Asymptotic Floquet states of non-Markovian systems

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We propose a method to find asymptotic states of a class of periodically modulated open systems which are outside the range of validity of the Floquet theory due to the presence of memory effects. The method is based on a Floquet treatment of the time-local, memoryless dynamics taking place in a minimally enlarged state space where the original system is coupled to auxiliary—typically nonphysical—variables. A projection of the Floquet solution into the physical subspace returns the sought asymptotic state of the system. The spectral gap of the Floquet propagator acting in the enlarged state space can be used to estimate the relaxation time. We illustrate the method with a modulated quantum random walk model.

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I. INTRODUCTION

Periodically driven systems can exhibit a spectrum of states which are unattainable in the static limit. This makes the idea of modulations appealing to various fields, ranging from dynamical chaos theory [1] and chemical kinetics [2] to neuroscience [3] and quantum physics [4–8]. In the latter field, periodic driving has been used to realize new topological states [9,10], engineer artificial gauge fields [11,12], and create so-called 'Floquet time crystals' [13–15].

Typically, one models periodically modulated systems via linear differential equations with time-periodic coefficients whose solution is provided by Floquet theory [16,17]. The key prerequisite for constructing such a model is the time-local character of the system dynamics, which means that the future of the system depends on its current state and not on its history. For a system with time-local and contractive (in terms of some proper norm) dynamics, the fate of the system is specified by the asymptotic state(s). A periodically modulated system interacting with a broadband environment in the Markovian limit evolves towards an asymptotic state which is periodic with the period of the modulations [18–20]. On the model level, this state represents a limit-cycle solution of the dissipative equations describing the system dynamics [17]. Very recently, an idea to combine modulations and dissipation to explore many-body quantum states [21,22] has emerged as a natural extension of the established Hamiltonian-oriented approach [7,12,23–25]. Still, the use of Floquet theory in the dissipative context implies that the equations used to describe the dynamics remain local in time [26–28].

What are the asymptotic Floquet states of systems governed by time-*non*local evolution equations? This question is of special relevance in the context of quantum non-Markovian dynamics, a topic being actively explored now in both the theoretical [29] and the experimental [30] domains. Here we present a method to get the answer for a broad class of periodically modulated systems whose evolution is governed by memory-kernel (MK) master equations. We show that the corresponding asymptotic solutions possess the periodicity of the modulations and have the form prescribed by the recently introduced generalized Floquet theorem [31].

II. METHOD

We consider a physical system, classical or quantum, whose state is described by an n-dimensional vector $\mathbf{x}(t)$ obeying a generalized master equation of the form

$$\dot{\mathbf{x}}(t) = \int_{t_0}^{t} dt' \mathbf{K}(t,t')\mathbf{x}(t') + \mathbf{z}(t), \tag{1}$$

with an integrable MK matrix $\mathbf{K}(t,t')$ and an asymptotically vanishing inhomogeneous term $\lim_{t\to\infty}\mathbf{z}(t) = \mathbf{0}$ [33]. The vector $\mathbf{x}(t)$ may describe, for example, an n-state classical system or the density matrix of a quantum system in a suitable representation [34,35]. We assume the MK to be biperiodic, i.e., $\mathbf{K}(t+T,t'+T) = \mathbf{K}(t,t')$, where T denotes the period of the modulation. The above requirements ensure that, in the limit $t \to \infty$, the action of the operator $\mathcal{L}_t\{\mathbf{x}(t)\} = \int_{t_0}^{t} dt'\mathbf{K}(t,t')\mathbf{x}(t') + \mathbf{z}(t)$ commutes with that of the one-period translation operator $\mathcal{S}_T\{\mathbf{x}(t)\} = \mathbf{x}(t+T)$, thus entailing the applicability of the generalized Floquet theorem to the asymptotic dynamics induced by Eq. (1) [36]. Note that the condition of biperiodicity is automatically satisfied if the MK depends exclusively on the difference $\tau = t - t'$ or if the MK is periodic with respect to t (t') and depends only on t (t') and τ.

Our approach to determining the asymptotic solution of Eq. (1) is based on the idea of embedding the system in an enlarged state space where the physical variable $\mathbf{x}(t)$ is coupled to an auxiliary vector variable $\mathbf{u}(t)$. The coupling is realized in such a way that the resulting extended system described by the new vector $\mathbf{v}^{\mathrm{T}}(t) = [\mathbf{x}^{\mathrm{T}}(t),\mathbf{u}^{\mathrm{T}}(t)]$ (concatenation) obeys a time-local equation satisfying the conditions of the standard Floquet theorem. The latter provides the solution for the state of the extended vector $\mathbf{v}(t)$ whose projection into the physical $\mathbf{x}$ subspace constitutes the asymptotic Floquet state of the system.

The concept of embedding is well known within the theory of classical stochastic processes [37–42] where it is related to the celebrated Erlang's method of stages [43]. Embedding schemes are also employed [44–46] in the recently established field of quantum non-Markovian processes [29,35,47]. However, in the above cases the embedding is *physical*, i.e., the new auxiliary variables (states or degrees of freedom) have the same physical meaning as the original variable(s). In other

words, the enlarged system is obtained by attaching ancillas to the original system (see, e.g., Ref. [45]). In the quantum case, such physical embedding might be very hard (or not possible) to construct for a given kernel, and even when constructed, it might be very complicated to deal with. Here we do not restrict ourselves to the physical embedding. Instead we propose a minimal enlargement of the system, by introducing a set of *non*physical auxiliary variables, which leads to a new set of equations, now local in time.

For the $n \times n$ MK matrix in Eq. (1) we consider the structure

$$\mathbf{K}(t,t') = \sum_{j=1}^{k} \Gamma_j e^{-\gamma_j (t-t')} \mathbf{E}_j(t) \mathbf{F}_j(t'), \tag{2}$$

with $\Gamma_j, \gamma_j \in \mathbb{C}$, and $\mathbf{E}_j(t) = \mathbf{E}_j(t+T), \mathbf{F}_j(t) = \mathbf{F}_j(t+T) \in \mathbb{C}^{n\times n}$. In the stationary case, $\mathbf{E}_j(t) \equiv \mathbf{E}_j$ and $\mathbf{F}_j(t') \equiv \mathbf{F}_j$; this form relates to Erlang's method of stages [48]. The structure given by Eq. (2) allows us to reproduce—exactly or arbitrary well—a large class of MKs, including oscillatory ones [49].

With the kernel (2), the time evolution given by Eq. (1) for the physical system described by $\mathbf{x}$ is equivalently obtained by solving a time-local set of equations in which the n-dimensional vector $\mathbf{x}$ is coupled to an auxiliary variable $\mathbf{u}$ of the dimension $p = n \times k$. The equations for the extended system read (from now on we set $t_0 = 0$)

$$\begin{aligned} \dot{\mathbf{x}}(t) &= -\mathbf{H}(t)\mathbf{u}(t), \\ \dot{\mathbf{u}}(t) &= -\mathbf{G}(t)\mathbf{x}(t) - \mathbf{A}\mathbf{u}(t), \end{aligned} \tag{3}$$

where $\mathbf{H}(t) = (\Gamma_1 \mathbf{E}_1(t), \ldots, \Gamma_k \mathbf{E}_k(t))$, $\mathbf{G}^{\mathrm{T}}(t) = (\mathbf{F}_1^{\mathrm{T}}(t), \ldots, \mathbf{F}_k^{\mathrm{T}}(t))$, and $\mathbf{A} = \mathrm{diag}(\gamma_1 \mathbf{1}^{n\times n}, \ldots, \gamma_k \mathbf{1}^{n\times n})$.

To assess the equivalence of Eqs. (1) and (3) we set $\mathbf{G}(t)\mathbf{x}(t) \equiv \mathbf{w}(t)$ and note that the equation for $\mathbf{u}$ reads, in Laplace space, $\mathbf{u}(\lambda) = [\lambda \mathbf{1} + \mathbf{A}]^{-1}\mathbf{u}(0) - [\lambda \mathbf{1} + \mathbf{A}]^{-1}\mathbf{w}(\lambda)$. Going back to the time domain, multiplying the resulting equation on the left by $-\mathbf{H}(t)$, and using the first of equations (3), we end up with Eq. (1) with $\mathbf{K}(t,t') = \mathbf{H}(t)e^{-\mathbf{A}(t-t')}\mathbf{G}(t')$, provided that the following relation is satisfied:

$$\mathbf{z}(t) = -\mathbf{H}(t)e^{-\mathbf{A}t}\mathbf{u}(0). \tag{4}$$

Thus, the requirement for the evolution of the physical part $\mathbf{x}(t)$ of the enlarged system to coincide with that given by Eq. (1) fixes the initial condition for the auxiliary variable $\mathbf{u}(t)$.

The system of equations (3) can be put in the compact form

$$\dot{\mathbf{v}}(t) = \mathbf{M}(t)\mathbf{v}(t), \tag{5}$$

where $\mathbf{v} = (x_1, \ldots, x_n, u_1, \ldots, u_{nk})^{\mathrm{T}}$ is the $(p = n + nk)$–dimensional vector describing the enlarged system, with the matrix $\mathbf{M}$ assuming the block structure

$$\mathbf{M}(t) = \begin{pmatrix} \mathbf{0} & -\mathbf{H}(t) \\ -\mathbf{G}(t) & \mathbf{A} \end{pmatrix}, \tag{6}$$

where $\mathbf{0} \in \mathbb{R}^{n\times n}$. A memory kernel of the form of Eq. (2), with T-periodic $\mathbf{E}(t)$ and $\mathbf{F}(t)$, entails T-periodicity for matrix $\mathbf{M}(t)$. This in turn ensures that Eq. (5) qualifies to invoke the Floquet theorem, which leads to a solution of the form $\mathbf{v}(t) = \tilde{\mathbf{S}}(t,0)e^{\mathbf{R}t}\mathbf{v}(0)$, where $\tilde{\mathbf{S}}(t,0)$ is a T-periodic $p \times p$ matrix and $\mathbf{R}$ a constant $p \times p$ matrix [17]. The projection of the vector $\mathbf{v}(t)$ onto the physical manifold, $\mathcal{P}_{\mathbf{x}}\mathbf{v}(t) = \mathbf{x}(t)$, yields, for the solution in the original state space,

$$\mathbf{x}(t) = \mathbf{S}(t,0)e^{\mathbf{R}t}\mathbf{v}(0), \tag{7}$$

with an $n \times p$ matrix $\mathbf{S} = \mathcal{P}_{\mathbf{x}}\tilde{\mathbf{S}}$. This is the form expected from the generalized Floquet theorem [31]. In the time-local limit, i.e., $\mathbf{K}(t,t') = \delta(t-t')\mathbf{K}_{\mathrm{t.l.}}(t)$, the solution reduces to the standard Floquet form, with $p = n$ [17].

Equation (5) can be solved by constructing the Floquet propagator $\mathbf{U}_T = \mathcal{T}\exp[\int_0^T \mathbf{M}(\tau)d\tau]$, where $\mathcal{T}$ denotes the time-ordering operator, and then finding its invariant, $\mathbf{U}_T\mathbf{y} = \mathbf{y}$ [17]. This vector yields the asymptotic solution at stroboscopic instants in time, i.e., $\mathbf{y} = \mathbf{v}^{\mathrm{a}}(sT)$, $s \in \mathbb{Z}$. The spectral properties of the propagator can be used to characterize the relaxation time towards the asymptotic state. A conventional candidate is the spectral gap [50], $g = 1 - |\lambda_m|$, with λ_m being the second largest (by absolute value) eigenvalue of $\mathbf{U}_T$ after $\lambda_1 = 1$. A straightforward diagonalization of the propagator and the use of the obtained λ_m as the quantifier of the relaxation speed are not suitable in our case. This is because this eigenvalue addresses the whole enlarged space, including the part which is not accessible with any physically meaningful initial condition. To address the physical manifold only, we suggest the Arnoldi iteration method, starting with the initial vector $\mathbf{v}^{\mathrm{T}}(0) = [\mathbf{x}^{\mathrm{T}}(0), \mathbf{u}^{\mathrm{T}}(0)]$, where $\mathbf{u}(0)$ satisfies Eq. (4), with consecutive diagonalization of the Hessenberg matrix [51].

III. APPLICATION: A PERIODICALLY DRIVEN QUANTUM RANDOM WALK

Here we apply the method described above to the non-Markovian master equation for a continuous-time quantum random walk model, which yields, by construction, a completely positive and trace-preserving (CPT) quantum evolution [32,34]. The model has a direct interpretation in terms of an operator generalization of a classical semi-Markov process, a multisite jump process defined by a transition matrix and a waiting-time distribution f [35,52]. This classical process is itself described by a generalized master equation of the form of Eq. (1) and turns out to be Markovian only for exponentially distributed waiting times τ between consecutive jumps, i.e., $f(\tau) = \lambda \exp(-\lambda\tau)$.

In our application, a qubit whose density matrix is denoted by $\rho(t)$ undergoes a trivial continuous background evolution (provided by the identity map $\mathbb{1}$), interrupted by the instantaneous actions of a CPT map $\mathcal{E}$ [32,34]. These 'collisions' occur at random instants in time, with the time intervals between consecutive collisions distributed according to a waiting-time distribution of the biexponential form

$$f(\tau) = \frac{2A}{a} e^{-\gamma\tau/2} \sinh(a\tau/2), \tag{8}$$

with $a = \sqrt{\gamma^2 - 4A} > 0$.

We generalize the original model [32,34] by assuming that the collision map itself periodically evolves in time, $\mathcal{E}(t+T) = \mathcal{E}(t)$ (see Fig. 1). The time-periodic CPT $\mathcal{E}(t)$ can be constructed as a convex combination of L CPT maps $\mathcal{E}(t) = \sum_s^L l_s(t)\mathcal{E}_s$, where $l_s(t) \geqslant 0$, $\sum_s^L l_s(t) = 1$ for $\forall t \in [0,T]$, and $l_s(t+T) = l_s(t)$. Note that, in order to get a nontrivial asymptotic state, $\rho^{\mathrm{a}}(t) \neq \mathbb{1}/2$, at least one of the

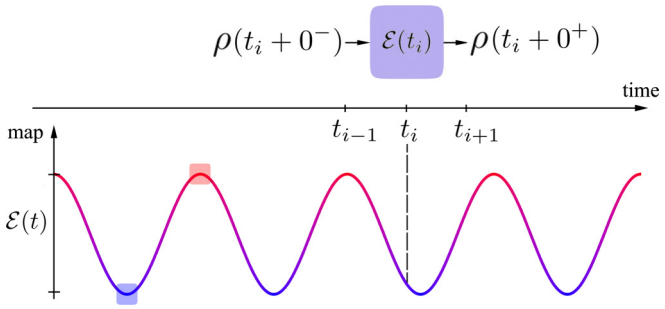

FIG. 1. Continuous-time quantum random walk with a time-periodic map. A qubit undergoes repeated instantaneous actions of a time-dependent map $\mathcal{E}(t)$ at random instants in time $\ldots t_{i-1}, t_i, t_{i+1} \ldots$. The time interval between consecutive collisions, $\tau_i = t_{i+1} - t_i$, is controlled by the waiting-time distribution $f(\tau)$. There is no evolution of the qubit between collisions. The map $\mathcal{E}(t)$ is obtained as a convex combination of constant CPT maps (in this case, of two maps labeled with colored squares) with time-periodic coefficients. Averaging over an infinite number of realizations of the process results in the master equation (12).

maps $\mathcal{E}_s$ has to be nonunital. We choose the time-periodic map

$$\mathcal{E}(t) = l_1(t)\mathcal{E}_1 + l_2(t)\mathcal{E}_2 + l_3(t)\mathcal{E}_3, \tag{9}$$

with $l_1(t) = \sin^2(\Omega t)$, $l_2 = \sin^2(\Omega t)\cos^2(\Omega t)$, and $l_3 = \cos^4(\Omega t)$, so that $T = \pi/\Omega$. The map $\mathcal{E}_1$ is the nonunital amplitude damping map defined by the following action on the 2×2 qubit density matrix [53]:

$$\mathcal{E}_1[\rho] = M_0 \rho M_0^\dagger + M_1 \rho M_1^\dagger, \tag{10}$$

where $M_0 = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$ and $M_1 = \begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix}$, with $b^2 = 1 - d^2$. The remaining two maps are $\mathcal{E}_{2(3)}[\rho] = M_{2(3)} \rho M_{2(3)}^\dagger$, with $M_{2(3)} = (\sigma_{y(x)} + \sigma_z)/\sqrt{2}$.

The density operator of the qubit is the average over all the possible realizations of the described process,

$$\rho(t) = g(t)\rho_0 + \sum_{n=1}^{\infty} \int_0^t dt_n \ldots \int_0^{t_2} dt_1 f(t - t_n) \times \mathcal{E}(t_n) \ldots \mathcal{E}(t_2) f(t_2 - t_1) \mathcal{E}(t_1) g(t_1) \rho_0, \tag{11}$$

where the function $g(t) = 1 - \int_0^t d\tau f(\tau)$ yields the probability that no collision has occurred up to time t. The resulting dynamics is equivalently obtained as the solution of the non-Markovian master equation

$$\dot{\rho}(t) = \int_0^t dt' \mathcal{K}(t - t') \mathcal{E}(t') \rho(t') + \mathcal{I}(t)\rho_0, \tag{12}$$

where $\mathcal{K}(t) = [\frac{d}{dt} f(t) + f(0)\delta(t)]\mathbb{1}$ and $\mathcal{I}(t) = \frac{d}{dt} g(t)\mathbb{1} = -f(t)\mathbb{1}$ (see the Appendix). Equation (12) is one of the few known instances of a well-defined MK quantum master equation. Within the path-integral formalism, evolution equations of the general form of (1) are also found for driven dissipative quantum systems [6,54].

In order to cast Eq. (12) in the form of Eq. (1), it is convenient to express the action of the quantum map $\mathcal{E}(t)$ on the qubit density matrix as the matrix multiplication of a four-dimensional vector $\boldsymbol{x}$ with a 4×4 matrix $\boldsymbol{\mathcal{E}}(t)$. The vector $\mathbf{x}$ has components $x_i = \mathrm{Tr}\{\rho\sigma_{i-1}\}/\sqrt{2}$ $(i = 1, \ldots, 4)$ with $\sigma_0 = \mathbb{1}$ and $\sigma_{1,2,3} \equiv \sigma_{x,y,z}$. In this four-dimensional representation, the time-periodic matrix $\boldsymbol{\mathcal{E}}(t)$ reads

$$\boldsymbol{\mathcal{E}}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & bf_1(t) + f_3(t) & 0 & f_2(t) \\ 0 & 0 & bf_1(t) - f_2(t) & f_3(t) \\ d^2 f_1(t) & f_2(t) & f_3(t) & b^2 f_1(t) \end{pmatrix}.$$

The four-dimensional vector $\boldsymbol{x}(t)$ obeys Eq. (1), with $\mathbf{z}(t) = -f(t)\boldsymbol{x}_0$, and a 4×4 kernel matrix of the form of Eq. (2), with $k = 2$, where $\mathbf{E}_1(t) = \mathbf{E}_2(t) = \mathbf{1}^{4\times4}$ and $\mathbf{F}_1(t) = \mathbf{F}_2(t) = \boldsymbol{\mathcal{E}}(t)$ and where $\Gamma_{1,2} = \pm A\gamma_\pm/a$ and $\gamma_{1,2} = (\gamma \pm a)/2$ [cf. Eq. (8)].

The embedding procedure with kernel (2), consisting of two terms, yields a 12-component vector $\mathbf{v} = (x_1, \ldots, x_4, u_1, \ldots, u_8)^{\mathrm{T}}$. Its evolution is governed by the time-local Eq. (5), with time-periodic matrix

$$\mathbf{M}(t) = \begin{pmatrix} \mathbf{0} & -\mathbf{H} \\ -\mathbf{G}(t) & \mathbf{A} \end{pmatrix}. \tag{13}$$

Here $\mathbf{0}$ is the null 4×4 matrix, $\mathbf{H} = (\Gamma_1\mathbf{1}, \quad \Gamma_2\mathbf{1})$,

$$\mathbf{G}(t) = \begin{pmatrix} \boldsymbol{\mathcal{E}}(t) \\ \boldsymbol{\mathcal{E}}(t) \end{pmatrix}, \quad \text{and } \mathbf{A} = \begin{pmatrix} \gamma_1\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \gamma_2\mathbf{1} \end{pmatrix}, \tag{14}$$

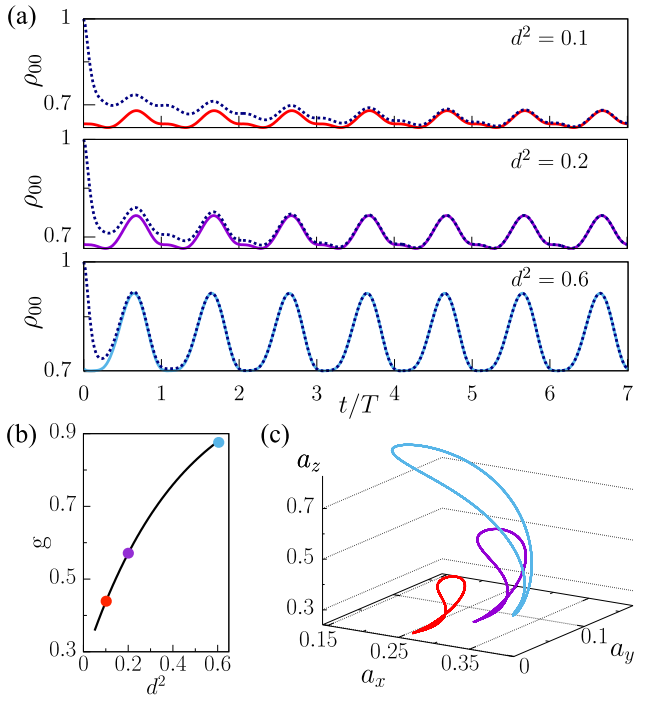

FIG. 2. Periodically modulated continuous-time quantum random walk of a qubit. Dynamics, spectral gap, and limit-cycle solution for three values of the parameter d^2 of the amplitude damping map $\mathcal{E}_1$ [see Eq. (10)] are shown. (a) Relaxation of the state $|0\rangle$ population obtained by numerical integration of Eq. (1) with initial condition $\rho(0) = |0\rangle\langle 0|$ (dotted lines), and asymptotic periodic states (solid lines) obtained from the Floquet propagator $\mathbf{U}_T$ of the embedded system. (b) Relevant spectral gap g of $\mathbf{U}_T$. (c) Limit cycle solutions for the Bloch vector. The waiting-time distribution, Eq. (8), has parameter $A = 0.24\gamma$. The modulation frequency is $\Omega = 0.1\gamma$.

where $\mathbf{1} \in \mathbb{R}^{4\times 4}$. Note that the embedding procedure yields both the transient and the asymptotic dynamics for the physical variable $\mathbf{x}(t)$ provided that the initial condition for the auxiliary vector, determined using Eq. (4), is

$$\mathbf{u}(0) = -\left(\frac{x_1(0)}{\gamma_1}, \ldots, \frac{x_4(0)}{\gamma_1}, \frac{x_1(0)}{\gamma_2}, \ldots, \frac{x_4(0)}{\gamma_2}\right)^{\mathrm{T}}.$$

In Fig. 2(a) we show the time evolution of $\rho_{00}(t) = \langle 0|\rho(t)|0\rangle$ obtained by the direct integration of Eq. (1), starting from $\rho(0) = |0\rangle\langle 0|$ (dashed lines). After several periods, the solutions land on the asymptotic limit cycles (solid lines) obtained from the Floquet propagator $\mathbf{U}_T$ with three Arnoldi iterations. The corresponding limit cycles of the Bloch vector of components $a_i = \mathrm{Tr}\{\rho\sigma_i\}$ $(i = x, y, z)$ are presented in Fig. 2(c). As shown in Fig. 2(b), the relevant spectral gap increases with the value of d^2, which in turn corresponds to a shorter time scale of relaxation towards the asymptotic state [see dashed lines in Fig. 2(a)].

IV. CONCLUSIONS

We have presented a method to find the asymptotic Floquet states for a class of periodically modulated systems governed by memory-kernel master equations. The method has been applied to a time-periodically modulated model of piecewise dynamics of a qubit, a quantum generalization of a classical semi-Markov process [47]. The asymptotic Floquet states are especially interesting in this context. In the stationary limit, the difference in non-Markovian and Markovian evolutions is noticeable only during the relaxation towards the asymptotic stationary state [29], which, for example, in the case of the qubit is a point inside (or on) the Bloch sphere. This point can be reached by following infinitely many trajectories, some of them corresponding to Markovian evolution and some not, so once the relaxation is over it is impossible to decide what kind of evolution the system has undergone. It is different when the qubit is periodically modulated because its asymptotic state represents a one-dimensional, time-parametrized manifold [see Fig. 2(c)]. This manifold is specific to the Liouville superoperator $\mathcal{L}$ and it could be that some Floquet states are not attainable with a Markovian $\mathcal{L}$. Outside of the quantum field, memristors [55] and metamaterials with memory [56,57] are considered now as perspective candidates for a new generation of nanoscale devices. They are typically modeled with Eq. (1); modulations can be introduced in these systems in different ways, thus creating room for new regimes.

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APPENDIX: DERIVATION OF EQ. (12)

Consider the general case in which the continuous background evolution between jumps is provided by some CPT map $\mathcal{F}(t)$. In the application we consider the case of a continuous-time quantum random walk [32] by setting $\mathcal{F}(t) \equiv \mathbb{1}$. The jumps are caused by the instantaneous actions of a CPT map $\mathcal{E}(t)$ at random instants in time distributed according to a waiting-time distribution $f(t)$. The starting point for deriving the generalized master equation (12) for density matrix $\rho(t)$ in the case of modulated piecewise dynamics, meaning that the map $\mathcal{E}$ is itself time dependent, is the sum over trajectories

$$\rho(t) = \mathcal{G}(t)\rho_0 + \sum_{n=1}^{\infty} \int_0^t dt_n \ldots \int_0^{t_2} dt_1 \tilde{\mathcal{F}}(t - t_n) \times \mathcal{E}(t_n) \ldots \mathcal{E}(t_2)\tilde{\mathcal{F}}(t_2 - t_1)\mathcal{E}(t_1)\mathcal{G}(t_1)\rho_0, \quad \text{(A1)}$$

where

$$\mathcal{G}(t) = g(t)\mathcal{F}(t),$$
$$\tilde{\mathcal{F}}(t_{j+1} - t_j) = f(t_{j+1} - t_j)\mathcal{F}(t_{j+1} - t_j). \quad \text{(A2)}$$

Here the function $g(t)$ gives the probability that no jump has occurred up to time t and is therefore defined by $g(t) = 1 - \int_0^t d\tau f(\tau)$.

In order to obtain the piecewise dynamics described by Eq. (A1) in the form of a master equation, we start by evaluating the series order by order in the number n of *jumps*, i.e., of actions of the map $\mathcal{E}$.

(i) Zero jumps ($n = 0$):

$$\rho^{(0)}(t) = \mathcal{G}(t)\rho_0. \quad \text{(A3)}$$

(ii) One jump ($n = 1$):

$$\rho^{(1)}(t) = \int_0^t dt_1 \tilde{\mathcal{F}}(t - t_1)\mathcal{E}(t_1)\mathcal{G}(t_1)\rho_0 = \int_0^t dt_1 \tilde{\mathcal{F}}(t - t_1)\mathcal{E}(t_1)\rho^{(0)}(t_1). \quad \text{(A4)}$$

(iii) Two jumps ($n = 2$):

$$\rho^{(2)}(t) = \int_0^t dt_2 \int_0^{t_2} dt_1 \tilde{\mathcal{F}}(t - t_2)\mathcal{E}(t_2) \times \tilde{\mathcal{F}}(t_2 - t_1)\mathcal{E}(t_1)\mathcal{G}(t_1)\rho_0 = \int_0^t dt_2 \tilde{\mathcal{F}}(t - t_2)\mathcal{E}(t_2) \times \int_0^{t_2} dt_1 \tilde{\mathcal{F}}(t_2 - t_1)\mathcal{E}(t_1)\rho^{(0)}(t_1) = \int_0^t dt_2 \tilde{\mathcal{F}}(t - t_2)\mathcal{E}(t_2)\rho^{(1)}(t_2), \quad \text{(A5)}$$

and so on.

We find the recursive relation

$$\rho^{(n)}(t) = \int_0^t dt' \tilde{\mathcal{F}}(t - t')\mathcal{E}(t')\rho^{(n-1)}(t') \quad (n \geqslant 1),$$
$$\rho^{(0)}(t) = \mathcal{G}(t)\rho_0. \quad \text{(A6)}$$

Summing the series we get

$$\rho(t) = \mathcal{G}(t)\rho_0 + \int_0^t dt' \tilde{\mathcal{F}}(t - t')\mathcal{E}(t') \sum_{n=1}^{\infty} \rho^{(n-1)}(t') = \mathcal{G}(t)\rho_0 + \int_0^t dt' \tilde{\mathcal{F}}(t - t')\mathcal{E}(t')\rho(t'). \quad \text{(A7)}$$

Finally, taking the time derivative of Eq. (A7) we obtain

$$\begin{aligned}\dot{\rho}(t) &= \frac{d}{dt}\mathcal{G}(t)\rho_0 + \int_0^t dt' \frac{\partial}{\partial t}\tilde{\mathcal{F}}(t-t')\mathcal{E}(t')\rho(t') \\ &\quad + \tilde{\mathcal{F}}(0)\mathcal{E}(t)\rho(t) \\ &= \int_0^t dt' \mathcal{K}(t-t')\mathcal{E}(t')\rho(t') + \mathcal{I}(t)\rho_0, \end{aligned} \tag{A8}$$

where

$$\begin{aligned}\mathcal{K}(t) &= \frac{d}{dt}\tilde{\mathcal{F}}(t) + \tilde{\mathcal{F}}(0)\delta(t), \\ \mathcal{I}(t) &= \frac{d}{dt}\mathcal{G}(t). \end{aligned} \tag{A9}$$

In the static case $\mathcal{E}(t) \equiv \mathcal{E}$, Eq. (A8) coincides with Eq. (7) in Ref. [34]. If instead we set $\mathcal{F}(t) \equiv \mathbb{1}$, the case considered in the application is recovered.

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