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Disorder-induced localization of excitability in an array of coupled lasers

M. Lamperti[1] and A. M. Perego[2,*]

[1]*Dipartimento di Fisica - Politecnico di Milano and IFN-CNR, via G. Previati 1/c, 23900 Lecco, Italy*

[2]*Aston Institute of Photonic Technologies, Aston University, Birmingham B4 7ET, United Kingdom*

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We report on the localization of excitability induced by disorder in an array of coupled semiconductor lasers with a saturable absorber. Through numerical simulations we show that the exponential localization of excitable waves occurs if a certain critical amount of randomness is present in the coupling coefficients among the lasers. The results presented in this Rapid Communication demonstrate that disorder can induce localization in lattices of excitable nonlinear oscillators, and can be of interest in the study of photonics-based random networks, neuromorphic systems, and, by analogy, in biology, in particular, in the investigation of the collective dynamics of neuronal cell populations.

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I. INTRODUCTION

The localization of waves in disordered systems is a fascinating phenomenon which can occur in many fields of physics. In a seminal paper published in 1958 [1], Anderson showed that electron transport in an atomic lattice can be inhibited in the presence of a sufficient amount of disorder, leading to an exponential localization of the electronic wave functions with a consequent transition of the system from metallic to insulator behavior. This phenomenon, differing substantially from the trivial localization due to a deep potential well, was shown to be related to a constructive quantum interference of the electronic wave functions induced by the scattering from the random impurities, and was later called *Anderson localization* in the literature. Localization can be induced either by disorder on the atomic site energy or by randomness affecting the hopping integrals between different lattice sites. The first case is the one originally considered by Anderson, and is referred to as diagonal localization, while the second case corresponds to so-called out-of-diagonal localization [2]. Disorder-induced localization has been studied deeply in solid state physics both theoretically and experimentally [3] but also in many other physical systems such as microwaves [4], ultrasounds [5], Bose-Einstein condensates [6], and light [7]. The signature of the localization of photons was first identified in pioneering works such as Refs. [8,9]. Later [10], it was shown that light traveling in a disordered two-dimensional photonic lattice can experience a transition from a transport regime to a localized one depending on the amount of randomness introduced in the system. Subsequent experiments on waveguide arrays have demonstrated that out-of-diagonal disorder can induce localization also in optical systems [11] and investigated the role of nonlinearity in the localization dynamics [12]. Despite the huge amount of literature devoted to the study of Anderson localization and to the fascinating field of disordered photonics [13], to the best of our knowledge, localization phenomena have yet to be studied in lattices of coupled excitable optical systems. In such systems, beside having an interest on their own, they could constitute an important ingredient for the emulation and understanding of similar phenomena occurring in populations of neurons. Excitability is a phenomenon that takes place when a dynamical system originally at a stationary state undergoes an *all-or-nothing* big excursion in phase space, after being triggered by a strong enough perturbation with at least one of its dynamical variable producing a spikelike pulse and subsequently relaxing back to equilibrium, until a new perturbation excites the process again. Although the most familiar example of excitable behavior is probably neuron cell activity [14,15], excitability has been diffusely studied in optics and, in particular, in laser systems, too [16–23]. Understanding the collective behavior and the dynamical properties of coupled excitable optical elements has inspired growing interest in recent years [24–28]. A well-known example of an excitable optical system is a semiconductor laser with an intracavity saturable absorber [16]. In this case, the laser is kept in the *off* solution but relatively close to threshold. Additive noise in the system provided by spontaneous emission or by some external perturbations can trigger the stimulated emission process, leading to a sudden gain depletion accompanied by the generation of a huge, spikelike, light pulse. After a so-called refractory time, related to the regeneration of the population inversion, the laser is ready to be excited again. We have recently shown that temporal and intensity dynamics in an array of coupled semiconductor lasers with an intracavity saturable absorber can cooperatively exhibit synchronization, when the coupling strength is large enough [28]. Indeed, in this strong-coupling regime, excitable waves can propagate through the array, leading to the synchronous behavior of the nonlinear oscillators. In this Rapid Communication we study numerically the effect of out-of-diagonal disorder in a one-dimensional array of coupled semiconductor lasers with a saturable absorber operating in the excitable regime. We demonstrate that when the amount of disorder exceeds a critical threshold, excitable waves cannot propagate freely in the lattice and excitability becomes exponentially localized in space.

II. THE MODEL

We consider a one-dimensional lattice where the lattice site is defined by a semiconductor laser with a saturable absorber described by the Yamada model [16,29]; each laser is then coupled locally to its nearest neighbors. The coupling

*peregoa@aston.ac.uk

considered here is a lossy one and corresponds physically to a nondelayed mutual injection between nearest-neighbor lasers, resulting in an effective discrete Laplacian operator which describes field diffusion across the array. For a population of n coupled lasers the dynamics of the ith oscillator is described by the following coupled nonlinear equations [28],

$$\begin{aligned}\dot{F}_i &= \frac{1}{2}(G_i - Q_i - 1)F_i + \sigma_i + \frac{K_{i,i+1}}{2}F_{i+1} + \frac{K_{i,i-1}}{2}F_{i-1}\\ &\quad -\left(\frac{K_{i,i-1}}{2} + \frac{K_{i,i+1}}{2}\right)F_i,\\ \dot{G}_i &= \gamma_i(A_i - G_i - I_iG_i),\\ \dot{Q}_i &= \gamma_i(B_i - Q_i - a_iQ_iI_i),\end{aligned} \tag{1}$$

where the time-dependent dynamical variables describing the ith laser are the complex electric field amplitude F_i, the inversion G_i, and the absorption Q_i. $I_i = |F_i|^2$ is the ith laser field intensity, σ_i is a delta-correlated complex Gaussian additive noise term with $\langle\sigma_i(t_1)\sigma_j(t_2)\rangle = \sqrt{2D}\delta(t_1 - t_2)\delta_{ij}$, γ_i is the absorber and gain decay rate, A_i is the bias current of the gain, a_i the differential absorption relative to the differential gain, and B_i the background absorption. The dot denotes a temporal derivative. $K_{i,i\pm1}$ describes local coupling between first neighbor lasers scaled to the intensity damping rate of the single uncoupled laser. Note that throughout the study only the case of reciprocal couplings $K_{i,i+1} = K_{i+1,i}$ $\forall i$, is considered. We have chosen the following parameters values, $A_i = 6.5$, $B_i = 5.8$, $a_i = 1.8$, and $\gamma_i = 10^{-3}$, $\forall i$; the noise strength D has been kept constant across all the array and periodic boundary conditions have been assumed. Disorder has been introduced into the system by letting the coupling $K_{i,i\pm1}$ vary randomly from laser to laser following a uniform distribution. In particular, we write $K_{i,i\pm1} = K_0 + \rho_{i,i\pm1}$, with K_0 an average coupling and $\rho_{i\pm1}$ a random number constant in time drawn from a uniform distribution in the interval $[-r, +r]$ with $\rho_{i,i+1} \neq \rho_{i,i-1}$ in general. In a solid-state physics analogy, this choice would correspond to a randomization of the hopping probability between neighbor sites of the atomic lattice.

III. DIFFUSIVE REGIME

In the absence of disorder in the coupling constants, the coupling term reduces to the Laplace discrete diffusion operator $K_0/2(F_{i+1} + F_{i-1} - 2F_i)$. In the strong-coupling regime, if without loss of generality we add noisy perturbations only to the central element of the array, when a sufficiently large perturbation is able to initiate the stimulated process, the laser will emit a giant pulse. Its energy is fed from the central laser to the two closest neighbors, which will also be able to "fire", generating the corresponding spikelike pulses. The process repeats and a wave of excitability propagates through the array. This is the diffusive, or ballistic, regime and constitutes the starting point for the localization phenomena (see Fig. 1). In this strong-coupling regime the lasers are synchronized in time: Qualitatively, the temporal interval within which all the lasers emit a pulse is much shorter than the single laser refractory time. Furthermore, the phases of the lasers appear to be locked or partially locked for a significant amount of time, as shown in Fig. 1(b).

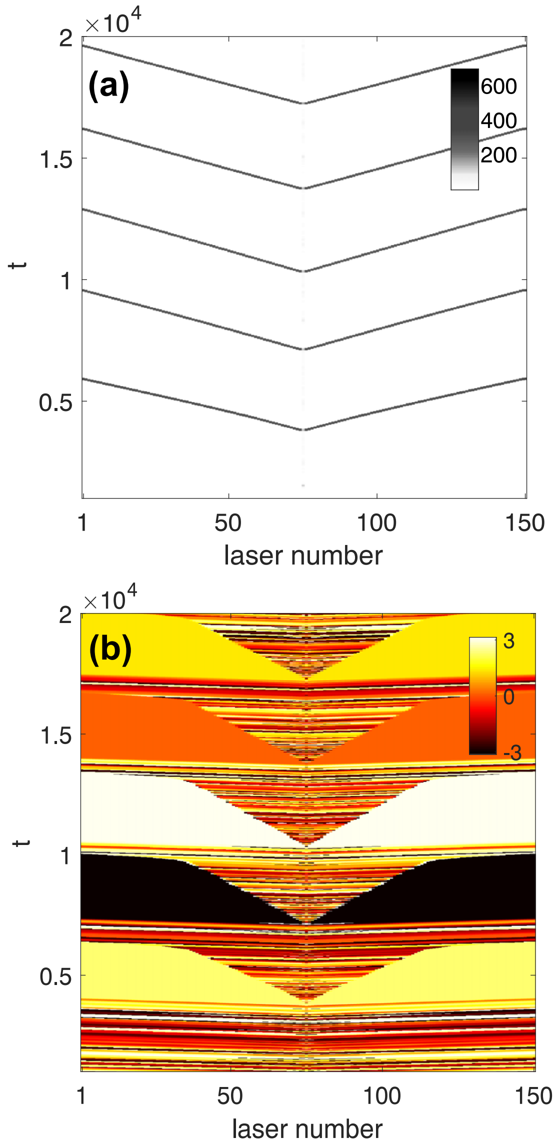

FIG. 1. The spatiotemporal dynamics corresponding to the diffusive regime: The excitability wave emanates from the center of the array where the intensity noise is added. Field intensity is plotted vs laser (x axis) and time (y axis) in (a), while in (b) the corresponding phase evolution is shown, demonstrating a substantial locking among the lasers during the firing events. Parameters used are $D = 0.1$ and $K_{i,i\pm1} = K_0 = 0.1$ $\forall i$.

IV. LOCALIZED REGIME

In the presence of disordered coupling terms in the array, when the excitability wave propagates, the left-right asymmetry induced by the randomness in the values of one laser's closest neighbor couplings leads to a preferential propagation direction of the wave, making one neighbor laser more likely to fire than the other one. This fact can lead ultimately to a multiple backscattering of the electric field with a consequent inhibition of wave propagation, and it results in an exponential localization of excitable behavior across the system if the strength of the randomness r is larger than a given threshold.

In Fig. 2(a) we have depicted an example of spatiotemporal dynamics for the coupled lasers in a localized regime.

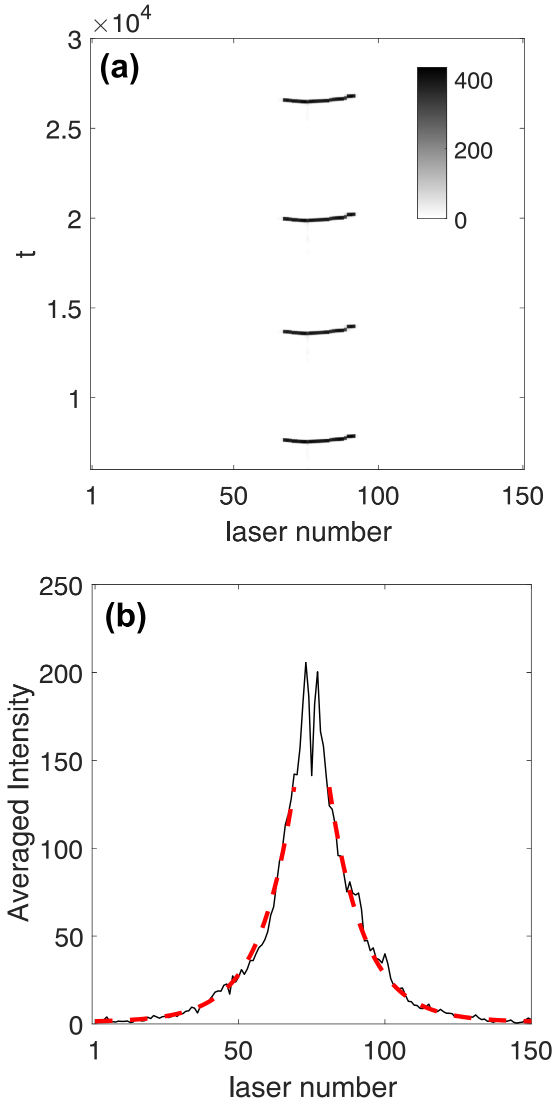

FIG. 2. In the presence of disorder, the excitable behavior of the laser array is localized as the spatiotemporal dynamics depicted in (a) shows. The average intensity across the array averaged over the "firing events" of 150 multiple realizations of disorder in the coupled lasers' system is fitted by an exponential function [dashed red line in (b)]. Parameters used are $r = 0.4$, $D = 0.1$, and $K_0 = 0.5$; $n = 150$ coupled lasers have been considered. We excluded from the fit the central laser as well as its three nearest neighbors on each side.

A substantial localization of the excitable behavior of the system takes place.

In order to characterize quantitatively the localization we have defined a "firing event" as a small time window located around the interval of time when the coupled lasers generate an excitable wave. Within the firing event temporal window we have recorded the maximum intensity emitted by each laser and averaged it over all firing events occurring during one simulation. After averaging the intensity distribution over many different realizations of the disorder in the system (i.e., different draws of the $K_{i,i\pm1}$'s for fixed r), we have then fitted the tails of such a resulting averaged intensity distribution with

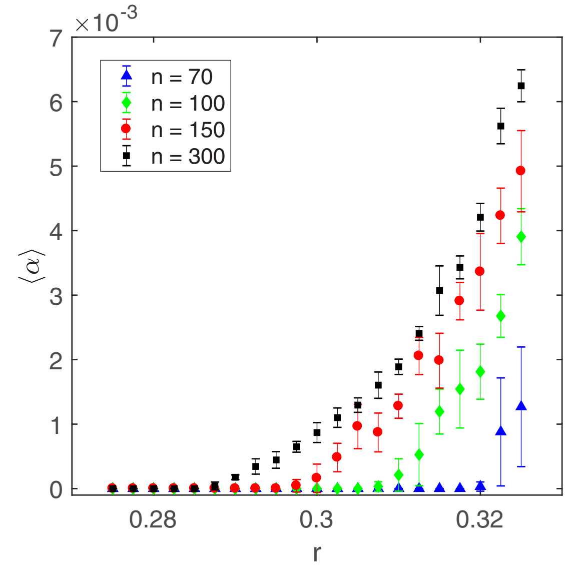

FIG. 3. The phase transition from a diffusive to localized regime is illustrated by plotting the average localization exponent $\langle\alpha\rangle$ and relative standard deviation vs the randomness strength r, for laser chains having different numbers of elements n (see legend). The amount of randomness necessary to achieve localization decreases by increasing the lattice size. Each point has been calculated averaging over five values of α obtained from 150 different realizations of disorder with the same strength r. The remaining parameters used are the same as in Fig. 2.

an exponential function [see Fig. 2(b)]

$$f = b + \exp(-\alpha|i - i_0|), \tag{2}$$

where i_0 is the position of the central laser. Repeating the above-mentioned procedure for five times allows one to obtain an average localization exponent $\langle\alpha\rangle$ and a relative standard deviation. The localization length can be hence defined as the inverse of $\langle\alpha\rangle$. Note that the method used is reliable and takes into account the fluctuations in the height of the different spikes emitted. Indeed, although in a single firing event some lasers may emit higher spikes than others, lasers that emit the most intense pulses change randomly (within the localization length) in successive firing events and in different realizations of the disorder, hence the average is justified. The dip in the average intensity [Fig. 2(b)] is due to a pulse reshaping dynamics: When the central laser fires, its pulse is rather noisy, as one could expect from an additive noise injection. As the pulse propagates outwards along the array, its shape changes, giving rise to higher pulses on the three nearest-neighboring lasers and then settling on a constant height afterwards, when a steady-state regime is attained. This occurs both in the diffusive and in the localized regime, and explains the presence of the dip in the data. Further details about such a collective pulse reshaping mechanism will be published elsewhere. In Fig. 3 the average localization exponent $\langle\alpha\rangle$ has been plotted versus the amount of disorder r for chains having different numbers of lasers n. Figure 3 shows that the system undergoes a transition from diffusive to

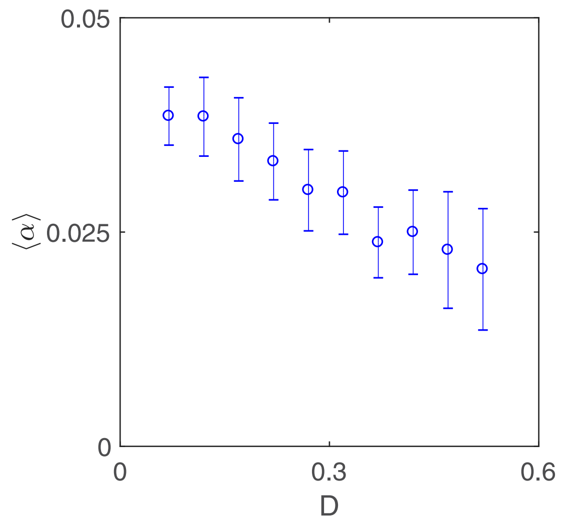

FIG. 4. The average localization exponent dependence on the noise strength D for $n = 150$ coupled lasers. Each point, and relative standard deviation, is the result of an average of over 20 different values of $\langle\alpha\rangle$, each one obtained through 150 realizations of the disorder and with the same value of D. Parameters used are $K_0 = 0.5$ and $r = 0.4$.

localized dynamics for $r > r_c$, where $r_c \approx 0.3$ is the critical point which depends on the length of the chain. Although the trend shown may allow one to imagine that, for arbitrarily large lattices, localization could occur with an arbitrary small noise strength ($r_c \to 0$), the latter fact cannot be stated yet with certainty and further investigations are needed in this direction. We have verified that for every value of the disorder strength r used, the lasers are always operating in the excitable regimes and excitability waves can propagate normally through the array if all the coupling coefficients are set identically equal to $K_0 \pm r$. As confirmation of this fact, Fig. 1 indeed shows that, for the case of $n = 150$, if all the lasers are identically coupled with a coupling constant smaller than the minimum considered in Fig. 3, the propagation of excitable waves is not inhibited. This check supports the fact that the observed localization phenomenon is due to a nontrivial dynamical scattering process, and it cannot be explained either by a purely particlelike dynamics or by some artificial local breaking of the links between neighbor elements of the array. It is important to stress that the synchronization of the laser is preserved in the localized regime, and indeed, the temporal interval within which all the lasers inside the localization length emit a pulse is much shorter than the single laser refractory time. The backscattering mechanism involved here is a complicated one, where phase and intensity dynamics coexist together with a major role played by the dissipation, and it is hence most likely the contribution of all these three players that determines the effective degradation of the diffusion of excitability.

Knowing the impact of spontaneous and/or injected noise on the behavior of complex nonlinear excitable systems is of primary importance in order to achieve the desired performances and to gain control over their dynamical behavior, as the paradigmatic case of coherence resonance in excitable lasers shows [16,28]. Consequently, we have characterized the dependence of the average localization exponent on the lasers' additive noise intensity D for fixed randomness strength r. The results of this analysis are summarized in Fig. 4 and show a decreasing trend of $\langle\alpha\rangle$ with increasing noise strength D.

As far as the terminology is concerned, it is important to mention the fact that the concept of Anderson localization, originally referring to the inhibition of electronic transport in an atomic lattice caused by disorder, is used in a broad sense in studies conducted in nonlinear optics. The system studied in this Rapid Communication presents substantial differences with respect to the original work by Anderson as well as with respect to some studies about the localization of light induced by disorder, in particular, here the dissipation plays a fundamental role. For this reason we do not think it is appropriate to call the phenomena discussed here *Anderson localization of excitability*, but simply *disorder-induced localization of excitability*.

V. CONCLUSIONS

In conclusion, we have demonstrated the *exponential localization of excitability* in a one-dimensional lattice of excitable lasers due to out-of-diagonal disorder, physically corresponding to random variations of the lasers' coupling coefficients. Our work suggests that disorder-induced localization of excitability can be potentially observed experimentally and deserves further investigation both in higher-dimensional excitable photonic systems as well as in other fields of science where excitability plays a major role, for instance, in populations of neurons. We believe that our results are especially relevant in the general study of signal transmission and control in networks of excitable systems where disorder is present as a natural feature, or vice versa, where it can be engineered to induce the manifestation of some particular phenomena. The field of neuromomorphic photonics is indeed a promising candidate platform where such fundamental effects can be studied and tailored, which hopefully could also provide insights for a cross-disciplinary interaction with computational neuroscience.

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