the Weyl characteristic function of the quantum state. For one-vibrational modes of a trapped ion this strategy was outlined in [54] and later generalized for a network of oscillators in [55]. The second strategy is more suitable in the context of optomechanical systems. It consists in using a CV probe entangled with the oscillators [56,57]. In this case an intracavity electromagnetic mode is coupled via radiation pressure to a mechanical mode through one mobile cavity mirror. The covariance matrix of the mechanical mode is inferred through homodyne detection of the leaking field of the cavity, which contains information about the intracavity mode and hence about the mechanical mode. In the context of the CV system corresponding to the spatial transverse modes of single photons, the best method available to determine the CM of a quantum state (Gaussian or not) was reported in [58].

Here we present a unified method to determine the covariance matrix of Gaussian states that can be implemented in any CV system. The tools involved in our method are unitary evolutions that preserve the Gaussianity of the evolved state and the total phase acquired by the state under such evolutions. The former corresponds to the metaplectic group of unitary operations $\hat{M}_\mathsf{S}$ generated by quadratic Hamiltonians in the position and momentum canonical conjugate operators [22,59], which are characterized by a symplectic matrix S [59]. The second tool is the total phase acquired by the Gaussian state $\hat{\rho}_G$ through the evolution, given by [60]

$$\phi = \arg[\mathrm{Tr}(\hat{M}_\mathsf{S}\hat{\rho}_G)]. \qquad (1)$$

This is a particular case of a general extension of the total phase $\phi = \arg[\langle\psi|\hat{U}|\psi\rangle]$, originally defined for pure states

$\hat{\rho} = |\psi\rangle\langle\psi|$, where it was defined as the sum of the geometric and dynamical phases of the evolution [61].

The feasibility of the method developed here relies basically on two main features. The first one is that the required unitary evolutions are one-mode metaplectic operators (such as rotations, shearings, and squeezings [19,59,62]) and a single two-mode rotation (i.e., a beam-splitter-like rotation [22]). The second one is that these evolutions imprint the information of the covariance matrix elements in the corresponding total phases ϕ. Hence, by means of a protocol experimentally favorable for measuring these total phases, the information of the full covariance matrix can be recovered, irrespective of the CV system involved. In particular, here we propose such experimental protocols in three paradigmatic CV systems: the quantized electromagnetic field, the vibrational modes in trapped ions, and the transverse spatial degrees of freedom of entangled single photons.

The work is structured as follows. In Sec. II we review the Weyl-Wigner formalism that will allow us to calculate the total phase acquired by Gaussian states under arbitrary metaplectic evolutions. The metaplectic group, with special attention paid to the Weyl and Wigner symbols of the metaplectic operators, is introduced in Sec. III. The total phase acquired by an n-mode arbitrary GS under metaplectic evolutions is calculated in Sec. IV. In Sec. V we present the strategies that allow for full determination of the covariance matrix (and hence the GS) through the implementation of appropriate metaplectic evolutions in different copies of the GS, plus further measurement of the corresponding acquired phases. Section V C is devoted to a brief outline of the main features of Gaussian entanglement and the applicability of our method to determine entanglement in bipartitions having $1 \times (n-1)$ modes in GS. The general experimental protocol aimed at measuring the total phase acquired by a general state under an arbitrary unitary evolution is described in Sec. VI. We also discuss specifically the case of metaplectic operations over GSs. In Sec. VII we describe the implementation of the protocol in the context of the spatial degrees of freedom of twin photons generated in the SPDC, trapped ions, and quantized modes of the electromagnetic field. Finally, a summary and discussion are provided in Sec. VIII.

II. WEYL-WIGNER FORMALISM

We consider a multipartite system composed of n bosonic modes, described through the column vector of operators denoted by $\hat{x} := (\hat{q}_1, \hat{p}_1, \ldots, \hat{q}_n, \hat{p}_n)^\top$, where $\hat{q}_j$ and $\hat{p}_j$ stand for the position and momentum operators, respectively, of the jth mode. The usual commutation relation between these operators can be succinctly written as $[\hat{x}_j, \hat{x}_k] = i\hbar\mathsf{J}_{jk}$, where $\hat{x}_{j\,(k)}$ is the jth (kth) component of $\hat{x}$, with J_{jk} being the elements of the $2n \times 2n$ symplectic matrix

$$\mathsf{J} = \bigoplus_{j=1}^{n} \mathsf{J}_2, \quad \mathsf{J}_2 \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad (2)$$

such that $\mathsf{J}^\top = -\mathsf{J} = \mathsf{J}^{-1}$. When dealing with only one mode, it is useful to define the two-component column vector of canonically conjugate operators $\hat{x}^{(j)} = (\hat{q}_j, \hat{p}_j)^\top$ such that $[\hat{x}_k^{(j)}, \hat{x}_l^{(j)}] = i\hbar(\mathsf{J}_2)_{kl}$.

An alternative description of n bosonic modes, more often used in the context of second quantization, resorts to the annihilation and creation operators of each mode, $\hat{a}_j := (1/\sqrt{2\hbar})(\hat{q}_j + i\,\hat{p}_j)$ and $\hat{a}_j^\dagger := (1/\sqrt{2\hbar})(\hat{q}_j - i\,\hat{p}_j)$, respectively. These operators satisfy the bosonic commutation relations $[\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk}$ and $[\hat{a}_j, \hat{a}_k] = [\hat{a}_j^\dagger, \hat{a}_k^\dagger] = 0$. We will resort to this description only to specify some quadratic Hamiltonians that will appear in the following sections.

The Weyl translation operator is defined as [19,59,62]

$$\hat{T}_\xi := \hat{T}_{\xi^{(1)}} \otimes \cdots \otimes \hat{T}_{\xi^{(n)}} = e^{(i/\hbar)\hat{x}^\top \mathsf{J}\xi}, \tag{3}$$

where we define the chord $\xi := (\xi_{q_1}, \xi_{p_1}, \ldots, \xi_{q_n}, \xi_{p_n})^\top$ and $\xi^{(j)} := (\xi_{q_j}, \xi_{p_j})$. Note that the chord is a column vector indicating the direction of the translation of the canonically conjugate operators, i.e., $\hat{T}_\xi^\dagger \hat{x} \hat{T}_\xi = \hat{x} + \xi\hat{1}$. Notice that the translation operator is unitary, so $\hat{T}_\xi^{-1} = \hat{T}_\xi^\dagger = \hat{T}_{-\xi}$.

The symplectic Fourier transform of $\hat{T}_\xi$ is known as the reflection operator [62,63], namely,

$$\hat{R}_x := \frac{1}{2^n}\hat{\mathbb{R}}_x = \frac{1}{2^n}(\hat{\mathbb{R}}_{x^{(1)}} \otimes \cdots \otimes \hat{\mathbb{R}}_{x^{(n)}})$$
$$= \int \frac{d\xi}{(4\pi\hbar)^n} e^{(i/\hbar)\xi^\top \mathsf{J}x}\hat{T}_\xi, \tag{4}$$

which is a Hermitian and unitary (hence involutory) operator, that is, $\hat{R}_x^2 = \hat{1}$. Here the center $x := (q_1, p_1, \ldots, q_n, p_n)^\top$ is a column vector in phase space indicating the reflection point, i.e., $\hat{R}_x \hat{x} \hat{R}_x = -\hat{x} + 2x\hat{1}$. We also define $x^{(j)} := (q_j, p_j)$.

An arbitrary operator $\hat{A}$ acting on the Hilbert space of the continuous-variable (n-mode) system can be uniquely expanded as a linear combination of either translation (3) or reflection (4) operators [59,62]. These expansions constitute, respectively, the Weyl and the Wigner representation of $\hat{A}$:

$$\hat{A} = \int \frac{d\xi}{(2\pi\hbar)^n} \mathcal{A}(\xi)\hat{T}_\xi, \tag{5a}$$

$$\hat{A} = \int \frac{dx}{(2\pi\hbar)^n} A(x)\hat{\mathbb{R}}_x. \tag{5b}$$

The coefficients $\mathcal{A}(\xi)$ and $A(x)$ are, respectively, the Weyl and the Wigner symbols of the operator $\hat{A}$, given by

$$\mathcal{A}(\xi) = \mathrm{Tr}(\hat{A}\hat{T}_\xi^\dagger), \quad A(x) = \mathrm{Tr}(\hat{A}\hat{\mathbb{R}}_x), \tag{6}$$

by virtue of [62]

$$\mathrm{Tr}(\hat{T}_\xi \hat{T}_{\xi'}^\dagger) = (2\pi\hbar)^n \delta(\xi' - \xi),$$
$$\mathrm{Tr}(\hat{\mathbb{R}}_x \hat{\mathbb{R}}_{x'}) = (2\pi\hbar)^n \delta(x' - x). \tag{7}$$

The Weyl and Wigner symbols are related to each other via a symplectic Fourier transform, viz.,

$$\mathcal{A}(\xi) = \int \frac{dx}{(2\pi\hbar)^n} A(x)e^{(i/\hbar)\xi^\top \mathsf{J}x}, \tag{8a}$$

$$A(x) = \int \frac{d\xi}{(2\pi\hbar)^n} \mathcal{A}(\xi)e^{(i/\hbar)x^\top \mathsf{J}\xi}. \tag{8b}$$

In particular, the Wigner function $W(x)$ of a quantum state is (a normalized version of) the Wigner symbol associated with

the corresponding density operator $\hat{\rho}$ [62–64], that is,

$$W(x) := \mathrm{Tr}\left[\frac{\hat{\rho}}{(2\pi\hbar)^n}\hat{\mathbb{R}}_x\right]. \tag{9}$$

Its symplectic Fourier transform is the Weyl symbol (or characteristic function) of $\hat{\rho}$ [62]:

$$\chi(\xi) = \int \frac{dx}{(2\pi\hbar)^n} W(x)e^{(i/\hbar)\xi^\top \mathsf{J}x} = \mathrm{Tr}\left[\frac{\hat{\rho}}{(2\pi\hbar)^n}\hat{T}_\xi^\dagger\right]. \tag{10}$$

Thus, for example, for a Gaussian state with null mean values, Eqs. (9) and (10) lead to

$$W_G(x) = \frac{1}{(2\pi\hbar)^n} \frac{\exp\left[-\frac{1}{2\hbar}x^\top \mathbf{V}^{-1}x\right]}{\sqrt{\det \mathbf{V}}} \tag{11}$$

and

$$\chi_G(\xi) = \frac{1}{(2\pi\hbar)^n} \exp\left[-\frac{1}{2\hbar}\xi^\top \mathsf{J}^\top \mathbf{V}\mathsf{J}\xi\right], \tag{12}$$

where $\mathbf{V}$ is the $2n \times 2n$ covariance matrix with elements

$$\mathbf{V}_{ij} = \frac{1}{2\hbar}\mathrm{Tr}[\hat{\rho}(\hat{x}_i\hat{x}_j + \hat{x}_j\hat{x}_i)]. \tag{13}$$

Notice that since the mean values of a general state $\hat{\rho}$ can be made equal to zero by simply performing a translation [that is, a local operation in each mode; see Eq. (3)] according to $\hat{T}_\xi^\dagger \hat{\rho} \hat{T}_\xi$, with $\xi = -\mathrm{Tr}(\hat{\rho}\hat{x}) = -[\mathrm{Tr}(\hat{\rho}\hat{x}_1), \ldots, \mathrm{Tr}(\hat{\rho}\hat{x}_{2n})]^\top$, we can restrict our attention to Gaussian states with null mean values, without loss of generality.

III. METAPLECTIC GROUP AND ITS WEYL-WIGNER REPRESENTATIONS

In this section we introduce the metaplectic operators associated with unitary evolutions under quadratic Hamiltonians and focus on their corresponding Weyl and Wigner symbols. These symbols will allow us to calculate the total phase corresponding to metaplectic evolutions over Gaussian states in the following.

Quadratic Hamiltonians are defined as those of the form

$$\hat{H} = \frac{\omega}{2}\hat{x}^\top \mathbf{H}\hat{x}, \tag{14}$$

where $\mathbf{H}$ is a $2n \times 2n$ symmetric real matrix known as the Hessian of $\hat{H}$ and ω is a real parameter. These Hamiltonians constitute the algebra $\mathfrak{mp}(2n,\mathbb{R})$ of the metaplectic group [19,22,59,62]. As is usual for Lie groups, when exponentiating elements in the algebra we obtain elements of the group

$$\hat{M}_\mathsf{S} := e^{-(i\omega t/2\hbar)\hat{x}^\top \mathbf{H}\hat{x}}. \tag{15}$$

Here the subindex S highlights the relation between the metaplectic operator $\hat{M}_\mathsf{S}$ in Eq. (15) and the matrix

$$\mathsf{S} := e^{\mathsf{J}\mathbf{H}\omega t}, \tag{16}$$

which is an element of the real symplectic group $\mathrm{Sp}(2n,\mathbb{R})$, defined as the set of matrices such that $\mathsf{S}^\top \mathsf{J}\mathsf{S} = \mathsf{J}$.

Note that $\mathsf{J}\mathbf{H}$ is an element of the symplectic algebra $\mathfrak{sp}(2n,\mathbb{R})$ that is in one-to-one correspondence with the element in Eq. (14) that belongs to the algebra $\mathfrak{mp}(2n,\mathbb{R})$. However, it may be that for some matrices $\mathsf{J}\mathbf{H}$ there are two values of ωt that give the same symplectic matrix S in Eq. (16).

This is a manifestation of the fact that the metaplectic group is a double covering group of the symplectic one, i.e., there are two metaplectic operators, namely, $\pm \hat{M}_{\mathsf{S}}$, associated with each symplectic matrix S [19,22]. Another peculiar characteristic of the metaplectic group is that, as occurs in the symplectic group, it is not an exponential group [22]. Thus, there are elements in $\mathrm{Mp}(2n,\mathbb{R})$ and in $\mathrm{Sp}(2n,\mathbb{R})$ that cannot be written as an exponentiation of some element in $\mathfrak{mp}(2n,\mathbb{R})$ and $\mathfrak{sp}(2n,\mathbb{R})$, respectively, but rather decompose into products of operators like that in Eq. (15). In this case the associated symplectic matrix is the product of symplectic matrices like those in Eq. (16), corresponding to each factor of the metaplectic decomposition. In fact, any symplectic matrix can be written as a product of another symplectic matrices in a nonunique way. This leads to different decompositions for the associated metaplectic operator. In particular, it will be useful for later purposes to resort to the factorization proved in [59] that establishes that every $\mathsf{S} \in \mathrm{Sp}(2n,\mathbb{R})$ can be written as

$$\mathsf{S} = \mathsf{S}'\mathsf{S}'', \tag{17}$$

where S' and S'' are symplectic matrices that are products of matrices of the form (16) and such that $\det(\mathsf{S}' + \mathsf{I}_{2n}) \neq 0$ and $\det(\mathsf{S}'' + \mathsf{I}_{2n}) \neq 0$ (that is, neither S' nor S'' has an eigenvalue equal to -1). The metaplectic operator corresponding to S as given by Eq. (17) can be chosen as

$$\hat{M}_{\mathsf{S}} = \pm \hat{M}_{\mathsf{S}'} \hat{M}_{\mathsf{S}''}. \tag{18}$$

The indeterminacy of the signal $\pm$ is removed once we specify the time dependence of the symplectic matrix $\mathsf{S} = \mathsf{S}(t)$ as we will see in what follows.

The Weyl and Wigner symbols of the metaplectic operator (15) are given, respectively, by [59,65]

$$\mathcal{M}_{\mathsf{S}}(\xi) = \frac{i^{\nu_{\mathsf{S}}^-} \exp\left[-\frac{i}{4\hbar}\xi^\top \mathsf{J}\mathbf{C}_{\mathsf{S}}^{-1}\mathsf{J}\xi\right]}{\sqrt{|\det(\mathsf{S} - \mathsf{I}_{2n})|}} \tag{19}$$

and

$$M_{\mathsf{S}}(x) = \frac{2^n i^{\nu_{\mathsf{S}}^+} \exp\left[-\frac{i}{\hbar}x^\top \mathbf{C}_{\mathsf{S}}x\right]}{\sqrt{|\det(\mathsf{S} + \mathsf{I}_{2n})|}}. \tag{20}$$

Here the symmetric matrix

$$\mathbf{C}_{\mathsf{S}} = -\mathsf{J}\frac{\mathsf{S} - \mathsf{I}_{2n}}{\mathsf{S} + \mathsf{I}_{2n}} \tag{21}$$

stands for the Cayley parametrization of S. Note that, depending on S, the above symbols may not be defined, since $\mathbf{C}_{\mathsf{S}}$ or its inverse may not exist. When both symbols $\mathcal{M}_{\mathsf{S}}(\xi)$ and $M_{\mathsf{S}}(x)$ in Eqs. (19) and (20) have no divergences, they are related by the symplectic Fourier transform and the index ν_{S}^+ is given by

$$\nu_{\mathsf{S}}^+ = \nu_{\mathsf{S}}^- + \tfrac{1}{2}\mathrm{sng}\mathbf{C}_{\mathsf{S}}(\mathrm{mod}4), \tag{22}$$

where $\mathrm{sng}\mathbf{X}$ is the number of positive eigenvalues minus the number of negative eigenvalues of the matrix $\mathbf{X}$ and ν_{S}^- is the Conley-Zehnder (CZ) index [59,62,65,66]. This index determines the sign of the metaplectic operator associated with the single matrix S. This can be summarized in the definition

$$\sqrt{\det(\mathsf{S} - \mathsf{I}_{2n})} := i^{-\nu_{\mathsf{S}}^-}\sqrt{|\det(\mathsf{S} - \mathsf{I}_{2n})|}, \tag{23}$$

where ν_{S}^- acquires the values $\{0,2\}$ if $\det(\mathsf{S} - \mathsf{I}_{2n}) > 0$ and $\{1,3\}$ if $\det(\mathsf{S} - \mathsf{I}_{2n}) < 0$. For an invertible $\mathbf{C}_{\mathsf{S}}$, $\tfrac{1}{2}\mathrm{sng}\mathbf{C}_{\mathsf{S}}$ is an

integer, thus ν_{S}^+ is also an integer number in the set $\{0,1,2,3\}$ [59], in accord with Eq. (22). Notice that the symbols in (19) and in (20) diverge, respectively, when an eigenvalue of S becomes 1 and -1. In this case the symbols do exist, yet they are not calculated via Eqs. (19) and (20), but instead using, for example, Eq. (25).

Here we are interested in metaplectic operators associated with a temporal evolution, so let us assume that S depends continuously on a real parameter t. An example is given in (16), where S belongs to a uniparametric subgroup of $\mathrm{Sp}(2n,\mathbb{R})$; however, in the general case $\mathsf{S} = \mathsf{S}(t)$ does not necessarily belong to any uniparametric subgroup. At each time t the metaplectic operator associated with S has a definite sign that can be traced out by continuity of the operator with respect to S, in accord with

$$\lim_{t \to 0^+} \mathsf{S} = \mathsf{I}_{2n} \implies \lim_{t \to 0^+} \hat{M}_{\mathsf{S}} = +\hat{1}. \tag{24}$$

This continuity property reflects in the behavior of the Weyl and Wigner symbols of $\hat{M}_{\mathsf{S}}$ through the indices $\nu_{\mathsf{S}}^{\pm}$, which must change accordingly whenever there exists a discontinuity of the symbol $\mathcal{M}_{\mathsf{S}}(\xi)$ or $M_{\mathsf{S}}(x)$ in Eqs. (19) and (20), that is, whenever S has an eigenvalue 1 or -1, respectively. For example, when $t = 0$ the Wigner symbol of the identity operator is $1(x) = 1$, so $\nu_{\mathsf{S}}^+ = 0$ for $t = 0$ and all $t > 0$ until an eigenvalue of S becomes -1, which occurs, say, at $t = t^*$. Then, as long as $\mathsf{S}(t^*)$ does not have an eigenvalue equal to 1, we can switch the representation and calculate the Weyl symbol of the metaplectic operator. The continuity of the symbols in the vicinity of $t = t^*$ is guaranteed by the relation in Eq. (22). If at some time t, $\mathsf{S}(t)$ has simultaneous eigenvalues 1 and -1, we rely on the decomposition in Eq. (18) and calculate the Wigner symbol of the composition such that $\hat{M}_{\mathsf{S}} = \hat{M}_{\mathsf{S}'}\hat{M}_{\mathsf{S}''}$ using the following expression [62]:

$$M_{\mathsf{S}}(x) = \int \frac{dx'}{(\pi\hbar)^n} \int \frac{dx''}{(\pi\hbar)^n} M_{\mathsf{S}'}(x') M_{\mathsf{S}''}(x'')$$
$$\times e^{(2i/\hbar)(x''-x)^\top \mathsf{J}(x'-x)}, \tag{25}$$

where $M_{\mathsf{S}'}(x)$ and $M_{\mathsf{S}''}(x)$ are the Wigner symbols of $\hat{M}_{\mathsf{S}'}$ and $\hat{M}_{\mathsf{S}''}$, respectively, whose structure is given in Eq. (20).

IV. TOTAL PHASE OF GAUSSIAN STATES UNDER METAPLECTIC EVOLUTIONS

Consider the unitary evolution $\hat{U} = e^{-(i/\hbar)\hat{H}t}$ generated by the Hamiltonian $\hat{H}$. As a quantum state $\hat{\rho}$ evolves accordingly, it acquires a total phase defined as [61]

$$\phi = \arg[\mathrm{Tr}(\hat{U}\hat{\rho})], \tag{26}$$

with the argument function defined as

$$\arg(x + iy) = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } x > 0 \\ \arctan\left(\frac{y}{x}\right) + \pi & \text{if } x < 0, \ y \geqslant 0 \\ \arctan\left(\frac{y}{x}\right) - \pi & \text{if } x < 0, \ y < 0 \\ +\frac{\pi}{2} & \text{if } x = 0, \ y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0, \ y < 0 \\ \text{undefined} & \text{if } x = y = 0. \end{cases} \tag{27}$$

This implies that $-\pi \leqslant \phi \leqslant \pi$. It is important to notice, for future analysis, that we always have $|\mathrm{Tr}(\hat{U}\hat{\rho})| \leqslant 1$.

In what follows we will calculate this phase for an initial n-mode arbitrary Gaussian state with null mean value $\hat{\rho} = \hat{\rho}_G$, subject to a unitary evolution generated by a generic metaplectic operator $\hat{U} = \hat{M}_{\mathsf{S}}(t)$, that is, a generic composition of evolutions like those in Eq. (15). As for the density operator, we can expand it in the Weyl representation, with the coefficients given by Eq. (12), or rather we can resort to its Wigner representation, and employ Eq. (11). Then, using Eqs. (5) for $\hat{A} = \hat{M}_{\mathsf{S}}$, we can write $\mathrm{Tr}(\hat{\rho}_G \hat{M}_{\mathsf{S}})$ as

$$\mathrm{Tr}(\hat{\rho}_G \hat{M}_{\mathsf{S}}) = \int d\xi \, \chi_G(\xi) \mathcal{M}_{\mathsf{S}}(-\xi) \tag{28a}$$

$$= \int dx \, W_G(x) M_{\mathsf{S}}(x). \tag{28b}$$

We now resort to Eqs. (11), (12), (19), and (20) and perform the Gaussian integrations to get

$$\mathrm{Tr}(\hat{\rho}_G \hat{M}_{\mathsf{S}}) = \frac{i^{\nu_{\mathsf{S}}^-}}{\sqrt{|\det(\mathsf{S} - \mathsf{I}_{2n})|\det\left(\mathbf{V} - \frac{i}{2}\mathbf{C}_{\mathsf{S}}^{-1}\right)}} \tag{29a}$$

$$= \frac{i^{\nu_{\mathsf{S}}^+}}{\sqrt{|\det(\mathsf{S} + \mathsf{I}_{2n})|\det\left(\frac{1}{2}\mathsf{I}_{2n} + i\mathbf{V}\mathbf{C}_{\mathsf{S}}\right)}}, \tag{29b}$$

where $\sqrt{z}$ is the square root with a positive real part of the complex number z. Notice that Eq. (29a) holds whenever S does not have an eigenvalue equal to 1, whereas Eq. (29b) is valid as long as none of the eigenvalues of S equals -1. If S possess eigenvalues 1 *and* -1, we can resort to the factorization in Eq. (17), decompose $\hat{M}_{\mathsf{S}}$ according to Eq. (18), and write

$$\mathrm{Tr}(\hat{\rho}_G \hat{M}_{\mathsf{S}}) = \mathrm{Tr}(\hat{\rho}_G \hat{M}_{\mathsf{S}'} \hat{M}_{\mathsf{S}''}) = \int dx \, W_G(x) M_{\mathsf{S}}(x), \tag{30}$$

where the symbol $M_{\mathsf{S}}(x)$ is given in Eq. (25). After performing the integration in (30), we obtain

$$\mathrm{Tr}(\hat{\rho}_G \hat{M}_{\mathsf{S}}) = \frac{i^{\nu_{\mathsf{S}'}^+ + \nu_{\mathsf{S}''}^+}}{\sqrt{|\det(\mathsf{S}' + \mathsf{I}_{2n})\det(\mathsf{S}'' + \mathsf{I}_{2n})|}} \frac{\sqrt{\det\left(\mathbf{V} + \frac{i}{2}\mathsf{J}\right)^{-1}}}{\sqrt{\det\left[\left(\mathbf{V} - \frac{i}{2}\mathsf{J}\right) - \left(\mathbf{V} + \frac{i}{2}\mathbf{C}_{\mathsf{S}'^{\top}}\right)\left(\mathbf{V} + \frac{i}{2}\mathsf{J}\right)^{-1}\left(\mathbf{V} + \frac{i}{2}\mathbf{C}_{\mathsf{S}''^{\top}}\right)\right]}}. \tag{31}$$

Finally, Eqs. (29a), (29b), and (31), together with Eq. (26), allow us to write the total phase $\phi_{\mathsf{S}}[\hat{\rho}_G]$ acquired by an arbitrary (null mean value) n-mode Gaussian state evolving under the metaplectic evolution $\hat{M}_{\mathsf{S}}$ as

$$\phi_{\mathsf{S}}[\hat{\rho}_G] = \frac{\pi}{2}\nu_{\mathsf{S}}^- - \frac{1}{2}\arg\left[\det\left(\mathbf{V} - \frac{i}{2}\mathbf{C}_{\mathsf{S}}^{-1}\right)\right] \tag{32a}$$

$$= \frac{\pi}{2}\nu_{\mathsf{S}}^+ - \frac{1}{2}\arg\left[\det\left(\frac{1}{2}\mathsf{I}_{2n} + i\mathbf{V}\mathbf{C}_{\mathsf{S}}\right)\right] \tag{32b}$$

$$= \frac{\pi}{2}(\nu_{\mathsf{S}'}^+ + \nu_{\mathsf{S}''}^+) - \frac{1}{2}\arg\left\{\det\left[\left(\mathbf{V} - \frac{i}{2}\mathsf{J}\right) - \left(\mathbf{V} + \frac{i}{2}\mathbf{C}_{\mathsf{S}'^{\top}}\right)\left(\mathbf{V} + \frac{i}{2}\mathsf{J}\right)^{-1}\left(\mathbf{V} + \frac{i}{2}\mathbf{C}_{\mathsf{S}''^{\top}}\right)\right]\right\}. \tag{32c}$$

Here Eq. (32a) holds whenever S does not have an eigenvalue equal to 1, Eq. (32b) whenever S does not have an eigenvalue equal to -1, and Eq. (32c) for any symplectic matrix S once the factorization in Eq. (17) is found. For a matrix S that does not have eigenvalues ± 1, Eqs. (32) above coincide.

V. DETERMINATION OF THE GAUSSIAN STATE THROUGH THE TOTAL PHASE

In what follows we exploit Eq. (32b) to design strategies that allow for the complete determination of the covariance matrix $\mathbf{V}$, hence the complete specification of an arbitrary Gaussian state $\hat{\rho}_G$, by the mere implementation of appropriate metaplectic evolutions over one and two modes, once an adequate measure of the total phase acquired in each evolution is performed. This section deals specifically with the development of the strategies (assuming that the total phases are known), whereas a particular experimental protocol for measuring such phases in the context of any CV system is left for Sec. V C.

Since the state $\hat{\rho}_G$ to be determined will evolve under suitable metaplectic evolutions, we will refer to it as the initial state. In addition, $\hat{\rho}_G$ is a quantum state in the interaction picture representation, in relation to a free evolution of the CV system represented by a metaplectic evolution (generically, though not necessarily, a harmonic one). Therefore, the total phase ϕ is the phase of the evolution in the interaction picture, associated exactly with the Hamiltonian in Eq. (14), without any constant factors added.

We start by writing the n-mode covariance matrix of the initial state $\hat{\rho}_G$ in the block form

$$\mathbf{V} = \begin{pmatrix} \mathbf{V}^{(1)} & \mathbf{E}^{(1,2)} & \cdots & \cdots & \cdots & \cdots & \mathbf{E}^{(1,n)} \\ \mathbf{E}^{(2,1)} & \ddots & \vdots & \cdots & \vdots & \vdots & \vdots \\ \vdots & \cdots & \mathbf{V}^{(j)} & \cdots & \mathbf{E}^{(j,k)} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \cdots & \mathbf{E}^{(k,j)} & \cdots & \mathbf{V}^{(k)} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \ddots & \vdots \\ \mathbf{E}^{(n,1)} & \cdots & \cdots & \cdots & \cdots & \cdots & \mathbf{V}^{(n)} \end{pmatrix}, \tag{33}$$

where $\mathbf{V}^{(i)}$ stands for the covariance matrix of the (initial) reduced ith-mode state $\hat{\varrho}^{(i)} \equiv \mathrm{Tr}_{\{l\}}(\hat{\rho}_G)$ (with $l = 1, \ldots, n$ such that $l \neq i$) and $\mathbf{E}^{(j,k)}$ denotes the intermodal correlation matrix between the modes j and k; note that since $\mathbf{V}$ is symmetric, $\mathbf{E}^{(k,j)} = (\mathbf{E}^{(j,k)})^\top$. With this notation, the covariance matrix of the two (j and k) modes, corresponding to the reduced state $\hat{\varrho}^{(jk)} = \mathrm{Tr}_{\{l\}}(\hat{\rho}_G)$, with $l = 1, \ldots, n$ such that $l \neq j, k$, reads

$$\mathbf{V}^{(j,k)} = \begin{pmatrix} \mathbf{V}^{(j)} & \mathbf{E}^{(j,k)} \\ \mathbf{E}^{(k,j)} & \mathbf{V}^{(k)} \end{pmatrix}. \tag{34}$$

Our method for determining the matrix $\mathbf{V}$ is based on determining first the elements of the matrices $\mathbf{V}^{(i)}$ and then the elements of the matrices $\mathbf{E}^{(j,k)}$ as follows.

A. Determination of the reduced single-mode covariance matrices of the initial state $\hat{\rho}_G$

In order to relate the total phase $\phi_{\mathsf{S}}[\hat{\rho}_G]$ with single-mode covariance matrices of the initial state $\hat{\rho}_G$, we apply local metaplectic operations with respect to the modes j and k, associated with symplectic matrices S of the form

$$\mathsf{S} = \mathsf{I}_{2j-2} \oplus \mathsf{S}^{(j)} \oplus \mathsf{I}_{2k-2j-2} \oplus \mathsf{S}^{(k)} \oplus \mathsf{I}_{2n-2k}. \tag{35}$$

Resorting to Eqs. (2) and (21) this gives

$$\mathbf{C}_{\mathsf{S}} = \mathbf{0}_{2j-2} \oplus \mathbf{C}_{\mathsf{S}^{(j)}} \oplus \mathbf{0}_{2k-2j-2} \oplus \mathbf{C}_{\mathsf{S}^{(k)}} \oplus \mathbf{0}_{2n-2k}, \tag{36}$$

where $\mathbf{0}_j$ is the $j \times j$ null matrix and $\mathbf{C}_{\mathsf{S}^{(j)}} = \mathsf{J}_2^{(j)}(\mathsf{I}_2 - \mathsf{S}^{(j)})(\mathsf{I}_2 + \mathsf{S}^{(j)})^{-1}$. The block matrix $\mathbf{V}^{(j)}$ (equivalently $\mathbf{V}^{(k)}$) can be selected by choosing $\mathsf{S}^{(k)} = \mathsf{I}_2$ (equivalently $\mathsf{S}^{(j)} = \mathsf{I}_2$) in Eq. (35). This corresponds to evolving the single mode j (or k) and consequently the total phase $\phi_{\mathsf{S}}[\hat{\rho}_G]$ depends only on the reduced state $\hat{\varrho}^{(j)}$ (equivalently $\hat{\varrho}^{(k)}$). As long as none of the eigenvalues of $\mathsf{S}^{(k)}$ (equivalently $\mathsf{S}^{(j)}$) equals -1, the total phase results in

$$\phi_{\mathsf{S}^{(k)}}[\hat{\varrho}^{(k)}] = \frac{\pi}{2}\nu_{\mathsf{S}^{(k)}}^+ - \frac{1}{2}\arg\left[\det\left(\frac{1}{2}\mathsf{I}_2 + i\mathbf{V}^{(k)}\mathbf{C}_{\mathsf{S}^{(k)}}\right)\right], \tag{37}$$

and similarly for the mode j. Naturally, Eq. (37) is the single-mode version of Eq. (32b), and since $\mathbf{V}^{(k)}$ is a 2×2 matrix, Eq. (37) reduces to

$$\begin{aligned} &\phi_{\mathsf{S}^{(k)}}[\hat{\varrho}^{(k)}] \\ &= \frac{\pi}{2}\nu_{\mathsf{S}^{(k)}}^+ - \frac{1}{2}\arg\left[\frac{1}{4} - \det(\mathbf{V}^{(k)}\mathbf{C}_{\mathsf{S}^{(k)}}) + \frac{i}{2}\mathrm{Tr}(\mathbf{V}^{(k)}\mathbf{C}_{\mathsf{S}^{(k)}})\right]. \end{aligned} \tag{38}$$

This expression allows us to determine the elements of the matrix

$$\mathbf{V}^{(k)} = \begin{pmatrix} a & c \\ c & b \end{pmatrix}, \tag{39}$$

with $a, b > 0$, once appropriate one-mode metaplectic evolutions are implemented and the phase $\phi_{\mathsf{S}^{(k)}}[\hat{\varrho}^{(k)}]$ is known.

In what follows we describe three strategies to do so. The single-mode transformations involved are rotation R,

squeezing Z, position shear F, and momentum shear M, corresponding to the Hamiltonians

$$\hat{H}_{\mathsf{R}} = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2), \tag{40a}$$

$$\hat{H}_{\mathsf{Z}_\varphi} = \frac{\hbar\omega}{2}(\hat{a}^{\dagger 2}e^{i\varphi} + e^{-i\varphi}\hat{a}^2), \tag{40b}$$

$$\hat{H}_{\mathsf{F}} = -\frac{\hbar\omega}{4}(\hat{a}^\dagger - \hat{a})^2, \tag{40c}$$

$$\hat{H}_{\mathsf{M}} = \frac{\hbar\omega}{4}(\hat{a}^\dagger + \hat{a})^2, \tag{40d}$$

respectively. All these metaplectic transformations are described in detail in the Appendix, where the total phases acquired by the reduced (single-mode) Gaussian state for each evolution are shown to be (here $\tau = \det\mathbf{V}^{(i)} = ab - c^2 \geqslant 1/4$ and $\beta = \mathrm{Tr}\mathbf{V}^{(k)} = a + b > 0$)

$$\phi_{\mathsf{R}} = \frac{\pi}{2}\nu_{\mathsf{R}}^+ - \frac{1}{2}\arg\left[\frac{1}{4} - \tau\tan^2\frac{\theta}{2} + \frac{i}{2}\beta\tan\frac{\theta}{2}\right] \tag{41}$$

whenever $\theta = \omega t \neq \pi, 3\pi$;

$$\begin{aligned} \phi_{\mathsf{Z}_\varphi} = -\frac{1}{2}\arg\bigg[&\frac{1}{4} + \tau\tanh^2\frac{\zeta}{2} \\ &+ i\left(\frac{a-b}{2}\cos\varphi + c\sin\varphi\right)\tanh\frac{\zeta}{2}\bigg], \end{aligned} \tag{42}$$

where $\zeta = \omega t$ is the squeezing parameter;

$$\phi_{\mathsf{F}} = -\frac{1}{2}\arg(1 + ibs), \tag{43}$$

where $s = \omega t \geqslant 0$; and

$$\phi_{\mathsf{M}} = -\frac{1}{2}\arg(1 + ias), \tag{44}$$

where $s = \omega t \geqslant 0$.

Under a particular metaplectic evolution, the initial single-mode Gaussian state $\hat{\varrho}^{(k)}$ acquires a total phase given by one of Eqs. (41)–(44). Such a phase depends on the evolution parameter as well as on the elements of $\mathbf{V}^{(k)}$. Therefore, a single evolution (hence knowledge of a single ϕ) does not suffice to invert the equations and completely determine all the elements of $\mathbf{V}^{(k)}$. Thus, a set of evolutions over the initial state (or rather a set of phases) is needed to determine $\mathbf{V}^{(k)}$.

For each evolution an acquired ϕ is determined, all of which depend on the same (initial) covariance matrix. Ultimately, when a sufficient number of phases are known, this allows for the inversion of the set of equations and the determination of all the elements of $\mathbf{V}^{(k)}$.

1. First strategy

This strategy is more suitable to be used in the determination of Gaussian states within the context of CV systems corresponding to vibrational modes of trapped ions. It involves the application of two different rotations and two different squeezing transformations.

For $0 \leqslant \theta < \pi$ we have $\nu_{\mathsf{R}}^+ = 0$ [see Eq. (A8)]. Then, according to Eq. (41), $\tan(-2\phi_{\mathsf{R}}) = \tan(\arg[z])$ with $z = \frac{1}{4} - \tau\tan^2\frac{\theta}{2} + \frac{i}{2}\beta\tan\frac{\theta}{2}$ and $\mathrm{Im}(z) \geqslant 0$. Therefore, z could be in the first or second quadrant of the complex plane. In both

cases we have

$$\tan(-2\phi_{\mathsf{R}}) = \frac{\text{Im}[z]}{\text{Re}[z]} = \frac{2\beta\tan\frac{\theta}{2}}{1 - 4\tau\tan^2\frac{\theta}{2}}. \tag{45}$$

Assume that two values of the total phase, namely, $\phi_{\mathsf{R}}' := \phi_{\mathsf{R}}(\theta')$ and $\phi_{\mathsf{R}}'' := \phi_{\mathsf{R}}(\theta'')$, are known, corresponding to two distinct rotation angles θ' and θ'', both in the interval $[0,\pi)$. Then, substituting these two values in Eq. (45), we can set up a linear system in the variables τ and β, whose solution is

$$\beta = \frac{\left(\cot^2\frac{\theta'}{2} - \cot^2\frac{\theta''}{2}\right)\tan(2\phi_{\mathsf{R}}')\tan(2\phi_{\mathsf{R}}'')}{2\cot\frac{\theta''}{2}\tan(2\phi_{\mathsf{R}}') - 2\cot\frac{\theta'}{2}\tan(2\phi_{\mathsf{R}}'')}, \tag{46a}$$

$$\tau = \frac{\cot\frac{\theta'}{2}\tan(2\phi_{\mathsf{R}}') - \cot\frac{\theta''}{2}\tan(2\phi_{\mathsf{R}}'')}{4\tan\frac{\theta'}{2}\tan(2\phi_{\mathsf{R}}') - 4\tan\frac{\theta''}{2}\tan(2\phi_{\mathsf{R}}'')}. \tag{46b}$$

On the other hand, the real part of the complex number z' in the argument function in Eq. (42) is always positive. So z' could be in the first or fourth quadrant of the complex plane. In both cases we have

$$\tan(-2\phi_{\mathsf{Z}_\varphi}) = \frac{\text{Im}[z']}{\text{Re}[z']} = \frac{2(a-b)\cos\varphi + 4c\sin\varphi}{1 + 4\tau\tanh^2\frac{\zeta}{2}}\tanh\frac{\zeta}{2}. \tag{47}$$

Once the value of τ has been obtained from Eq. (46b), it only remains to perform a squeezing transformation (with squeezing parameter ζ) and determine the value of the phase $\phi_{\mathsf{Z}_\varphi}$ to obtain c from Eq. (47) with $\varphi = \pi/2$:

$$c = -\frac{1 + 4\tau\tanh^2\frac{\zeta}{2}}{4\tanh\frac{\zeta}{2}}\tan(2\phi_{\mathsf{Z}_{\pi/2}}). \tag{48}$$

Finally, applying a second squeezing transformation with the same squeezing parameter ζ, but now with $\varphi = 0$, we get from Eq. (47) the following value for $\gamma := b - a$:

$$\gamma = \frac{1 + 4\tau\tanh^2\frac{\zeta}{2}}{2\tanh\frac{\zeta}{2}}\tan(2\phi_{\mathsf{Z}_0}). \tag{49}$$

Then, as we have $a + b = \beta$ and $b - a = \gamma$, we can calculate $a = (\beta - \gamma)/2$ and $b = (\beta + \gamma)/2$ and in this way completely determine the covariance matrix (39).

2. Second strategy

This strategy is more suitable to be used in the determination of Gaussian states within the context of CV systems corresponding to quadrature modes of the quantized electromagnetic field. It relies on three different rotations and a squeezing transformation, but here only the total phases corresponding to the rotations have to be determined, as we will see.

First, the trace β and the determinant τ of the matrix (39) are calculated by performing two different rotations, exactly as we did in Eqs. (46a) and (46b). Then a unitary evolution corresponding to a squeezing transformation is performed over the state $\hat{\rho}_G$ with Hamiltonian (40b) setting $\varphi = 0$, so the evolved state will be $\hat{\rho}_G' = \hat{M}_{\mathsf{Z}_0}\hat{\rho}_G\hat{M}_{\mathsf{Z}_0}^\dagger$. The corresponding

symplectic transformation is given by (A10), leading to a covariance matrix of the evolved state equal to $\mathbf{V}' = \mathbf{Z}_0\mathbf{V}\mathbf{Z}_0^\top$.

Now a third rotation of an angle $\theta''' \in [0,\pi)$ is performed to obtain the phase using (45) for the new (squeezed) state $\hat{\rho}_G'$. Thus, defining $\phi_{\mathsf{R}}''' := \phi_{\mathsf{R}}(\theta''')$, one gets

$$\tan(2\phi_{\mathsf{R}}''') = \frac{2\beta'\tan\frac{\theta}{2}}{4\tau\tan^2\frac{\theta}{2} - 1}, \tag{50}$$

where we have used $\det\mathbf{V}' = \det\mathbf{V} = \tau$ and $\beta' := \text{Tr}\mathbf{V}' = \beta\cosh(2\zeta) - 2c\sinh(2\zeta)$. Solving Eq. (50) for c, one finds

$$c = \frac{1 - 4\tau\tan^2\frac{\theta}{2}}{4\tan\frac{\theta}{2}\sinh(2\zeta)}\tan(2\phi_{\mathsf{R}}''') + \frac{\beta}{2}\text{cotanh}(2\zeta). \tag{51}$$

With this, and β and τ given by Eqs. (46a) and (46b), respectively, the system of equations $\beta = a + b$ and $\tau = ab - c^2$ can be solved to get $a = \sqrt{\beta - \tau - c^2}$ and $b = \beta - \sqrt{\beta - \tau - c^2}$. It is worth noting that, in this strategy, it is necessary to determine only phases associated with rotations. The evolution phase corresponding to the intermediate application of a squeezing transformation does not need to be determined.

3. Third strategy

Although the first and second strategies can be implemented in the determination of Gaussian states within the context of CV systems corresponding to spatial transverse degrees of freedom of single photons, this third strategy would be experimentally less demanding in this particular system. It involves the implementation of two squeezing plus a coordinate or momentum shear transformation.

Let us assume that two values of the total phase, namely, $\phi_{\mathsf{Z}}' := \phi_{\mathsf{Z}_{\pi/2}}(\zeta')$ and $\phi_{\mathsf{Z}}'' := \phi_{\mathsf{Z}_{\pi/2}}(\zeta'')$, are known, corresponding to squeezing transformations for two distinct values of the squeezing parameter ζ and $\varphi = \pi/2$. Then, from Eq. (47) we can set up a linear system of equations with unknown variables c and τ, whose solution is

$$c = \frac{\frac{1}{4}\left[\tanh^2\frac{\zeta''}{2} - \tanh^2\frac{\zeta'}{2}\right]\tan(2\phi_{\mathsf{Z}}')\tan(2\phi_{\mathsf{Z}}'')}{\tanh\frac{\zeta''}{2}\tanh\frac{\zeta'}{2}\left[\tanh\frac{\zeta''}{2}\tan(2\phi_{\mathsf{Z}}') - \tanh\frac{\zeta'}{2}\tan(2\phi_{\mathsf{Z}}'')\right]}, \tag{52a}$$

$$\tau = \frac{\frac{1}{4}\tanh\frac{\zeta'}{2}\tan(2\phi_{\mathsf{Z}}'') - \frac{1}{4}\tanh\frac{\zeta''}{2}\tan(2\phi_{\mathsf{Z}}')}{\tanh\frac{\zeta''}{2}\tanh\frac{\zeta'}{2}\left[\tanh\frac{\zeta'}{2}\tan(2\phi_{\mathsf{Z}}') - \tanh\frac{\zeta''}{2}\tan(2\phi_{\mathsf{Z}}'')\right]}. \tag{52b}$$

In order to determine the matrix elements a and b we need to perform either a position or a momentum shear and use Eq. (43) or (44). Thus, for example, if we perform a position shear so that ϕ_{F} is known, Eq. (43) leads to

$$b = -\frac{1}{s}\tan(2\phi_{\mathsf{F}}). \tag{53}$$

Then, once b, c, and τ are known, it is straightforward to determine a according to $a = (\tau + c^2)/b$. Alternatively, we can perform a momentum shear transformation, determine the

total phase ϕ_M, and resort to Eq. (44) to get

$$a = -\frac{1}{s}\tan(2\phi_M). \quad (54)$$

Once a, c, and τ are known, we can obtain the matrix element b according to $b = (\tau + c^2)/a$.

B. Determination of the two-mode intermodal correlation matrices of the initial state $\hat{\rho}_G$

The preceding section provided a method for determining the one-mode covariance matrices $\mathbf{V}^{(i)}$ with $i = j,k$. Here we will assume that these two matrices are already known and develop a strategy for determining the elements of a generic two-mode correlation matrix

$$\mathbf{E}^{(j,k)} = \begin{pmatrix} v & w \\ y & z \end{pmatrix}. \quad (55)$$

The method exhibits the same spirit as that for determining $\mathbf{V}^{(k)}$ in the sense that it resorts to the implementation of appropriate metaplectic evolutions to extract information regarding the matrix elements of $\mathbf{E}^{(j,k)}$. However, it differs from the strategies of Sec. V A in that here, before the determination of the evolution phases, an extra two-mode and single-mode rotations must be implemented.

We first apply a (nonlocal) two-mode rotation, corresponding to the Hamiltonian $\hat{H}^{(j,k)} = (\hbar\omega/2)(\hat{a}_k^\dagger \hat{a}_j + \hat{a}_k \hat{a}_j^\dagger)$ (when the continuous-variable system refers to the quantized electromagnetic fields, $\hat{H}^{(j,k)}$ represents the beam splitter evolution over modes j and k). If the rotation is performed by an angle $\theta = \omega t = \pi/2$, the symplectic matrix associated with the n-mode transformation reads

$$\mathbf{S} = \mathbf{I}_{2j-2} \oplus \mathbf{S}^{(j,k)} \oplus \mathbf{I}_{2n-2k}, \quad (56)$$

with the $(2k - 2j + 2) \times (2k - 2j + 2)$ matrix $\mathbf{S}^{(j,k)}$ given by

$$\mathbf{S}^{(j,k)} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{I}_2 & \mathbf{0}_2 & \cdots & \mathbf{0}_2 & \mathbf{J}_2 \\ \mathbf{0}_2 & \mathbf{I}_2 & \cdots & \mathbf{0}_2 & \mathbf{0}_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0}_2 & \mathbf{0}_2 & \cdots & \mathbf{I}_2 & \mathbf{0}_2 \\ \mathbf{J}_2 & \mathbf{0}_2 & \cdots & \mathbf{0}_2 & \mathbf{I}_2 \end{pmatrix}. \quad (57)$$

This matrix corresponds to the unitary metaplectic evolution $\hat{M}_{\mathbf{S}^{(j,k)}}$ corresponding to a two-mode rotation by angle $\omega t = \pi/2$. The total (n-mode) metaplectic evolution associated with $\mathbf{S}$ in Eq. (56) is thus $\hat{M}_{\mathbf{S}} = \hat{1}_{j-1} \otimes \hat{M}_{\mathbf{S}^{(j,k)}} \otimes \hat{1}_{n-k}$, where $\hat{1}_{j-1}$ is the identity operator acting on the first $j - 1$ modes and the evolved state is $\hat{\rho}_G' = \hat{M}_{\mathbf{S}} \hat{\rho}_G \hat{M}_{\mathbf{S}}^\dagger$.

Let $\mathbf{V}'^{(j,k)}$ denote the covariance matrix of the reduced two-mode evolved state $\hat{\varrho}'^{(jk)} = \mathrm{Tr}_{\{l\}}(\hat{\rho}_G')$, with $l = 1, \ldots, n$ such that $l \neq j,k$. Such a matrix is related to the original (nonevolved) covariance matrix $\mathbf{V}^{(j,k)}$ in (34) according to

$$\mathbf{V}'^{(j,k)} = \frac{1}{2}\begin{pmatrix} \mathbf{I}_2 & \mathbf{J}_2 \\ \mathbf{J}_2 & \mathbf{I}_2 \end{pmatrix} \mathbf{V}^{(j,k)} \begin{pmatrix} \mathbf{I}_2 & \mathbf{J}_2 \\ \mathbf{J}_2 & \mathbf{I}_2 \end{pmatrix}^\top. \quad (58)$$

The diagonal blocks of the above matrix are the single-mode covariance matrices given by

$$\mathbf{V}'^{(j)} = \mathbf{V}^{(j)} + \mathbf{J}_2 \mathbf{E}^{(k,j)} - \mathbf{E}^{(j,k)} \mathbf{J}_2 - \mathbf{J}_2 \mathbf{V}^{(k)} \mathbf{J}_2, \quad (59a)$$

$$\mathbf{V}'^{(k)} = \mathbf{V}^{(k)} - \mathbf{J}_2 \mathbf{V}^{(j)} \mathbf{J}_2 - \mathbf{E}^{(k,j)} \mathbf{J}_2 + \mathbf{J}_2 \mathbf{E}^{(j,k)}, \quad (59b)$$

corresponding to the reduced single-mode states $\hat{\varrho}^{(i)} \equiv \mathrm{Tr}_{\{l\}}(\hat{\rho}_G')$ (with $l = 1, \ldots, n$ such that $l \neq i = j,k$). Solving Eqs. (59) for $\mathbf{E}^{(j,k)}$, we obtain

$$2\mathbf{J}_2 \mathbf{E}^{(k,j)} \mathbf{J}_2 + 2\mathbf{E}^{(j,k)} = \mathbf{V}'^{(j)} \mathbf{J}_2 - \mathbf{J}_2 \mathbf{V}'^{(k)}. \quad (60)$$

Once the matrices $\mathbf{V}'^{(j)}$ and $\mathbf{V}'^{(k)}$ are determined using some strategy, as explained above, Eq. (60) becomes a linear system for the matrix elements of $\mathbf{E}^{(j,k)}$ in (55). From this system, one is able to obtain

$$w = \frac{1}{4}[\mathbf{V}'^{(j)} \mathbf{J}_2 - \mathbf{J}_2 \mathbf{V}'^{(k)}]_{1,2}, \quad (61a)$$

$$y = \frac{1}{4}[\mathbf{V}'^{(j)} \mathbf{J}_2 - \mathbf{J}_2 \mathbf{V}'^{(k)}]_{2,1}. \quad (61b)$$

In order to determine v and z, we perform an additional (single-mode local) operation over the mode j of the evolved state $\hat{\rho}_G'$, with the symplectic matrix $\mathbf{J}_2^{(j)}$, so that

$$\mathbf{S}' = \mathbf{I}_{2j-2} \oplus [\mathbf{S}^{(j,k)}(\mathbf{J}_2 \oplus \mathbf{I}_{2k-2j})] \oplus \mathbf{I}_{2n-2k}, \quad (62)$$

with $\mathbf{S}^{(j,k)}$ given in Eq. (57). The single-mode operation $\hat{1}_{j-1} \otimes \hat{M}_{\mathbf{J}_2^{(j)}} \otimes \hat{1}_{n-j}$ is implemented via a rotation with the angle $\theta = \pi/2$ (see the Appendix, Sec. 1) and the evolved state is $\hat{\rho}_G'' = \hat{M}_{\mathbf{S}''} \hat{\rho}_G' \hat{M}_{\mathbf{S}''}^\dagger$, where now $\hat{M}_{\mathbf{S}''} = \hat{1}_{j-1} \otimes \hat{M}_{\mathbf{J}_2^{(j)}} \otimes \hat{1}_{n-j}$. Denoting by $\mathbf{V}''^{(i)}$ the covariance matrix of mode i after the evolution, we proceed as we did to arrive at Eqs. (59) and get

$$\mathbf{V}''^{(j)} = \mathbf{J}_2[\mathbf{V}^{(j)} + \mathbf{V}^{(k)} + \mathbf{E}^{(j,k)} + \mathbf{E}^{(k,j)}]\mathbf{J}_2^\top, \quad (63a)$$

$$\mathbf{V}''^{(k)} = \mathbf{V}^{(j)} - \mathbf{E}^{(k,j)} - \mathbf{E}^{(j,k)} + \mathbf{V}^{(k)}. \quad (63b)$$

Solving Eqs. (63) for $\mathbf{E}^{(j,k)}$, we are led to the linear system

$$-2(\mathbf{E}^{(k,j)} + \mathbf{E}^{(j,k)}) = \mathbf{J}_2 \mathbf{V}''^{(j)} \mathbf{J}_2 + \mathbf{V}''^{(k)}, \quad (64)$$

from which we obtain the matrix elements

$$v = -\frac{1}{4}[\mathbf{J}_2 \mathbf{V}''^{(j)} \mathbf{J}_2 + \mathbf{V}''^{(k)}]_{1,1}, \quad (65a)$$

$$z = -\frac{1}{4}[\mathbf{J}_2 \mathbf{V}''^{(j)} \mathbf{J}_2 + \mathbf{V}''^{(k)}]_{2,2}. \quad (65b)$$

Therefore, the elements w, y, v, and z of the matrix $\mathbf{E}^{(j,k)}$ are written in terms of single-mode CMs, which can be determined using some strategy developed in Sec. V A.

Gathering results, we have provided a method that allows us to determine any (all) two-mode covariance matrices (34) with due implementation of one- and two-mode metaplectic evolutions. By applying the method repeatedly (varying j and k), the complete covariance matrix (33) of an arbitrary Gaussian state can be determined.

Finally, it is important to notice that the strategies developed here, involving the unitary operations as simply as possible, are suitable for several paradigmatic CV systems. In general, the same procedure can be applied with any combination of

metaplectic evolutions (or equivalently quadratic Hamiltonians) and the same results are obtained with alternative designed strategies, as long as the new set of evolutions allows one to extract the covariance matrix **V** from Eqs. (32)

C. Determination of entanglement in pure Gaussian states

Though all the informational properties of a GS are contained in its covariance matrix, in certain cases partial information of the full CM suffices to extract information regarding the entanglement between the modes. For example, in the case of n-mode pure Gaussian states $\hat{\rho}_G^{\text{pure}}$, the amount of entanglement in an arbitrary bipartition $A|B$ with $n_A \times n_B$ modes (such that $n_A + n_B = n$) can be computed resorting only to the covariance matrix of any of the reduced Gaussian states, namely, $\hat{\varrho}_A$ or $\hat{\varrho}_B$ [24]. In this regard, our method, like other methods in general, allows us to determine such reduced covariance matrices. However, it has the advantage that it provides an experimentally favorable way to determine the purity of the reduced single-mode states [which allows us to measure the entanglement in bipartitions having $1 \times (n-1)$ modes] without the need to determine the full reduced covariance matrix.

The purity of the reduced ith-mode Gaussian state $\hat{\varrho}^{(i)}$ is given by

$$\text{Tr}(\hat{\varrho}^{(i)})^2 = \frac{\hbar}{2\sqrt{\det \mathbf{V}^{(i)}}} = \frac{\hbar}{2\sqrt{\tau}} \tag{66}$$

and can be determined, according to Eq. (46b), from the knowledge of the total phases associated with only two local rotations, which determine the value of τ. Once τ is known, the amount of entanglement $\mathcal{E}$ between the ith and the remaining $n-1$ modes can be computed using the pure-state Rényi entropy of entanglement [24,67]

$$\mathcal{S}_\alpha(\hat{\varrho}^{(i)}) = \frac{\ln[\text{Tr}(\hat{\varrho}^{(i)})^\alpha]}{1-\alpha}, \tag{67}$$

with $\alpha = 2$, which gives

$$\mathcal{E} = \mathcal{S}_2(\hat{\varrho}^{(i)}) = (1/2)\ln(\tau) - \ln(\hbar/2). \tag{68}$$

VI. MEASUREMENT PROTOCOL OF THE TOTAL PHASE

In this section we describe our protocol to measure both the real and the imaginary parts of $\text{Tr}(\hat{\rho}\hat{U})$, in order to compute ϕ resorting to Eq. (26). The main idea is to entangle the n-mode system in an arbitrary state $\hat{\rho}$ with a qubit ancilla, using the conditional evolution

$$\hat{U}^{(c)} \equiv \exp\left[-i\frac{t}{2\hbar}(\hat{1}+\hat{\sigma}_3)\otimes \hat{H}\right], \tag{69}$$

with $\hat{H}$ an arbitrary Hamiltonian acting on the n-mode system. We use $|j,\pm\rangle$ to denote the eigenstates of the Pauli operators $\hat{\sigma}_j$ ($j=1,2,3$), so that $\hat{\sigma}_j|j,\pm\rangle = \pm|j,\pm\rangle$.

Initially, the n-mode system and the ancilla are assumed to be in the separable state $|1,+\rangle\langle 1,+|\otimes \hat{\rho}$, with $|1,+\rangle = (1/\sqrt{2})(|3,+\rangle + |3,-\rangle)$. The reduced state of the qubit ancilla after the evolution of the complete (n-mode plus ancilla)

system is thus given by

$$\begin{aligned}
\hat{\rho}_q &= \text{Tr}_n[\hat{U}^{(c)}|1,+\rangle\langle 1,+|\otimes \hat{\rho}\hat{U}^{(c)\dagger}] \\
&= \tfrac{1}{2}[|3,+\rangle\langle 3,+| + \text{Tr}_n(\hat{\rho}\hat{U})|3,+\rangle\langle 3,-| \\
&\quad + \text{Tr}_n(\hat{\rho}\hat{U}^\dagger)|3,-\rangle\langle 3,+| + |3,-\rangle\langle 3,-|], \tag{70}
\end{aligned}$$

where $\text{Tr}_n(\cdots)$ denotes the trace over the n-mode system and $\hat{U} = \exp(-i\frac{t}{\hbar}\hat{H})$ is a unitary evolution acting only on the n-mode system.

Then a $\pi/2$ rotation $\hat{U}_{\pi/2}(\vartheta)$ around an axis in the equator of the Bloch sphere that makes an angle ϑ with the x axis is performed on the qubit ancilla. From the probability measurement of the qubit's populations $P_\pm(\vartheta) := \text{Tr}_q[|3,\pm\rangle\langle 3,\pm|\hat{U}_{\pi/2}(\vartheta)\hat{\rho}_q\hat{U}_{\pi/2}^\dagger(\vartheta)]$ we get

$$P_-(\vartheta) - P_+(\vartheta) = \text{Im}[e^{i\vartheta}\text{Tr}(\hat{\rho}\hat{U})]. \tag{71}$$

Thus, by choosing qubit rotations with $\vartheta = 0$ and $\vartheta = \pi/2$ we obtain the imaginary and real parts, respectively, of $\text{Tr}(\hat{\rho}\hat{U})$ and hence the total phase $\phi = \arg[\text{Tr}(\hat{\rho}\hat{U})]$ can be determined.

A. Total phase acquired by an evolved reduced state in an n-mode system

Let us now assume that $\hat{H}$ in Eq. (69) has the form $\hat{H} = \hat{H}_A \otimes \hat{1}_B$, where $\hat{H}_A$ is a Hamiltonian acting on subsystem A consisting of m modes and $\hat{1}_B$ is the $(n-m)\times(n-m)$ identity operator acting on subsystem B. In this case $\hat{U}$ reduces to $\hat{U} = \exp(-i\frac{t}{\hbar}\hat{H}_A)$ and we should write $\text{Tr}(\hat{\rho}\hat{U}) = \text{Tr}_A(\hat{\varrho}\hat{U})$ in Eq. (71), with $\hat{\varrho}$ the reduced (m-mode) state $\hat{\varrho} = \text{Tr}_B\hat{\rho}$.

In order to measure the phase acquired by the arbitrary reduced state $\hat{\varrho}$ we need to entangle only the m modes of interest with the qubit ancilla through the conditional evolution (69), with $\hat{H} = \hat{H}_A \otimes \hat{1}_B$. This is the strategy required to measure the total phases that allow us to determine the covariance matrix of an arbitrary Gaussian state $\hat{\varrho}_G = \text{Tr}_B\hat{\rho}_G$, considering only (as has been shown in Sec. V) metaplectic evolutions of the form in (15), such that $\hat{H} = \hat{H}_A = \omega(\hat{x}_A^\top \mathbf{H}\hat{x}_A)/2$ with $x_A^\top := (q_1, p_1, \ldots, q_m, p_m)$. In fact, it is worth noting that, according to the method described in Sec. V, we only need to implement conditional evolutions over one mode, viz.,

$$\hat{U}^{(c)} = \exp\left[-\frac{it}{2\hbar}(\hat{1}+\hat{\sigma}_3)\otimes \hat{1}_{j-1}\otimes \hat{H}^{(j)}\otimes \hat{1}_{n-j}\right], \tag{72}$$

where $\hat{H}^{(j)}$ is one of the Hamiltonians in Eqs. (40). This is so because the two-mode rotation and the additional single-mode operation, described in Sec. V B and needed to determine the 2×2 intermodal correlation matrix $\mathbf{E}^{(i,k)}$, do not need to be applied conditionally to the state of the qubit ancilla.

VII. ONE-MODE CONDITIONAL METAPLECTIC EVOLUTIONS IN SEVERAL CV SYSTEMS

The feasibility of our method for determining Gaussian states depends on the possibility of implementing one-mode conditional evolutions such as that in Eq. (72), with $\hat{H}^{(j)}$ one of the Hamiltonians in Eqs. (40). Here we describe how these conditional evolutions can be implemented in the context of three CV systems: (i) the transverse spatial degree of freedom of single photons, (ii) the vibrational modes in trapped ions, and (iii) the quadrature modes of the quantized electromagnetic field.
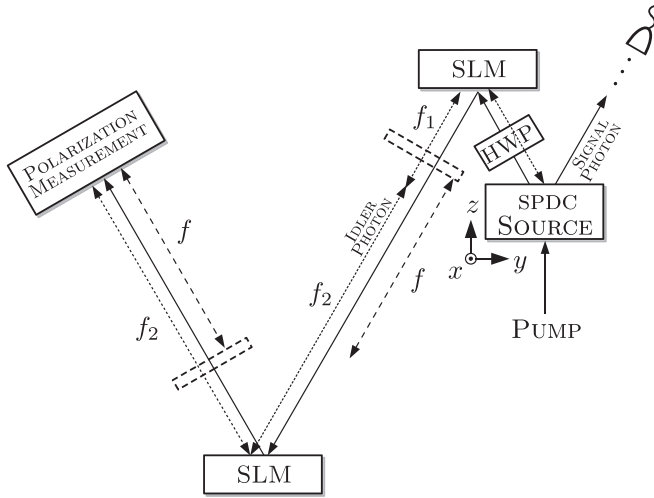
FIG. 1. Experimental setup to implement the conditional squeezing transformation on the transverse spatial degree of freedom of single photons, for example, generated with the SPDC process. Here the conditional squeezing transformation is implemented on the idler photon. On the signal photon it has to be implemented in an optical image system in both transverse degrees of freedom (not shown) before the coincidence measurement of the twin photons. The dashed rectangles correspond to cylindrical lenses of focal length f, acting in the transverse y degree of freedom. The acronym HWP denotes the half waveplate and SLM the spatial light modulator. See the text for details.
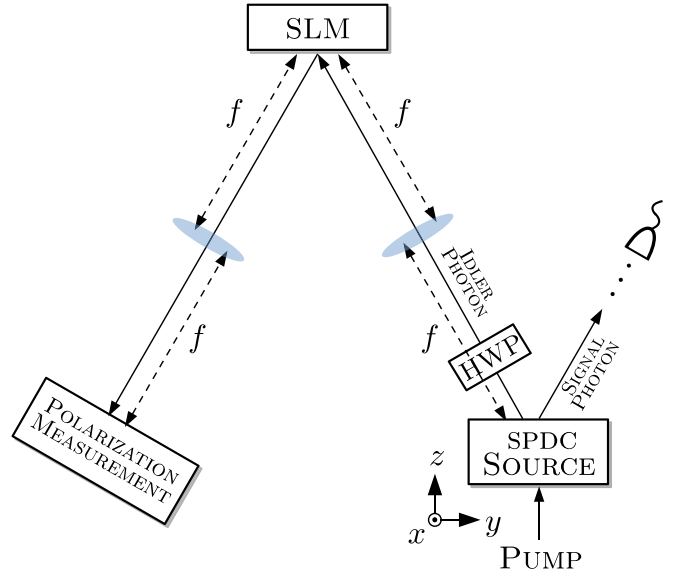

FIG. 2. Experimental setup to implement the conditional position shear transformation on the transverse spatial degree of freedom of single photons, for example, generated with the SPDC process. Here the conditional position shear transformation is implemented on the idler photon. On the signal photon it has to be implemented in an optical image system in both transverse degrees of freedom (not shown) before the coincidence measurement of the twin photons. The shaded ovals correspond to spherical lenses with focal length f. See the text for details.

A. Transverse spatial degrees of freedom of single photons

We consider first the implementation of single-mode conditional metaplectic evolutions in the CV system corresponding to the transverse spatial degrees of freedom (TSDF) of single photons propagating in the paraxial approximation. This is the CV systems of twin photons generated in SPDC [29,51,52]. Highly entangled Gaussian states can be generated with twin photons since, to a good approximation, generalization of two-mode squeezed states can be performed [51,53].

In order to determine Gaussian states in the TSDF of single photons, the less demanding experimental strategy is to implement the one-mode metaplectic operations described in the third strategy in Sec. V A, involving squeezing and position shear transformations. We identify the qubit ancilla with the polarization degrees of freedom of the single photon. As customary, we associate the horizontally (x direction) and vertically (y direction) polarized linear states with $|H\rangle := |3,+\rangle$ and $|V\rangle := |3,-\rangle$. The linearly polarized states rotated $45°$ in the counterclockwise direction with respect to x and y are identified, respectively, with $|+45°\rangle := |1,+\rangle$ and $|-45°\rangle := |1,-\rangle$. Finally, the states $|R\rangle := |2,+\rangle$ and $|L\rangle := |2,-\rangle$ are put into correspondence with the right- and left-circularly polarized states. With these identifications, measuring the polarization is equivalent to measuring the Pauli observables $\hat{\sigma}_j$ corresponding to the qubit-ancilla polarization degrees of freedom. The conditional evolutions, entangling the polarization and the transverse spatial degrees of freedom, are implemented using a spatial light modulator (SLM) that imprints a phase only on the horizontal polarization component (transverse spatial x direction). It is worth noting that the qubit

rotation that leads to Eq. (71) corresponds, in this context, to mapping one orientation of linear polarization to another or to a circular one.

In Fig. 1 we show the experimental setup for the implementation of the conditional evolution corresponding to squeezing transformation. The half waveplate rotates $45°$ the initial polarization originally in the x direction. The first and second SLMs, with focal lengths f_1 and f_2, respectively, implement thin lenses in the x direction, whereas they act only as mirrors in the y direction. Each of these lenses implements thus a Fourier transform in the x spatial degree of freedom [68] ($\pi/2$ rotation in Lohmann's type-I optical configuration [69]), whereas the combination produces a squeezing transformation with squeezing parameter $\zeta = f_2/f_1$ [52]. The dashed rectangles represent cylindrical lenses with focal length f that implement an optical image system in the y degree of freedom yet do not affect the evolution in the x degree of freedom. Notice that the total distance of propagation of the single photon until it enters the polarization measurement optical circuit is $4f = 2f_2 + 2f_1$.

In Fig. 2 we show the experimental setup that implements the conditional evolution corresponding to position shear transformation. The evolution associated with a shear in position over the single photon is $\langle p|\Phi'_G\rangle = e^{-i\frac{\omega t}{2}p^2}\langle p|\Phi_G\rangle$, where $\langle p|\Phi_G\rangle$ stands for the wave function in the transverse momentum representation at the source plane $z = 0$. Thus, for a shear in position we have to map the momentum wave function $\langle p|\Phi_G\rangle$ at $z = 0$ to the position wave function $\langle x'|\Phi_G\rangle$ in the far field, that is, when $x' = p$, where x' is the transverse spatial position of the single photon at a distance

z' equal to the distance of the SLM from the source. Then the phase $e^{-i(\omega t/2)p^2}$ is imprinted by the SLM. The map that changes the representation can be accomplished by a Fourier transform with Lohmann's type-I optical configuration such that $z' = 2f$, with f the focal length of the spherical lens. The second Fourier transform with identical optical configuration maps the wave function back to the position representation.

B. Vibrational modes in trapped ions

In order to determine Gaussian states in the vibrational modes of trapped ions [48], one-mode conditional rotation and squeezing transformations are needed. This constitutes the first strategy in Sec. V A.

We consider a system of ions confined in an elliptical trap. To a good approximation, the quantized motion of each ion's center of mass along the confined spatial dimensions can be described by a quantum harmonic oscillator [70]. We define $\hat{U}_0 := \bigotimes_{j=1}^{n} \hat{M}_{\mathsf{R}^{(j)}}$, the unitary free evolution of all the harmonic motions corresponding to local metaplectic rotations $\hat{M}_{\mathsf{R}^{(j)}}$ in each vibrational mode of the system. In this way, the Gaussian state to be determined can be written in the interaction picture with respect to the free evolution as $\hat{\bar{\rho}}_G := \hat{U}_0^\dagger \hat{\rho}_G \hat{U}_0$ and the engineered metaplectic evolution (in the same representation), needed for the determination of $\hat{\bar{\rho}}_G$, is written as $\hat{U}_I := \hat{U}_0^\dagger \hat{U} = \exp(-i\frac{t}{\hbar}\hat{1}_{j-1} \otimes \hat{H}_I^{(j)} \otimes \hat{1}_{n-j})$. With this notation, the trace that appears in Eq. (71) reads $\mathrm{Tr}[\hat{\bar{\rho}}_G \hat{U}_I]$.

The qubit ancilla in each single vibrational mode corresponds to two specific electronic states of each ion, namely, $|g\rangle := |3,-\rangle$ and $|e\rangle := |3,+\rangle$. We are interested in a type of laser excitation in which only the motional degree of freedom is excited conditioned on the occupation of the excited level, that is, if the ion is in the state $|g\rangle$ nothing happens, whereas if the ion is in the state $|e\rangle$ its vibrational motion is excited. This can be accomplished with a Raman excitation of one motional sideband via the virtual excitation to an auxiliary upper electronic state $|\text{aux}\rangle$, with $E_{|\text{aux}\rangle} > E_{|e\rangle}$ [71,72]. The interaction Hamiltonian, in the interaction picture, that describes the effective action of the laser over the jth motional degree of freedom (corresponding to a particular ion) is [71,72]

$$\hat{H}_I^{(j)} = \tfrac{1}{2}\hbar|\Omega_0|e^{i\varphi}\hat{f}_k(\hat{a}_j^\dagger \hat{a}_j, \eta)\hat{a}_j^k + \text{H.c.}, \qquad (73)$$

where $\hat{a}_j$ is the annihilation operator of the jth vibrational mode considered, $\Omega_0 = |\Omega_0|e^{i\varphi}$ is the effective Raman Rabi frequency, k corresponds to the excitation of the kth upper motional sideband (blue sideband transition), η is the Lamb-Dicke parameter, and $\hat{f}_k(\hat{a}_j^\dagger \hat{a}_j, \eta)$ is a Hermitian operator function that strongly depends on η [71]. Here we assume that each ion can be addressed individually.

The conditional rotation on the jth vibrational mode occurs when the carrier sideband $k = 0$ is excited and η is not extremely small so we have $\hat{f}_0(\hat{a}_j^\dagger \hat{a}_j, \eta) \approx A_0 + A_1\hat{a}_j^\dagger \hat{a}_j$ [72]. Thus, by choosing $\varphi = 0$, the Hamiltonian in Eq. (73) can be approximated by $\hat{H}_I^{(j)} \approx \hbar\omega\hat{a}_j^\dagger \hat{a}_j$, with $\omega = (1/2)A_1|\Omega_0|$. The conditional squeezing transformation of the jth vibrational mode can be implemented for very small values of the Lamb-Dicke parameter, that is, for $\eta \ll 1$, when the second blue sideband is excited, $k = 2$, so $\hat{f}_0(\hat{a}_j^\dagger \hat{a}_j, \eta) \approx A_0$, and
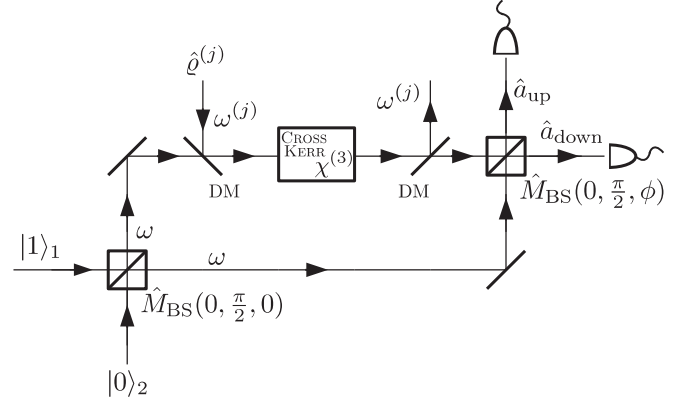
FIG. 3. Experimental setup of the measurement protocol of the total phase of a rotation of the reduced one-mode Gaussian state $\hat{\varrho}^{(j)}$ corresponding to the jth mode of a multimode Gaussian state. The initial qubit-ancilla state corresponds to the mode-entangled state of one photon (of frequency ω) in the interferometer after the first beam splitter. The jth mode of frequency $\omega^{(j)}$ is injected and extracted from one of the arms of the interferometer through suitable dichroic mirrors (DM). The rotation over $\hat{\varrho}^{(j)}$ is implemented by the Kerr medium conditioned to the one-photon occupation of the upper arm or the interferometer. Finally, the rotation of the qubit ancilla is performed by the second beam splitter and the measurement of the number of photons at the output modes determines, through Eq. (76), the total phase $\phi_{\mathsf{R}^{(j)}}$ once we choose $\varphi = 0$ and $\varphi = \pi/2$.

the Hamiltonian in Eq. (73) becomes approximately $\hat{H}_I^{(j)} \approx \frac{\hbar\omega}{2}(\hat{a}_j^{\dagger 2}e^{i\varphi} + \hat{a}_j^2 e^{-i\varphi})$, with $\omega = A_0|\Omega_0|$. In both cases (conditional rotation and conditional squeezing) the measurement protocol of the total phase must be initiated with the particular ion in the electronic state $|1,+\rangle := (1/\sqrt{2})(|e\rangle + |g\rangle)$. In addition, after the conditional evolutions, a $\pi/2$ rotation on the electronic states has to be implemented in order to obtain the probabilities in Eq. (71) through the measurement of the population of the excited state $|e\rangle$.

C. Quadrature modes of the quantized electromagnetic field

In the quadrature modes of the quantized electromagnetic field highly entangled multimode Gaussian states can be generated, for example, in an optical frequency comb generated by a SPOPO [17,45,46]. This is the specific CV system that we will consider in this section, and the most suitable strategy to determine the Gaussian state involves the one-mode conditional rotations described in the second strategy in Sec. V A. In a way analogous to the CV system of vibrational modes of ions, here we will determine the Gaussian state $\hat{\bar{\rho}}_G := \hat{U}_0^\dagger \hat{\rho}_G \hat{U}_0$ at the output of the SPOPO crystal, in the interaction picture with respect to the harmonic free evolution $\hat{U}_0 := \bigotimes_{j=1}^{n} \hat{M}_{\mathsf{R}^{(j)}}$ of the fields.

The experimental setup of the whole measurement protocol is sketched in Fig. 3. The ancilla system is composed of one photon in the two-mode output of the first beam splitter, in the state

$$|\Phi\rangle := \hat{M}_{\mathrm{BS}}(0,\pi/2,0)|1\rangle_1 |0\rangle_2$$
$$= (1/\sqrt{2})(|0\rangle_1 |1\rangle_2 + |1\rangle_1 |0\rangle_2), \qquad (74)$$

where the beam-splitter metaplectic operator is defined as

$$\hat{M}_{\mathrm{BS}}(\psi,\theta,\phi) = e^{-i\psi\hat{L}_z}e^{-i\theta\hat{L}_y}e^{-i\phi\hat{L}_z}, \quad (75)$$

with $\hat{L}_z := (1/2)(\hat{a}_1^\dagger\hat{a}_1 - \hat{a}_2^\dagger\hat{a}_2)$ and $\hat{L}_y := (i/2)(\hat{a}_1^\dagger\hat{a}_2 - \hat{a}_2^\dagger\hat{a}_1)$.

The conditional rotation is implemented with a cross-Kerr nonlinear medium characterized by a third-order electric susceptibility $\chi^{(3)}$ and with an interaction Hamiltonian between the modes given by $\hat{H}_K = \hbar\kappa\hat{a}_1^\dagger\hat{a}_1\hat{b}_j^\dagger\hat{b}_j$, where $\hat{a}_1$ $(\hat{a}_1^\dagger)$ and $\hat{b}_j$ $(\hat{b}_j^\dagger)$ are the annihilation (creation) operators in the modes with frequencies ω and $\omega^{(j)}$, respectively. After the interaction of the modes 1 and j at the Kerr medium we trace out mode j. Then modes 1 and 2 (both of frequency ω) enter in a second beam splitter characterized by the metaplectic operator $\hat{M}_{\mathrm{BS}}(0,\pi/2,\phi)$. From photocounting measurements at the output ports of the second beam splitter we get

$$\langle\hat{n}_{\mathrm{up}}\rangle - \langle\hat{n}_{\mathrm{down}}\rangle = \mathrm{Re}[e^{-i\phi}\mathrm{Tr}(\hat{\varrho}^{(j)}\hat{M}_{\mathsf{R}^{(j)}})], \quad (76)$$

where $\hat{M}_{\mathsf{R}^{(j)}} = e^{-i\theta\hat{b}_j^\dagger\hat{b}_j}$, $\theta = \kappa t$ with t the interaction time of the fields inside the Kerr medium, and $\hat{\varrho}^{(j)}$ is the determined reduced one-mode state. By setting $\phi = 0$ and $\phi = \pi/2$ we obtain the real and imaginary parts of the trace in Eq. (76) and therefore the total phase $\phi_{\mathsf{R}^{(j)}} = \arg[\mathrm{Tr}(\hat{\varrho}^{(j)}\hat{M}_{\mathsf{R}^{(j)}})]$. It is important to notice that any value of $\theta = \kappa t > 0$ serves to determine $\hat{\varrho}^{(j)}$ following the steps described in Sec. V. Thus, the current technological limitation of very small values of the coupling constant κ in Kerr media is not a problem in our scheme.

According to the second strategy described in Sec. V A, it still remains to implement a squeezing transformation over the one-mode reduced state $\hat{\varrho}^{(j)}$ before the determination of the total phase associated with a rotation. This squeezing transformation over the jth mode does not need to be implemented conditionally to the state of the ancilla. Therefore, we resort to the same experimental setup as before, but now introduce a type-I nonlinear crystal characterized by a $\chi^{(2)}$ electric susceptibility before the implementation of the conditional rotation with the Kerr medium (see Fig. 4). A pump laser beam of frequency $2\omega^{(j)}$ [73] enters, together with the field mode of frequency $\omega^{(j)}$ in the quantum state $\hat{\varrho}^{(j)}$, in a nonlinear $\chi^{(2)}$ crystal. In the approximation where the pump on the crystal is treated classically, the down-conversion Hamiltonian of the degenerate type-I crystal is $\hat{H} \approx \frac{\hbar\omega}{2}(\hat{b}_j^{\dagger 2}e^{i\varphi} + \hat{b}_j^2 e^{-i\varphi})$, which squeezes the quantum state $\hat{\varrho}^{(j)}$ of the stimulation field on the crystal. Typically, when the nonlinear crystal characterized by $\chi^{(2)}$ is outside a cavity, the squeezing parameter $\zeta = \omega t$ (with t the interaction time inside the crystal) is very small. This, however, does not represent a limitation in our protocol, since any squeezing parameter $\zeta > 0$ serves for the determination of the Gaussian state $\hat{\varrho}^{(j)}$.

VIII. CONCLUSION

We have designed an experimentally favorable method to determine Gaussian states of n-mode bosonic systems through the determination of its full covariance matrix, once the first moments of the state are experimentally determined and local translations are implemented so as to make the state one with null mean values. In particular, we constructed three strategies to determine the one-mode reduced covariance matrices, based
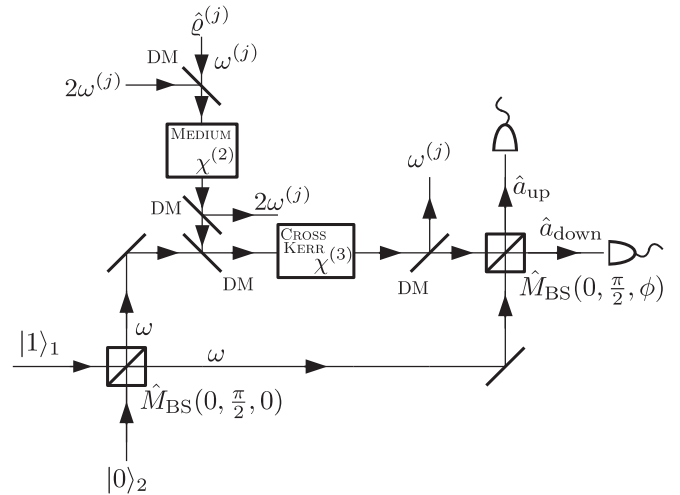

FIG. 4. Same experimental setup as in Fig. 3, but now before the conditional rotation in the Kerr medium a squeezing transformation over the state $\hat{\varrho}^{(j)}$ is performed. This is done with a stimulated degenerate parametric down-conversion in a type-I nonlinear crystal, characterized by a second-order electric susceptibility $\chi^{(2)}$. Thus, the down-conversion process pumped by the field of frequency $2\omega^{(j)}$ is stimulated by the mode of frequency $\omega^{(j)}$ in the quantum state $\hat{\varrho}^{(j)}$. The converted fields are in the same mode $\omega^{(j)}$ producing a squeezing effect on $\hat{\varrho}^{(j)}$.

on the knowledge of the total phases acquired under specific one-mode metaplectic transformations that include rotations, squeezing, and shears in position or momentum.

Each strategy is more suitable to be implemented in one of the three CV systems considered: the vibrational modes of trapped ions, the transverse spatial degrees of freedom of entangled single photons, and the quadrature degrees of freedom of n-mode quantized electromagnetic fields. Some of the one-mode transformations in each strategy must be implemented conditionally to the state of an ancilla qubit that, when measured, allows one to extract the total phase of each evolution, which bears the information of the matrix elements of the reduced single-mode covariance matrices. The same method used to determine the single-mode reduced covariance matrices is applied in order to determine each pair of two-mode intermodal correlation matrices after the application of a single beam-splitter-like two-mode rotation, plus an additional single-mode rotation that does not need to be applied conditionally to an ancilla's state. Further, the method proposed here is suitable for determining and quantifying entanglement in bipartitions having $1 \times (n-1)$ modes associated with pure Gaussian states, via the measurement of only two total phases associated with two local rotations.

The strategy proposed here represents an alternative to homodyne detection in the quadrature mode CV system of the quantized electromagnetic field in which a local oscillator is not necessary and the detector used in order to measure the one-photon qubit ancilla is a click-nonclick detector. Our strategy shows advantages in CV systems in which the quadrature measurement is not directly accessible, as, for instance, in vibrational modes of trapped ions and generically in networks of massive oscillators. In such systems the existing

strategies [54,55] (like ours) involve the entanglement with a qubit ancilla and consist in a qubit measurement from which it is possible to determine the phase-space values of the Weyl characteristic function of the GS. The determination of the covariance matrix in these strategies requires knowing a considerable number of phase-space points of the Weyl characteristic function around the origin, which in turn requires many measurements of the qubit ancilla. In contrast, our strategy involves only a few measurements of the qubit ancilla.

Optomechanical systems are also CV systems in which the quadrature measurement is not directly accessible and for which our strategy could offer advantages over the existing methods [56,57]. In these systems the CM of the mechanical mode is indirectly determined by measuring the leaking field of the cavity, which is entangled with an oscillating mirror. It is worth noting that the determination of the CM through this method is considerably noisy. In this regard, our strategy might represent a less noisy alternative that could deserve further investigation.

The method advanced in [58] to determine the CV corresponding to the spatial transverse modes of single photons can be applied in any quantum state (Gaussian or not). This strategy shares with ours the fact that it resorts to a controlled unitary operator implemented by a SLM and that the ancilla qubit to be measured (in order to extract the second-order moments of the $\hat{x}$ and $\hat{p}$ operators that build the CM) is the polarization state of the photon. However, the drawback of this strategy is that it rests on a very precise alignment between the region in the SLM where the unitary operation is implemented and the region of effective support of the quantum state of the photon in the spatial transverse degrees of freedom. In contrast, our strategy is free of this alignment problem.

APPENDIX: TOTAL PHASE OF METAPLECTIC EVOLUTIONS OVER A SINGLE-MODE GAUSSIAN STATE

In this appendix we calculate the total phase acquired when different metaplectic evolutions of interest are performed over a single-mode Gaussian state. In this case the (one-mode) covariance matrix is of the form (39). For such 2×2 matrices, Eq. (32b) reduces to

$$\phi_{\mathsf{S}}[\hat{\rho}_G] = \frac{\pi}{2}\nu_{\mathsf{S}}^+ - \frac{1}{2}\arg\left[\frac{1}{4} - \det(\mathbf{V}\mathbf{C_S}) + \frac{i}{2}\mathrm{Tr}(\mathbf{V}\mathbf{C_S})\right], \tag{A1}$$

which is equivalent to the phase given in Eq. (38).

1. Rotations

The unitary dynamics of a rotation is performed by the Hamiltonian of a harmonic oscillator $\hat{H}_{\mathsf{R}} = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2)$, corresponding to Eq. (14) with the Hessian $\mathbf{H_R} = \mathsf{I}_2$. The corresponding symplectic matrix and its Cayley parametrization are

$$\mathsf{R} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad \mathbf{C_R} = \tan\left(\frac{\theta}{2}\right)\mathsf{I}_2, \tag{A2}$$

with $\theta = \omega t$. The function $\mathrm{sng}\mathbf{C_R}$ is thus

$$\mathrm{sng}\mathbf{C_R} = -\mathrm{sng}(\mathsf{J}\mathbf{C_R}^{-1}\mathsf{J}) = \begin{cases} 2, & 0 < \theta < \pi \\ -2, & \pi < \theta < 2\pi \\ 2, & 2\pi < \theta < 3\pi \\ -2, & 3\pi < \theta < 4\pi. \end{cases} \tag{A3}$$

In order to determine the index ν_{R}^+ for all θ, we proceed as follows. First, according to the lines below Eq. (24), we fix the index ν_{R}^+ equal to 0 for $\theta = 0$. The continuity of the Wigner symbol (20) in θ then allows us to set $\nu_{\mathsf{R}}^+ = 0$ for $\theta \in [0,\pi)$. For $\theta = \pi$, R has eigenvalues equal to -1 and the symbol (20) diverges. To surmount this difficulty we resort to the Weyl symbol (19) before the divergence, that is, at $\theta = \pi^-$. Thus, using Eq. (22) we write

$$\nu_{\mathsf{R}(\pi^-)} = \nu_{\mathsf{R}(\pi^-)}^+ - \tfrac{1}{2}\mathrm{sng}\mathbf{C}_{\mathsf{R}(\pi^-)} = -1(\mathrm{mod}4) = 3. \tag{A4}$$

Due to the continuity of (19) we have $\nu_{\mathsf{R}(\theta)}^- = \nu_{\mathsf{R}(\pi^-)}^-$ for $\theta \in (0,2\pi)$. For $\theta = \pi^+$ we employ again the Wigner symbol (20) and use

$$\nu_{\mathsf{R}(\pi^+)}^+ = \nu_{\mathsf{R}(\pi^+)}^- + \tfrac{1}{2}\mathrm{sng}\mathbf{C}_{\mathsf{R}(\pi^+)} = -2(\mathrm{mod}4) = 2. \tag{A5}$$

Since (20) is continuous in $(\pi,3\pi)$ we set $\nu_{\mathsf{R}(\theta)}^+ = \nu_{\mathsf{R}(\pi^+)}^+$ for $\theta \in (\pi,3\pi)$. At $\theta = 3\pi$, the Wigner symbol exhibits a second divergence, so as before we resort to the Weyl representation at $\theta = 3\pi^-$, thus getting

$$\nu_{\mathsf{R}(3\pi^-)}^- = \nu_{\mathsf{R}(3\pi^-)}^+ - \tfrac{1}{2}\mathrm{sng}\mathbf{C}_{\mathsf{R}(3\pi^-)} = -3(\mathrm{mod}4) = 1 \tag{A6}$$

and $\nu_{\mathsf{R}(\theta)}^- = \nu_{\mathsf{R}(3\pi^-)}^-$ for $\theta \in (2\pi,4\pi)$, due to the continuity of the Weyl symbol in that interval. Finally, the Wigner symbol in the interval $\theta \in (3\pi,4\pi]$ has the index

$$\nu_{\mathsf{R}(3\pi^+)}^+ = \nu_{\mathsf{R}(3\pi^+)}^- + \tfrac{1}{2}\mathrm{sng}\mathbf{C}_{\mathsf{R}(3\pi^+)} = -4(\mathrm{mod}4) = 0. \tag{A7}$$

Gathering results, we are led to

$$\nu_{\mathsf{R}}^+ = \begin{cases} 0, & 0 \leqslant \theta < \pi \\ 2, & \pi < \theta < 3\pi \\ 0, & 3\pi < \theta \leqslant 4\pi, \end{cases} \quad \nu_{\mathsf{R}}^- = \begin{cases} 3, & 0 < \theta < 2\pi \\ 1, & 2\pi < \theta < 4\pi. \end{cases} \tag{A8}$$

We now resort to Eq. (38), write $\tau = \det \mathbf{V} = ab - c^2$, and get the result in Eq. (41).

2. Squeezing

The dynamics associated with a squeezing is now determined by the quadratic Hamiltonian

$$\hat{H}_{\mathsf{Z}_\varphi} = i\frac{\hbar\omega}{2}(\hat{a}^{\dagger 2}e^{i\varphi} + e^{-i\varphi}\hat{a}^2),$$

with the Hessian

$$\mathbf{H}_{Z_\varphi} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix}. \tag{A9}$$

The associated symplectic matrix and its Cayley parametrization are given, respectively, by

$$Z_\varphi = \begin{pmatrix} \cosh\zeta + \sin\varphi \sinh\zeta & -\cos\varphi \sinh\zeta \\ -\cos\varphi \sinh\zeta & \cosh\zeta - \sin\varphi \sinh\zeta \end{pmatrix},$$

$$\mathbf{C}_{Z_\varphi} = \tanh\tfrac{\zeta}{2} \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix}, \tag{A10}$$

with $\zeta = \omega t$. The eigenvalues of $\mathbf{C}_{Z_\varphi}$ are $\pm\tanh\tfrac{\zeta}{2}$, hence $\mathrm{sng}\mathbf{C}_{Z_\varphi} = 0$ and $\nu_{Z_\varphi}^- = \nu_{Z_\varphi}^+$. According to the condition (24), we have $\nu_{Z_\varphi}^+ = 0$ for $\zeta = \omega t = 0$. Moreover, since the symbol (20) has no divergences, we have $\nu_{Z_\varphi}^- = \nu_{Z_\varphi}^+ = 0$ for all ζ and φ. Therefore, since Z_φ has positive eigenvalues, Eq. (38) gives, for the total phase under squeezing, the result in Eq. (42).

3. Coordinate shear

This transformation corresponds to the Hamiltonian $\hat{H}_F = -\frac{\hbar\omega}{4}(\hat{a}^\dagger - \hat{a})^2$ with the Hessian given by

$$\mathbf{H}_F = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{A11}$$

The symplectic matrix and its Cayley parametrization are, respectively,

$$F = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \quad \mathbf{C}_F = \begin{pmatrix} 0 & 0 \\ 0 & s/2 \end{pmatrix}, \tag{A12}$$

where $s = \omega t \geqslant 0$. The index ν_F^+ is null by (24) and the symbol in (20) never diverges although the symbol in (19) does not exist for any value of s. Thus, by Eq. (38), the total phase is the result in Eq. (43).

4. Momentum shear

This transformation corresponds to the Hamiltonian $\hat{H}_M = \frac{\hbar\omega}{4}(\hat{a}^\dagger + \hat{a})^2$ characterized by the following Hessian:

$$\mathbf{H}_M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{A13}$$

In this case the symplectic matrix and its Cayley parametrization are given, respectively, by

$$M = \begin{pmatrix} 1 & 0 \\ -s & 1 \end{pmatrix}, \quad \mathbf{C}_M = \begin{pmatrix} s/2 & 0 \\ 0 & 0 \end{pmatrix}, \tag{A14}$$

where $s = \omega t \geqslant 0$. By the same reasoning as in the previous example, the index ν_M^+ is null, thus, using Eq. (38), we obtain the result in Eq. (44).

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