Photon-pair generation in a lossy microring resonator. I. Theory

Paul M. Alsing[1] and Edwin E. Hach, III[2]

[1]*Air Force Research Laboratory, Information Directorate, 525 Brooks Road, Rome, New York 13411, USA*

[2]*Rochester Institute of Technology, School of Physics and Astronomy, 85 Lomb Memorial Drive, Rochester, New York 14623, USA*

(Received 24 May 2017; revised manuscript received 27 June 2017; published 27 September 2017)

We investigate entangled photon-pair generation in a lossy microring resonator using an input-output formalism based on the work of M. G. Raymer and C. J. McKinstrie [Phys. Rev. A **88**, 043819 (2013)] and P. M. Alsing *et al.* [Phys. Rev. A **95**, 053828 (2017)] that incorporates circulation factors that account for the multiple round-trips of the fields within the cavity. We consider the nonlinear processes of spontaneous parametric down-conversion and spontaneous four-wave mixing, and we compute the generated biphoton signal-idler state from a single-bus microring resonator, along with the generation, coincidence-to-accidental, and heralding efficiency rates. We compare these generalized results to those obtained by previous works employing the standard Langevin input-output formalism.

DOI: 10.1103/PhysRevA.96.033847

I. INTRODUCTION

Over the last decade, advances in chip-based fabrication have made micron-scale, high-quality-factor Q integrated optical microring resonators (mrr's) coupled to an external bus ideal sources of entangled photon-pair generation, requiring only microwatts of pump power [1–6]. Such high-Q microring resonators exhibit nonlinear optical properties allowing for biphoton generation arising from the $\chi^{(2)}$ processes of spontaneous parametric down-conversion (SPDC) and the $\chi^{(3)}$ processes of spontaneous four-wave mixing (SFWM). Much theoretical research has been devoted to studying the generation of entangled photon pairs within cavities and mrr's in the weak pump field driving limit [2,7–14] and, more recently, in the strong pump field regime [15] where higher-order nonlinear effects such as self-phase and cross-phase modulation become important.

The predominant method of analysis for analyzing a driven cavity or mrr is the standard Langevin input-output formalism [16–20], which allows one to express the output field in terms of the intracavity and external driving fields. This formalism is valid in the high-cavity-Q limit, near cavity resonances, but does not adequately address processes throughout the entire free spectral range of the cavity. In this work we investigate entangled photon-pair generation in a microring resonator using a recent input-output formalism based on the work of Raymer and McKinstrie [21] and Alsing *et al.* [22] that incorporates the circulation factors that account for the multiple round-trips of the fields within the cavity. We consider biphoton-pair generation within the mrr via both SPDC and SFWM and compute the generated two-photon signal-idler intracavity and output states from a single-bus (all-through) microring resonator. We also compute the two-photon generation, coincidence-to-accidental, and heralding efficiency rates. We compare our results to related calculations [7,10,13] obtained using the standard Langevin input-output formalism.

This paper is organized as follows. In Sec. II we derive and solve the equations of motion for the pump, signal, and idler fields within a mrr coupled to a single external bus using a combination of the formalism of Raymer and McKinstrie [21] and Alsing *et al.* [22]. We consider the weak, nondepleted pump field limit where higher-order nonlinear processes such as self-phase and cross-phase modulations effects are neglected. We also examine the commutators of the quantum noise fields introduced to account for internal propagation loss which need not commute within the mrr, a phenomena noted previously by Barnett *et al.* [23] and Huang and Agarwal [24] in their studies of circulating cavity fields. In contrast to the standard Langevin approach, we show these commutators, which can be uniquely solved for by requiring the unitarity of the input and output fields, contain pump-dependent contributions. In Sec. III we compute the output biphoton state and calculate its generation rate, along with the coincidence-to-accidental and heralding efficiency rates. In Sec. IV we compute the biphoton state generated within the mrr since it is the state most often calculated in the literature and affords the most straightforward comparison. Again, we calculate the biphoton generation, coincidence-to-accidental, and heralding efficiency rates. We investigate how the mrr self-coupling (bus-bus, mrr-mrr) and internal propagation loss affect these rates. In Sec. V we summarize our results and indicate avenues for future research. In the Appendix we examine our expressions for the output fields and for rates derived from them in the high-cavity-Q limit where the standard Langevin input-output formalism is valid and compare them with prior works in the literature.

II. SPDC AND SPFM PROCESSES INSIDE A (SINGLE-BUS) MICRORING RESONATOR

A. Preliminaries

In this section we examine the nonlinear processes of SPDC and SFWM inside a single-bus mrr of length $L = 2\pi R$, as illustrated in Fig. 1. Here, a is the intracavity field which is coupled to a waveguide bus with input field a_{in} and output field a_{out}. The parameters ρ_a, τ_a are the beam-splitter-like self-coupling and cross-coupling strengths, respectively, of the bus to the mrr such that $|\rho_a|^2 + |\tau_a|^2 = 1$. $z = 0_+$ is the point just inside the mrr which cross couples to the input field a_{in}, and $z = L_-$ is the point after one round-trip in the mrr that cross couples to the output field a_{out}.

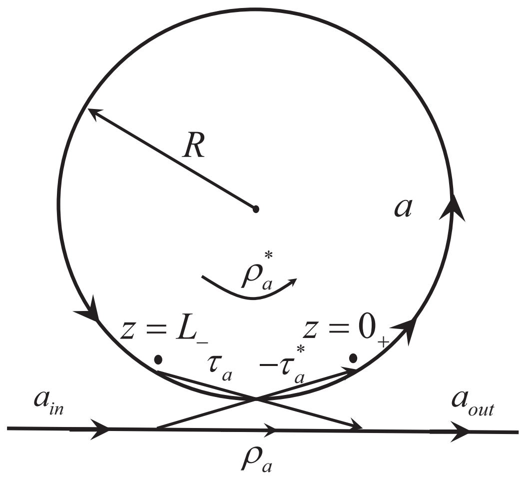

FIG. 1. A single-bus (all-through) microring resonator (mrr) of length $L = 2\pi R$ with intracavity field a, coupled to a waveguide bus with input field $a_{\rm in}$ and output field $a_{\rm out}$. ρ_a and τ_a are the beam-splitter-like self-coupling and cross-coupling strengths, respectively, of the bus to the mrr such that $|\rho_a|^2 + |\tau_a|^2 = 1$. $z = 0_+$ is the point just inside the mrr which cross couples to the input field $a_{\rm in}$, and $z = L_-$ is the point after one round-trip in the mrr that cross couples to the output field $a_{\rm out}$.

In the work of Raymer and McKinstrie [21] (abbreviated as RM) the cavity field a satisfies a traveling-wave Maxwell ordinary differential equation in the absence of internal propagation loss given by

$$(\partial_t + v_a\,\partial_z)\,a(z,t) = \alpha_{\rm polz}\,P(z,t), \tag{1}$$

where $a(z,t)$ is the ring-resonator cavity field (in the time domain), v_a is the group velocity, $P(z,t)$ is the polarization, and $\alpha_{\rm polz}$ is a coupling constant. The carrier wave frequency has been factored out so that all frequencies are relative to the optical carrier frequency. The input coupling and periodicity of the cavity are captured by the boundary conditions

$$a(0_+,t) = \rho_a\,a(L_-,t) + \tau_a\,a_{\rm in}(t), \tag{2a}$$

$$a_{\rm out}(t) = \tau_a\,a(L_-,t) - \rho_a\,a_{\rm in}(t), \tag{2b}$$

where we have taken the beam-splitter-like self-coupling ρ_a (buss-bus, mrr-mrr) and cross coupling τ_a (bus-mrr) to be real for simplicity and the minus sign in Eq. (2b) accounts for the π change in phase arising from the "reflection" of the input field off the higher index of the refraction mrr to the output (bus) field. The input and output fields satisfy the free-field commutators

$$[a_{\rm in}(t),a^\dagger_{\rm in}(t')] = \delta(t-t') = [a_{\rm out}(t),a^\dagger_{\rm out}(t')]. \tag{3}$$

The output field $a_{\rm out}(\omega)$ is easily solved from Eqs. (1), (2a), and (2b) in the Fourier domain, yielding the unimodular transfer function $G_{\rm out,in}(\omega)$ defined by

$$\begin{aligned} a_{\rm out}(\omega) &\equiv G_{\rm out,in}(\omega)\,a_{\rm in}(\omega),\\ G_{\rm out,in}(\omega) &= e^{i\omega T_a}\left[\frac{1-\rho_a\,e^{-i\omega T_a}}{1-\rho_a\,e^{i\omega T_a}}\right],\\ |G_{\rm out,in}(\omega)| &= 1. \end{aligned} \tag{4}$$

Note that in the classical case (see, e.g., Yariv [25] and Rabus [26]) one obtains the result with phenomenological loss factor $0 \leqslant \alpha_a \leqslant 1$,

$$\begin{aligned} a_{\rm out}(\omega) \equiv G^{(\alpha)}_{\rm out,in}(\omega)\,a_{\rm in}(\omega) &= \left(\frac{\alpha_a\,e^{i\theta_a} - \rho_a}{1-\rho_a^*\,\alpha_a\,e^{i\theta_a}}\right)a_{\rm in}(\omega),\\ \left|G^{(\alpha)}_{\rm out,in}(\omega)\right| &\leqslant 1, \end{aligned} \tag{5}$$

which is the same coefficient that appears in the quantum derivation with loss [see Eq. (13g) in Alsing *et al.* [22] (abbreviated as AH) with $\rho_a \to \tau_a$ being real] and $\theta_a = \omega\,T_a$. The lossless case corresponds to $\alpha_a \to 1$.

For the quantum derivation including internal propagation loss (generalizing the lossless mrr considerations begun in [27]), AH [22] used an expression by Loudon [28,29] for the attenuation loss of a traveling wave, modeled from a continuous set of beams splitters acting as scattering centers [28,29],

$$\begin{aligned} a(L_-,\omega) &= e^{i\xi_a(\omega)L}\,a(0_+,\omega)\\ &\quad + i\sqrt{\Gamma_a(\omega)}\int_0^L dz\,e^{i\xi_a(\omega)(L-z)}\,s(z,\omega), \end{aligned} \tag{6}$$

where $e^{i\xi_a L} \equiv \alpha_a\,e^{i\,\theta_a}$, with $\alpha_a = e^{-\frac{1}{2}(\Gamma_a/v_a)\,L}$ and $\theta_a = (\omega\,n(\omega)/c)\,L = \omega\,T_a$, and Γ_a incorporates both coupling and internal propagation losses. Here, $s(z,\omega)$ are the noise scattering operators that give rise to the internal loss and satisfy $[s(z,\omega),s^\dagger(z',\omega')] = \delta(z-z')\,\delta(\omega-\omega')$. AH explicitly showed that $a(L_-,\omega)$ in Eq. (6) satisfied $[a(L_-,\omega),a^\dagger(L_-,\omega')] = \delta(\omega-\omega')$. By tracking the infinite number of round-trip circulations of the cavity field in the single-bus mrr, AH derived the expression (with $\tau \to \rho_a$ and $\kappa \to \tau_a$ in [22])

$$\begin{aligned} a_{\rm out}(\omega) &= \left(\frac{\rho_a - \alpha_a\,e^{i\theta_a}}{1-\rho_a^*\,\alpha_a\,e^{i\theta_a}}\right)a_{\rm in}(\omega) - i\,|\tau_a|^2\,\sqrt{\Gamma_a}\sum_{n=0}^{\infty}(\rho_a)^n\\ &\quad\times\int_0^{(n+1)L} dz\,e^{i\xi_a(\omega)[(n+1)L-z]}\hat{s}(z,\omega). \end{aligned} \tag{7}$$

AH showed by explicit calculation that the output field satisfies $[a_{\rm out}(\omega),\,a^\dagger_{\rm out}(\omega)] = \delta(\omega-\omega')$. In general, this allows one to write

$$\begin{aligned} a_{\rm out}(\omega) &= G_{\rm out,in}(\omega)\,a_{\rm in} + H_{\rm out,in}(\omega)\,f_a(\omega),\\ |H_{\rm out,in}(\omega)| &= \sqrt{1-|G_{\rm out,in}(\omega)|^2}, \end{aligned} \tag{8}$$

which defines the quantum noise operator $f_a(\omega)$ from the unitary requirement of the preservation of the free-field output commutator. In the Appendix we examine $G_{\rm out,in}(\omega)$ and $H_{\rm out,in}(\omega)$ in the high-cavity-Q limit, where the standard Langevin input-output formalism is valid.

B. Derivation of output operators from input and noise operators

For the consideration of nonlinear biphoton-pair generation, we now consider three intracavity fields circulating within the mrr: the signal field $a(z,t)$, the idler field $b(z,t)$, and the pump field $c(z,t)$. As in the previous section, we work in the interaction picture where the carrier frequencies

ω_d for $d \in \{a,b,c\}$ have been removed, so that the fields are slowly varying in time. In the interaction picture the nonlinear Hamiltonian for these processes are taken to be

$$\mathcal{H}^{NL}_{\rm spdc} = g_{\rm spdc}\,(c\,a^\dagger\,b^\dagger + \text{H.a.}),\quad \omega_c = \omega_a + \omega_b, \tag{9a}$$

$$\mathcal{H}^{NL}_{\rm sfwm} = g_{\rm sfwm}\,(c^2\,a^\dagger\,b^\dagger + \text{H.a.}),\quad 2\,\omega_c = \omega_a + \omega_b. \tag{9b}$$

Each field $d(z,t)$ for $d \in \{a,b,c\}$ satisfies the equation of motion and input-output boundary conditions

$$(\partial_t + v_d\,\partial_z)\,d(z,t) = -i\,[d(z,t),\mathcal{H}^{NL}] - \frac{\gamma'_d}{2}\,d(z,t) + \alpha_{\rm polz}\,F_d(z,t), \tag{10a}$$

$$d(0_+,t) = \rho_d\,d(L_-,t) + \tau_d\,d_{\rm in}(t), \tag{10b}$$

$$d_{\rm out}(t) = \tau_d\,d(L_-,t) - \rho_d\,d_{\rm in}(t), \tag{10c}$$

where we have included the internal mrr propagation loss given by the rate γ'_d. We also allow for different group velocities $v_d(\omega) = c/n_d(\omega)$ for each mode d, leading to different round-trip times $T_d = L/v_d$ for $k \in \{a,b,c\}$. Junction coupling losses between the ring resonator and the bus are taken into account by later defining the self-coupling loss γ_d via $\rho_d \equiv e^{-\gamma_d\,T_d/2}$ [21].

In the above $F_d(z,t)$ are the noise operators inside the ring resonator, and $\alpha_{\rm polz}$ is a coupling constant of the internal modes a,b to the polarization field, giving rise to internal loss (see RM [21]). While the noise operators could be derived directly as in AH [22] by tracking the multiple round-trips of each field d through the mrr, here, we have opted for the Langevin-based approach indicated (but not explored) in RM [21]. Here, we differ from RM by not explicitly indicating the value of the commutation relations for the noise operators $F_d(z,t)$, preferring instead to compute their values later by the causality condition that the output fields $d_{\rm out}$ of the above coupled set of equations satisfy the free-field canonical commutation relations, given that the input fields $d_{\rm in}$ do. The particular value of the commutators is important when we compute the reduced density matrix ρ_{ab} for the two-photon output signal-idler state. For now the noise operators F_d are simply carried along through the computation.

The above equations are most easily solved in the frequency domain [30] [using $d(z,\omega) = \int_{-\infty}^{\infty} dt\,d(z,t)\,e^{i\omega t}$ for $d \in \{a,b,c\}$ and $f_d(z,\omega) = \int_{-\infty}^{\infty} dt\,F_d(z,t)\,e^{i\omega t}$]. Here, the interaction Hamiltonians are given by

$$\mathcal{H}^{NL}_{\rm spdc} = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,g_{\rm spdc}(\omega)\,[c(z,\omega)\,a^\dagger(z,\omega)\,b^\dagger(z,\omega) + \text{H.a.}],\quad g_{\rm spdc}(\omega) = \frac{3\,(\hbar\,\omega_c)^{3/2}\,\chi^{(2)}}{4\epsilon_0\,\bar{n}^4\,V_{\rm ring}}, \tag{11a}$$

$$\mathcal{H}^{NL}_{\rm sfwm} = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,g_{\rm sfwm}(\omega)\,[c^2(z,\omega)\,a^\dagger(z,\omega)\,b^\dagger(z,\omega) + \text{H.a.}],\quad g_{\rm sfwm}(\omega) = \frac{3(\hbar\,\omega_c)^2\,\chi^{(3)}}{4\epsilon_0\,\bar{n}^4\,V_{\rm ring}},\quad . \tag{11b}$$

where the values of $g_{\rm spdc}(\omega)$ and $g_{\rm sfwm}(\omega)$ [12,13] depend on the volume $V_{\rm ring}$ of the ring mode and the nonlinear susceptibilities $\chi^{(2)}$ and $\chi^{(3)}$ are accessed uniformly in the ring. Here, $\bar{n}$ is the average index of refraction of the ring (assumed constant), and ϵ_0 is the permittivity of free space.

In the undepleted pump approximation, employed here, the equation of motion for the pump mode c satisfies Eq. (1) [with $a \to c$ and $P(z,t) = 0$], and Hamiltonian terms such as $-i\,g_{\rm spdc}\,a\,b$ and $-i\,g_{\rm sfwm}\,c^\dagger\,a\,b$ are considered to be small and hence are dropped along with the noise term f_c [31]. This equation is then classical, and the value of the lossless pump inside the ring becomes

$$\langle c(0_+,\omega)\rangle = \frac{\tau_c}{1-\rho_c\,e^{i\omega T_c}}\,\langle c_{\rm in}(\omega)\rangle,\quad \langle c(L_-,\omega)\rangle = \frac{\tau_c\,e^{i\omega T_c}}{1-\rho_c\,e^{i\omega T_c}}\,\langle c_{\rm in}(\omega)\rangle, \tag{12}$$

where the angle brackets indicate that we are dealing with a c-number field value. Outside the ring, the pump field is given by

$$\langle c_{\rm out}(\omega)\rangle \equiv G_{c_{\rm out}\,c_{\rm in}}(\omega)\,\langle c_{\rm in}(\omega)\rangle,\quad G_{c_{\rm out}\,c_{\rm in}}(\omega) = e^{i\omega T_c}\left[\frac{1-\rho_c\,e^{-i\omega T_c}}{1-\rho_c\,e^{i\omega T_c}}\right]. \tag{13}$$

Therefore, in the Hamiltonian we replace the operator c_p by $\langle c(z,\omega)\rangle$ and write

$$\mathcal{H}^{NL} = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,g(\omega)\,[\alpha_p(z,\omega)\,a^\dagger(z,\omega)\,b^\dagger(z,\omega) + \alpha_p^*(z,\omega)\,a(z,\omega)\,b(z,\omega)], \tag{14a}$$

$$g(\omega) = g_{\rm spcd}(\omega),\qquad \alpha_p(z,\omega) = \langle c(z,\omega)\rangle, \tag{14b}$$

$$g(\omega) = g_{\rm sfwm}(\omega),\quad \alpha_p(z,\omega) = \langle c^2(z,\omega)\rangle. \tag{14c}$$

Thus, for both nonlinear processes the signal and idler modes satisfy the equation of motion in the frequency domain,

$$(-i\,\omega + v_a\,\partial_z)\,a(z,\omega) = -i\,g\,\alpha_p(z,\omega)\,b^\dagger(z,\omega) - \frac{\gamma'_a}{2}\,a(z,\omega) + \alpha_{\rm polz}\,f_a(z,\omega), \tag{15a}$$

$$(-i\,\omega + v_b\,\partial_z)\,b^\dagger(z,\omega) = i\,g\,\alpha_p^*(z,\omega)\,a(z,\omega) - \frac{\gamma'_b}{2}\,b^\dagger(z,\omega) + \alpha_{\rm polz}\,f_b(z,\omega). \tag{15b}$$

Equation (15a) has the formal solution

$$a(L_-,\omega) = a(0_+,t)\, e^{i\xi_a L} + \int_0^L dz'\, [(-ig\alpha_P/v_a)\, b^\dagger(z',\omega) + (\alpha_{\rm polz}/v_a)\, \tilde{f}_a(z',\omega)]\, e^{i\,\xi_a(L-z')}, \tag{16}$$

where $\xi_a = (\omega + i\,\gamma_a'/2)/v_a$ so that $i\,\xi_a L = (i\omega - \gamma_a'/2)\,T_a$. For fast molecular damping we approximate the last term by setting $z' \to L$ (the upper limit of the integral) and factoring out $\alpha_p(L_-,\omega)\, b^\dagger(L_-,\omega)$ from the integral. The remaining integral yields $[-i\, g\, \alpha_p(L_-,\omega)/v_a] \int_0^L dz'\, e^{i\,\xi_a(L-z')} = [-ig\alpha_p(L_-,\omega)/v_a](1 - e^{i\xi_a L})/(-i\xi_a) \to -ig\,\alpha_p(L_-,\omega)\,T_a \equiv -i\, r_a(\omega)$, defining the dimensionless pump parameter $r_a(\omega) = g\,\alpha_p(\omega)\,T_a$. Thus, we write Eq. (16) as

$$a(L_-,\omega) = a(0_+,t)\, e^{i\xi_a L} - i\, r_a(\omega)\, b^\dagger(L_-,t) + f_a(\omega), \quad f_a(\omega) \equiv (\alpha_{\rm polz}/v_a) \int_0^L dz'\, \tilde{f}_a(z',\omega)\, e^{i\,\xi_a(L-z')}. \tag{17a}$$

Similarly, Eq. (15b) yields

$$b^\dagger(L_-,\omega) = b^\dagger(0_+,t)\, e^{i\xi_b L} + i\, r_b(\omega)\, a(L_-,t) + f_b^\dagger(\omega),\; f_b^\dagger(\omega) = (\alpha_{\rm polz}/v_a) \int_0^L dz'\, \tilde{f}_b^\dagger(z',\omega)\, e^{i\,\xi_b(L-z')}, \tag{17b}$$

where we have defined $f_a(\omega)$ and $f_b^\dagger(\omega)$ and used the notation $r_a(\omega) \equiv g\,\alpha_p(L_-,\omega)\,T_a$ and $r_b(\omega) \equiv g\,\alpha_p^*(L_-,\omega)\,T_b$. We can therefore put equations of motion and boundary conditions for the signal a and idler b modes in matrix form as

$$M\,\vec{a}(L_-,\omega) = P_\xi\, \vec{a}(0_+,\omega) + \vec{f}(\omega), \tag{18a}$$

$$\vec{a}(0_+,\omega) = T_\rho\, \vec{a}(L_-,\omega) + X_\tau\, \vec{a}_{\rm in}(\omega), \tag{18b}$$

$$\vec{a}_{\rm out}(\omega) = X_\tau\, \vec{a}(L_-,\omega) - T_\rho\, \vec{a}_{\rm in}(\omega), \tag{18c}$$

where we have defined

$$M = \begin{pmatrix} 1 & i\,r_a \\ -i\,r_b & 1 \end{pmatrix}, \quad P_\xi = \begin{pmatrix} e^{i\xi_a L} & 0 \\ 0 & e^{i\xi_b L} \end{pmatrix}, \quad T_\rho = \begin{pmatrix} \rho_a & 0 \\ 0 & \rho_b \end{pmatrix}, \quad X_\tau = \begin{pmatrix} \tau_a & 0 \\ 0 & \tau_b \end{pmatrix}, \tag{19}$$

and

$$\vec{a}(\omega) = \begin{pmatrix} a(\omega) \\ b^\dagger(\omega) \end{pmatrix}, \quad \vec{a}_{\rm in}(\omega) = \begin{pmatrix} a_{\rm in}(\omega) \\ b_{\rm in}^\dagger(\omega) \end{pmatrix}, \quad \vec{a}_{\rm out}(\omega) = \begin{pmatrix} a_{\rm out}(\omega) \\ b_{\rm out}^\dagger(\omega) \end{pmatrix}, \quad \vec{f}(\omega) = \begin{pmatrix} f_a(\omega) \\ f_b^\dagger(\omega) \end{pmatrix}. \tag{20}$$

Here, T_ρ represents the through coupling "reflection" from the input bus off the ring resonator to the output bus (and the self-coupling within the mrr), while X_τ represents the cross-coupling "transmission" between the bus and the ring resonator. The term P_ξ represents the round-trip phase accumulation and intrinsic loss within the ring resonator, and we define $e^{i\xi_k L} \equiv \alpha_k\, e^{i\theta_k}$ with $\alpha_k = e^{-\gamma_k' T_k/2}$ and $\theta_k = \omega\, T_k$, $r_a = g\,\alpha_p\, T_a$, and $r_b = g\,\alpha_p^*\, T_b$.

A substitution of $\vec{a}(0_+,\omega)$ from Eq. (18b) into the right-hand side of Eq. (18a) allows for the solution of the intracavity field (just before exit) $\vec{a}(L_-,\omega)$ in terms of $\vec{a}_{\rm in}(\omega)$ and $\vec{f}(\omega)$. A subsequent substitution of this solution for $\vec{a}(L_-,\omega)$ into the right-hand side of Eq. (18c) produces the desired output field $\vec{a}_{\rm out}(\omega)$ in terms of the input field $\vec{a}_{\rm in}(\omega)$ and noise operators $\vec{f}(\omega)$. After some lengthy but straightforward algebra, the output fields can be expressed in terms of the input fields as

$$\vec{a}_{\rm out}(\omega) = G(\omega)\, \vec{a}_{\rm in}(\omega) + H(\omega)\, \vec{f}(\omega), \tag{21}$$

where

$$G(\omega) = [X_\tau (M - P_\xi\, T_\rho)^{-1}] P_\xi\, X_\tau - T_\rho \equiv \begin{pmatrix} G_{aa}(\omega) & G_{ab}(\omega) \\ G_{ba}(\omega) & G_{bb}(\omega) \end{pmatrix} \tag{22a}$$

$$\equiv \frac{1}{D} \begin{pmatrix} (e^{i\xi_a L} - \rho_a)(1 - \rho_b\, e^{i\xi_b L}) + r_a\, r_b\, \rho_a & -i\, r_a\, \tau_a\, \tau_b\, e^{i\xi_b L} \\ i\, r_b\, \tau_b\, \tau_a\, e^{i\xi_a L} & (e^{i\xi_b L} - \rho_b)(1 - \rho_a\, e^{i\xi_a L}) + r_a\, r_b\, \rho_b \end{pmatrix}, \tag{22b}$$

with

$$D = (1 - \rho_a\, e^{i\xi_a L})(1 - \rho_b\, e^{i\xi_b L}) - r_a\, r_b, \qquad r_a = g\alpha_P T_a, \qquad r_b = g\alpha_P^* T_b, \tag{23a}$$

$$\alpha_k = e^{-\gamma_k'/2\, T_k}, \quad \theta_k = \omega\, T_k \quad \text{for} \quad k \in \{a,b\} \tag{23b}$$

and

$$H(\omega) = X_\tau (M - P_\xi\, T_\rho)^{-1} \equiv \begin{pmatrix} H_{aa}(\omega) & H_{ab}(\omega) \\ H_{ba}(\omega) & H_{bb}(\omega) \end{pmatrix} = \frac{1}{D} \begin{pmatrix} \tau_a\,(1 - \rho_b e^{i\xi_b L}) & -i\, r_a\, \tau_a \\ i\, r_b\, \tau_b & \tau_b\,(1 - \rho_a e^{i\xi_a L}) \end{pmatrix}. \tag{24}$$

Note that to lowest order in $|g\alpha_p|$, we have $1/D \approx S_a\, S_b$, where $S_k = \frac{1}{1-\rho_k\, e^{i\xi_k L}} = \sum_{n=0}^\infty (\rho_k\, e^{i\xi_k L})^n \equiv \sum_{n=0}^\infty (\rho_k\, \alpha_k\, e^{i\,\theta_k})^n$ for $k \in \{a,b\}$ are the geometric series factors resulting from the round-trip circulations of the internal fields $k \in \{a,b\}$ inside the ring resonator. For a typical ring resonator of radius $R = 20\ \mu$m and pump laser power of 1 mW ($\chi^{(2)} \sim 2\times10^{-12}$ m/V, $\alpha_p \sim 10^3$ V/m) and round-trip times of $T_k \sim 1$ ps, we have $r_p \sim 10^{-5}$ [32].

A comparison of the matrix forms of $G(\omega)$ in Eq. (22a) and $H(\omega)$ in Eq. (24) reveals the useful relationship

$$G(\omega) = H(\omega)\, P_\xi(\omega)\, X_\tau(\omega) - T_\rho(\omega). \tag{25}$$

In the Appendix we examine $G(\omega)$ and $H(\omega)$ in the high-cavity-Q limit defined by $\rho_k \equiv e^{-\gamma_k T_k/2} \to 1$, $\omega T_k \ll 1$, where the standard Langevin approximation [17,20] is valid, and compare our results with recent related work [10] using the latter formulation.

C. Commutators of the noise operators

1. Linear equations determined by causality

The commutation relations between the noise operators are fundamentally determined by the canonical commutators of the free input and output fields. Given that the input fields satisfy $[a_{\rm in}(\omega),a^\dagger_{\rm in}(\omega')] = [b_{\rm in}(\omega),b^\dagger_{\rm in}(\omega')] = \delta(\omega-\omega')$ and that they each commute with the noise operators $f_a(\omega)$, $f_b(\omega)$ (via causality), one must also have that $[a_{\rm out}(\omega),a^\dagger_{\rm out}(\omega')] = [b_{\rm out}(\omega),b^\dagger_{\rm out}(\omega')] = \delta(\omega-\omega')$. Using Eq. (25), this unitary requirement determines the set of linear equations

$$[a_{\rm out}(\omega),a^\dagger_{\rm out}(\omega')] = \delta(\omega-\omega') \Rightarrow |H_{aa}|^2\, C_{aa} - |H_{ab}|^2\, C_{bb} + 2{\rm Re}(H_{aa}\, H^*_{ab}\, D_{ab}) = 1 - (|G_{aa}|^2 - |G_{ab}|^2), \tag{26a}$$

$$[b_{\rm out}(\omega),b^\dagger_{\rm out}(\omega')] = \delta(\omega-\omega') \Rightarrow -|H_{ba}|^2\, C_{aa} + |H_{bb}|^2\, C_{bb} + 2{\rm Re}(H_{ba}\, H^*_{bb}\, D_{ab}) = 1 - (|G_{bb}|^2 - |G_{ba}|^2), \tag{26b}$$

$$[a_{\rm out}(\omega),b_{\rm out}(\omega')] = 0 \Rightarrow H_{aa}\, H^*_{ba}\, C_{aa} - H_{ab}\, H^*_{bb}\, C_{bb} + H_{aa}\, H^*_{bb}\, D_{ab} + H_{ab}\, H^*_{ba}\, D^*_{ab} = G_{ab}\, G^*_{bb} - G_{aa}\, G^*_{ba}, \tag{26c}$$

$$[a_{\rm out}(\omega),b^\dagger_{\rm out}(\omega')] = 0 \Rightarrow \det(H)\, C_{ab} = 0 \tag{26d}$$

for the four constants C_{aa}, C_{bb}, C_{ab}, D_{ab} defined by the commutation relations

$$[f_a(\omega), f^\dagger_a(\omega')] = C_{aa}\, \delta(\omega-\omega'), \quad [f_b(\omega), f^\dagger_b(\omega')] = C_{bb}\, \delta(\omega-\omega'), \tag{27a}$$

$$[f_a(\omega), f^\dagger_b(\omega')] = C_{ab}\, \delta(\omega-\omega'), \quad [f_a(\omega), f_b(\omega')] = D_{ab}\, \delta(\omega-\omega'). \tag{27b}$$

Since $\det(H) \neq 0$, Eq. (26d) reveals that $C_{ab} = 0$. The first three equations are four equations in the four (real) unknowns C_{aa}, C_{bb}, ${\rm Re}(D_{ab})$, ${\rm Im}(D_{ab})$, which therefore have a unique solution. Note that in the standard Langevin approach [17,20] one assumes the canonical values $C_{aa} = C_{bb} = 1$ and $C_{ab} = D_{ab} = 0$. But the standard input-output formalism (here, valid near resonances of the ring resonator) was explicitly constructed so these canonical values identically satisfy the above set of linear equations (see, for example, the G and H matrices used in Tsang [10] and Barzanjeh *et al.* [33]). These special commutator values are not necessarily valid in general, and in particular $D_{ab} \neq 0$, as pointed out in the works of Barnett *et al.* [23] and Huang and Agarwal [24]. In general, the values of C_{aa}, C_{bb}, D_{ab} must be computed from Eqs. (26a), (26b), and (26c). The values of these commutators not only are important for the self-consistency of the theory but are also relevant when one computes the accidental singles rate upon the loss of either the generated signal or idler photon due to noise in the ring resonator. However, the values of these commutators do not affect the two-photon portion of the total state (see the next section) where neither a signal nor an idler photon is absorbed within the mrr.

2. Exact solution of the commutator equations

Using the expressions in Eq. (22a) for $G(\omega)$ and Eq. (24) for $H(\omega)$, a long but straightforward calculation results in the following simple *exact* solutions for the commutator equations (26a)–(26c):

$$C_{kk}(\omega) = 1 - \alpha_k^2 - |r_k|^2 = 1 - e^{-\gamma'_k T_k} - |g\alpha_p\, T_k|^2 \underset{{\rm high}\,Q}{\longrightarrow} \gamma'_k\, T_k - |g\alpha_p\, T_k|^2, \quad k \in \{a,b\}, \tag{28a}$$

$$D_{ab} = i\,(r^*_b - r_a) = i\, g\, \alpha_p\, (T_b - T_a), \tag{28b}$$

where for $C_{aa}(\omega)$ and $C_{bb}(\omega)$ we have also indicated their values in the high-cavity-Q limit. We note that $C_{kk}(\omega)$ for $k \in \{a,b\}$ contains a power-dependent correction $|r_k|^2 = |g\,\alpha_p\, T_k|^2$ of higher order than the leading-order term $1-\alpha_k^2$, which approaches $\gamma'_k\, T_k$ in the high-Q limit. If we were to redefine the noise operators as $f_k(\omega) \equiv (T_k)^{1/2}\, f'_k(\omega)$, then $C_{kk}(\omega) \equiv T_k\, \tilde{C}_{kk}$, where $\tilde{C}_{kk} = [f'_k(\omega), f'^\dagger_k(\omega)] \approx \gamma'_k\, \delta(\omega-\omega')$ to lowest order in $|r_k|^2$. This is the scaling employed by Raymer and McKinstrie [21] for the intracavity fields when comparing the operator equations of motions in the high-Q limit to the standard Langevin approach.

Other authors (see, e.g., [10,33]) using the standard Langevin approach often simply state from the outset that $[f''_k(\omega), f''^\dagger_k(\omega')] = \delta(\omega-\omega')$, with the assumption that all cross commutators are zero (i.e., $C_{ab} = D_{ab} = 0$), invoking independent noise sources. One could, of course, obtain this form of the diagonal commutators by redefining $f_k(\omega) = (C_{kk})^{1/2}\, f''_k(\omega)$ [with an appropriate rescaling of $H(\omega)$]. However, even in the high-Q limit, we see from Eq. (28b) that the cross commutator $[f_a(\omega), f_b(\omega)] = D_{ab}\, \delta(\omega-\omega')$ remains nonzero (though small), unless we assume equal group velocities (index of refractions) for both the signal (a) and idler (b) modes so that $T_a = T_b$.

III. THE OUTPUT TWO-PHOTON SIGNAL-IDLER STATE

Inside the ring resonator the Hamiltonian in frequency space is

$$\mathcal{H}^{NL} = \int_{0_+}^{L_-} dz \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, g(\omega)\,(\alpha_p(z,\omega)\, a^\dagger(z,\omega)\, b^\dagger(z,\omega) + \alpha_p^*(z,\omega)\, a(z,\omega)\, b(z,\omega)). \tag{29}$$

For a weak driving field $\alpha_p(\omega) = |\alpha_p(\omega)|\, e^{i\theta_p(\omega)}$, the two-photon state inside the mrr is given by

$$|\Psi(T_{ab})\rangle_{ab} = e^{-i\,\mathcal{H}^{NL}\,T_{ab}}\,|\Psi\rangle_{\rm in} \approx (1 - i\,\mathcal{H}^{NL}\,T_{ab})|{\rm vac}\rangle \tag{30a}$$

$$= \left\{1 - i\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\, r_{ab}(\omega)\,[e^{i\theta_p(\omega)}\,a^\dagger(L_-,\omega)\,b^\dagger(L_-,\omega) + e^{-i\theta_p(\omega)}\,a(L_-,\omega)\,b(L_-,\omega)]\right\}|{\rm vac}\rangle, \tag{30b}$$

where we have taken $T_{ab} = \sqrt{T_a T_b} = L/\sqrt{v_a\, v_b}$, assuming the group velocities of the generated signal and idler photons v_a, v_b are not too different and $|r_{ab}(\omega)| \equiv |g(\omega)\,\alpha_p(L_-,\omega)\,T_{ab}|$.

For simplicity, in Eq. (30b) we have assumed perfect phase matching and zero dispersion. In general [7,13,15,32,34], when the field operators inside the mrr are decomposed in terms of their spatial Fourier components, the spatial integral produces a phase-matching contribution term, $\int_{0_+}^{L_-} dz \exp\{i[k_p(\omega_p) - k_a(\omega_a) - k_b(\omega_b)]\,z\}$ for SPDC and $\int_{0_+}^{L_-} dz \exp\{i[2\,k_p(\omega_p) - k_a(\omega_a) - k_b(\omega_b)]\,z\}$ for SFWM, yielding oscillatory sinc function contributions over the longitudinal momentum-conservation mismatch within the mrr. Further, dispersion effects within the mrr could be accounted for by Taylor expanding $k(\omega_k) = \omega_k\, n_k(\omega_k)/c$ about central frequencies $\omega_{k,0}$ for $k \in \{p,a,b\}$ to either first or second order. While these spatially modulating sinc factors (which are unity for perfect phase matching) and dispersion effects are important to account for in actual physical devices, we will ignore them here in this work for ease of exposition.

The output state $|\Psi\rangle_{\rm out}$ is obtained from the internal $|\Psi(T_{ab})\rangle_{ab}$ as the Heisenberg operators evolve from inside the mrr to the output bus. Making the substitutions $a(L_-,\omega) \to a_{\rm out}(\omega)$ and $b(L_-,\omega) \to b_{\rm out}(\omega)$ in Eq. (30b) and inserting the expressions for $a_{\rm out}(\omega)$ and $b_{\rm out}(\omega)$ from Eq. (21) into Eq. (30b), we obtain the output state

$$|\Psi\rangle_{\rm out} = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,|\Psi^{(2)}(\omega)\rangle_{ab}\,|{\rm vac}\rangle_{\rm env} - i\,|r_{ab}(\omega)|\left[\left|\phi_a^{(1)}(\omega)\right\rangle_a |0\rangle_b\, f_b^\dagger(\omega)|{\rm vac}\rangle_{\rm env} + |0\rangle_a \left|\varphi_b^{(1)}(\omega)\right\rangle_b f_a^\dagger(\omega)\,|{\rm vac}\rangle_{\rm env} + |0\rangle_{ab}\,|\Phi^{(2)}(\omega)\rangle_{\rm env}\right], \tag{31}$$

where the vacuum state is given by $|{\rm vac}\rangle = |0\rangle_{ab}|{\rm vac}\rangle_{\rm env} = |0\rangle_a\,|0\rangle_b\,|{\rm vac}\rangle_{\rm env}$ such that $a_{\rm in}|0\rangle_a = b_{\rm in}|0\rangle_b = f_a|0\rangle_{\rm env} = f_b|0\rangle_{\rm env} = 0$. The states in Eq. (31) are given by

$$|\Psi^{(2)}(\omega)\rangle_{ab} = [2\pi\,\delta(\omega) - i\,|r_{ab}(\omega)|\,C_{\rm vac}(\omega)]\,|{\rm vac}\rangle_{ab} - i\,|r_{ab}(\omega)|\,\psi_{ab}^{(2)}(\omega)\,a_{\rm in}^\dagger(\omega)\,b_{\rm in}^\dagger(\omega)\,|0\rangle_{ab}, \tag{32a}$$

$$C_{\rm vac}(\omega) = e^{i\theta_p(\omega)}\,[G_{ab}^*(\omega)\,G_{bb}(\omega) + H_{ab}^*(\omega)\,H_{bb}(\omega)\,C_{bb}(\omega)] + e^{-i\theta_p(\omega)}\,[G_{aa}(\omega)\,G_{ba}^*(\omega) + H_{aa}(\omega)\,H_{ba}^*(\omega)\,C_{aa}(\omega)] \tag{32b}$$

$$\psi_{ab}^{(2)}(\omega) = e^{i\theta_p(\omega)}\,G_{aa}^*(\omega)\,G_{bb}(\omega) + e^{-i\theta_p(\omega)}\,G_{ab}(\omega)\,G_{ba}^*(\omega), \tag{32c}$$

$$\left|\phi_a^{(1)}(\omega)\right\rangle_a = [e^{i\theta_p(\omega)}\,G_{aa}^*(\omega)\,H_{bb}(\omega) + e^{-i\theta_p(\omega)}\,G_{ba}^*(\omega)\,H_{ab}(\omega)]\,a_{\rm in}^\dagger(\omega)\,|0\rangle_a \equiv \phi_a^{(1)}(\omega)\,|1_\omega\rangle_a, \tag{32d}$$

$$\left|\varphi_b^{(1)}(\omega)\right\rangle_b = [e^{i\theta_p(\omega)}\,G_{bb}(\omega)\,H_{aa}^*(\omega) + e^{-i\theta_p(\omega)}\,G_{ab}(\omega)\,H_{ba}^*(\omega)]\,b_{\rm in}^\dagger(\omega)\,|0\rangle_b \equiv \varphi_b^{(1)}(\omega)\,|1_{-\omega}\rangle_b, \tag{32e}$$

$$\left|\Phi^{(2)}(\omega)\right\rangle_{\rm env} = [e^{i\theta_p(\omega)}\,H_{aa}^*(\omega)\,H_{bb}(\omega)\,f_a^\dagger(\omega)\,f_b^\dagger(\omega) + e^{-i\theta_p(\omega)}\,H_{ab}(\omega)\,H_{ba}^*(\omega)\,f_b^\dagger(\omega)\,f_a^\dagger(\omega)]\,|{\rm vac}\rangle_{\rm env}. \tag{32f}$$

In the above, $C_{\rm vac}(\omega)$ is the first-order [in $|r_{ab}(\omega)|$] correction to the signal-idler vacuum state, and $|r_{ab}(\omega)|\,\psi_{ab}^{(2)}(\omega)$ is the two-photon wave function. From Eqs. (32c) and (A6) we observe that to zeroth order in $|g\,\alpha_p\,T_{ab}|$, the output two-photon state $\psi_{ab}^{(2)}(\omega) \approx e^{i\theta_p(\omega)}\,G_{aa}^*(\omega)\,G_{bb}(\omega)$ involves the frequency-dependent shifts of the input fields to the output fields. The second term in Eq. (32c), $e^{-i\theta_p(\omega)}\,G_{ab}(\omega)\,G_{ba}^*(\omega) \propto |g\,\alpha_p\,T_{ab}|^2$, represents a higher-order pump-dependent correction to $\psi_{ab}^{(2)}(\omega)$ involving the product of Lorentzian line-shape factors $\sqrt{\gamma_k}/(s + \Gamma_k/2)$ (where $s \equiv -i\omega$ can be considered a Laplace-transform-solution variable [10]), relating the fields $(\vec{a})_k$ inside the cavity to the input fields $(\vec{a}_{\rm in})_k$.

We are interested in the reduced density matrix of the signal-idler system obtained from $\rho_{ab} = {\rm Tr}_{\rm env}[|\Psi\rangle_{\rm out}\langle\Psi|]$. To trace over the environment we use the fact that ${\rm Tr}_{\rm env}[f_{i'}^\dagger(\omega')|{\rm vac}\rangle_{\rm env}\langle{\rm vac}|\,f_i(\omega)] = {}_{\rm env}\langle{\rm vac}|\,f_i(\omega)\,f_{i'}^\dagger(\omega')\,|{\rm vac}\rangle_{\rm env} = C_{i\,i'}\,\delta(\omega-\omega')$, where we have used $f_{i'}\,f_i^\dagger = [f_{i'},f_i^\dagger] + f_i^\dagger\,f_{i'}$. Similarly, ${\rm Tr}_{\rm env}[f_{i'}^\dagger(\omega')\,f_{j'}^\dagger(\omega')\,|{\rm vac}\rangle_{\rm env}\langle{\rm vac}|\,f_i(\omega)\,f_j(\omega)] = {}_{\rm env}\langle{\rm vac}|\,f_i(\omega)\,f_j(\omega)\,f_{i'}^\dagger(\omega')\,f_{j'}^\dagger(\omega')\,|{\rm vac}\rangle_{\rm env} = [C_{i\,i'}(\omega)\,C_{j\,j'}(\omega) + C_{i\,j'}(\omega)\,C_{j\,i'}(\omega)]\,\delta(\omega-\omega')$. Using the additional fact that

$C_{ab}(\omega) = 0$ from Eq. (26d), we have

$$\rho_{ab} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, |\Psi^{(2)}(\omega)\rangle_{ab}\langle\Psi^{(2)}(\omega)| + |r_{ab}(\omega)|^2 \Bigg(\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, R_0(\omega)|0\rangle_{ab}\langle 0|, \tag{33a}$$

$$+ \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \big|\phi_a^{(1)}(\omega)\big|^2\, C_{bb}(\omega)\, |1_\omega,0\rangle_{ab}\langle 1_\omega,0| + \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, C_{aa}(\omega)\, \big|\varphi_b^{(1)}(\omega)\big|^2\, |0,1_\omega\rangle_{ab}\langle 0,1_\omega| \Bigg),$$

$$|\Psi^{(2)}(\omega)\rangle_{ab} = [2\,\pi\,\delta(\omega) - i\,|r_{ab}(\omega)|\,C_{\mathrm{vac}}(\omega)]\,|0\rangle_{ab} - i\,|r_{ab}(\omega)|\,\psi_{ab}^{(2)}(\omega)\,|1_\omega,1_{-\omega}\rangle_{ab}, \tag{33b}$$

$$R_0(\omega) = C_{ab}(\omega)\,C_{ab}(\omega)\,|e^{i\theta_p(\omega)}\,H_{aa}^*(\omega)\,H_{bb}(\omega) + e^{-i\theta_p(\omega)}\,H_{ab}(\omega)\,H_{ba}^*(\omega)|^2. \tag{33c}$$

In the above, $|\Psi^{(2)}(\omega)\rangle_{ab}$ is the two-photon signal-idler state including the vacuum. The two-photon generation rate [7,10] is given by $R_{ab}(\omega) = |r_{ab}(\omega)|^2\,|\psi_{ab}^{(2)}(\omega)|^2$. The second line of Eq. (33a) gives the single-photon contributions due to the loss of an idler (leftmost term) or signal (rightmost term) photon with singles rates $|r_{ab}(\omega)|^2\,|\phi_a^{(1)}(\omega)|^2\,C_{bb}(\omega)$ and $|r_{ab}(\omega)|^2\,C_{aa}(\omega)\,|\varphi_b^{(1)}(\omega)|^2$, respectively, where the effect of the noise commutators is explicitly evident. We can therefore write

$$\rho_{ab} = \mathrm{Tr}_{\mathrm{env}}[|\Psi\rangle_{\mathrm{out}}\langle\Psi|] = \sum_{k=0,1,2} p_k\,\rho_{ab}^{(k)}, \quad \mathrm{Tr}_{ab}\big[\rho_{ab}^{(k)}\big] = 1, \quad \sum_{k=0,1,2} p_k = 1. \tag{34}$$

Here, $\rho_{ab}^{(k)}$, with $k \in \{0,1,2\}$, represents the k *system*-photon (i.e., signal-idler) portion of the reduced density matrix ρ_{ab}. One can then define the output coincidence-to-accidental rate (CAR) [10] as

$$R_{\mathrm{CAR}}^{(\mathrm{out})}(\omega) = \frac{\big|\psi_{ab}^{(2)}(\omega)\big|^2}{\big|\phi_a^{(1)}(\omega)\big|^2\,C_{bb}(\omega) + C_{aa}(\omega)\,\big|\varphi_b^{(1)}(\omega)\big|^2}$$

$$\xrightarrow[O(|g\,\alpha_p|)]{\text{high } Q} \frac{|G_{aa}^*(\omega)\,G_{bb}(\omega)|^2}{|G_{aa}^*(\omega)\,\tilde{H}_{bb}(\omega)|^2 + |\tilde{H}_{aa}^*(\omega)\,G_{bb}(\omega)|^2} = \left(\frac{\gamma_a\,\gamma_a'}{\omega^2 + (\Delta_a/2)^2} + \frac{\gamma_b\,\gamma_b'}{\omega^2 + (\Delta_b/2)^2}\right)^{-1} \tag{35}$$

and the output heralding efficiency [10] of, say, the idler photon by the measurement of a signal photon as

$$R_{\mathrm{herald}}^{(\mathrm{out})}(\omega) = \frac{\big|\psi_{ab}^{(2)}(\omega)\big|^2}{\big|\phi_a^{(1)}(\omega)\big|^2\,C_{bb}(\omega) + \big|\psi_{ab}^{(2)}(\omega)\big|^2}$$

$$\xrightarrow[O(|g\,\alpha_p|)]{\text{high} Q} \frac{|G_{aa}^*(\omega)\,G_{bb}(\omega)|^2}{|G_{aa}^*(\omega)\,\tilde{H}_{bb}(\omega)|^2 + |G_{aa}^*(\omega)\,G_{bb}(\omega)|^2} = \left(1 + \frac{\gamma_b\,\gamma_b'}{\omega^2 + (\Delta_b/2)^2}\right)^{-1}, \tag{36}$$

where we have used $H_{kk}(\omega)\,C_{kk} \to H_{kk}(\omega)\,(1-\alpha_k^2)^{1/2} \to \gamma_k'\,H_{kk}(\omega) = \tilde{H}_{kk}(\omega)$ in the high-cavity-Q limit. Note that in the first lines in Eqs. (35) and (36) a common factor of $|r_{ab}(\omega)|^2 = |g\,\alpha_p\,T_{ab}|^2$ in the numerator and denominator has been canceled.

IV. THE TWO-PHOTON SIGNAL-IDLER STATE INSIDE THE mrr

It is noteworthy to investigate the state of the two-photon state *inside* the mrr cavity since it is this state which is most often computed in other treatments [7,10] (with the output field usually given as simply $\sqrt{\gamma_a\,\gamma_b}$ times the input field; see, e.g., [7,17]).

For a weak driving field $\alpha_p(\omega) = |\alpha_p(\omega)|\,e^{i\theta_p(\omega)}$ the two-photon state inside the mrr is given by Eq. (30a), which for convenience we restate:

$$|\Psi(T_{ab})\rangle_{ab} = e^{-i\,\mathcal{H}^{NL}\,T_{ab}}\,|\Psi\rangle_{\mathrm{in}} \approx (1 - i\,\mathcal{H}^{NL}\,T_{ab})|\mathrm{vac}\rangle$$

$$= \left\{1 - i\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,r_{ab}(\omega)\,[e^{i\theta_p(\omega)}\,a^\dagger(L_-,\omega)\,b^\dagger(L_-,\omega) + e^{-i\theta_p(\omega)}\,a(L_-,\omega)\,b(L_-,\omega)]\right\}|\mathrm{vac}\rangle. \tag{37}$$

Using the output boundary condition (18c) and Eq. (25) relating the output fields to the input fields, we obtain

$$\vec{a}(L_-,\omega) = \big[X_\tau^{-1}\,H(\omega)\,P_\chi\,X_\tau\big]\,\vec{a}_{\mathrm{in}} + \big[X_\tau^{-1}\,H(\omega)\big]\,\vec{f}(\omega) \equiv G^{(L)}(\omega)\,\vec{a}_{\mathrm{in}}(\omega) + H^{(L)}(\omega)\,\vec{f}(\omega), \tag{38}$$

with [employing the expression for $H(\omega)$ in Eq. (24)]

$$G^{(L)}(\omega) = \frac{1}{D(s)}\begin{pmatrix} \tau_a\,(1 - \rho_b e^{i\xi_b L})\,e^{i\xi_a L} & -i\,r_a\,\tau_b\,e^{i\xi_b L} \\ i\,r_b\,\tau_a\,e^{i\xi_a L} & \tau_b\,(1 - \rho_a e^{i\xi_a L})\,e^{i\xi_b L} \end{pmatrix} \tag{39a}$$

$$\xrightarrow[O(|g\,\alpha_p|)]{\text{high} Q} \begin{pmatrix} \frac{\sqrt{\gamma_a}}{s+\Gamma_a/2} & -i\,r_a\,\big(\frac{1}{s+\Gamma_a/2}\big)\big(\frac{\sqrt{\gamma_b}}{s+\Gamma_b/2}\big)\,e^{i\xi_b L} \\ -i\,r_b\,\big(\frac{\sqrt{\gamma_a}}{s+\Gamma_a/2}\big)\big(\frac{1}{s+\Gamma_b/2}\big)\,e^{i\xi_a L} & \frac{\sqrt{\gamma_b}}{s+\Gamma_b/2}, \end{pmatrix}, \tag{39b}$$

where in Eq. (39b) we have used $e^{i\xi_k L} \approx 1$, and

$$H^{(L)}(\omega) = \frac{1}{D(s)} \begin{pmatrix} (1-\rho_b e^{i\xi_b L}) & -i\, r_a \\ i\, r_b & (1-\rho_a e^{i\xi_a L}) \end{pmatrix} \equiv \tilde{H}^{(L)}(\omega)\, \Lambda_\alpha^{-1}(\omega),\ \Lambda_\alpha \equiv \begin{pmatrix} \left(1-\alpha_a^2\right)^{1/2} & 0 \\ 0 & \left(1-\alpha_b^2\right)^{1/2} \end{pmatrix}, \tag{40a}$$

$$\underset{O(|g\,\alpha_p|)}{\overset{\text{high}\varrho}{\longrightarrow}} \begin{pmatrix} \frac{1}{s+\Gamma_a/2} & -i\, r_a \left(\frac{1}{s+\Gamma_a/2}\right)\left(\frac{1}{s+\Gamma_b/2}\right) e^{i\xi_b L} \\ -i\, r_b \left(\frac{1}{s+\Gamma_a/2}\right)\left(\frac{1}{s+\Gamma_b/2}\right) e^{i\xi_a L} & \frac{1}{s+\Gamma_b/2}, \end{pmatrix}, \tag{40b}$$

where we have similarly defined $\tilde{H}^{(L)}(\omega)$ as in Eq. (A4). The calculation of the wave function $|\Psi(T_{ab})\rangle_{ab}$ and reduced density matrix $\rho_{ab}(T_{ab})$ inside the mrr at $z = L_-$ proceeds identically as in Sec. IV except for the replacement of $G(\omega) \to G^{(L)}(\omega)$ and $H(\omega) \to H^{(L)}(\omega)$ in Eqs. (31) and (33a), respectively. Analogous to Eq. (32c), the two-photon wave function inside the mrr is given by $|r_{ab}(\omega)|$ times

$$\psi^{(2)}_{ab,\,mrr}(\omega) = e^{i\theta_p(\omega)}\, G^{(L)*}_{aa}(\omega)\, G^{(L)}_{bb}(\omega) + e^{-i\theta_p(\omega)}\, G^{(L)}_{ab}(\omega)\, G^{(L)*}_{ba}(\omega). \tag{41}$$

To zeroth order in $|g\,\alpha_p\, T_{ab}|$, the first term gives $\psi^{(2)}_{ab,\text{mrr}}(\omega) \approx G^{(L)*}_{aa}(\omega)\, G^{(L)}_{bb}(\omega) = [\sqrt{\gamma_a}/(s+\Gamma_a/2)][\sqrt{\gamma_b}/(s+\Gamma_b/2)]$, the product of the standard Langevin input-output theory Lorentzian line-shape factors for each field a,b. This is the typical expression found in other works computing the two-photon state inside a cavity or mrr [7,10,11,35]. The starting point for many such calculations invoking the standard Langevin input-output formalism [17,20] begins with the statement that the (generic) free-field operator $a(\omega)$ is modified inside the cavity or mrr by the change in the density of states, which is accounted for by the substitution $a(\omega) \to \sqrt{\gamma_a}\, a(\omega)/(s+\Gamma_a/2)$. Again, the second term, $e^{-i\theta_p(\omega)}\, G^{(L)}_{ab}(\omega)\, G^{(L)*}_{ba}(\omega)$, in $\psi^{(2)}_{ab,\,\text{mrr}}(\omega)$ represents a second-order (in $|g\,\alpha_p|$) pump-dependent correction. Inside the mrr cavity, the expression for the CAR is

$$\begin{aligned} R^{(\text{mrr})}_{\text{CAR}}(\omega) &= \frac{\left|\psi^{(2)}_{ab,\,\text{mrr}}(\omega)\right|^2}{\left|\phi^{(1)}_{a,\,\text{mrr}}(\omega)\right|^2 C_{bb}(\omega) + C_{aa}(\omega)\left|\varphi^{(1)}_{b,\,\text{mrr}}(\omega)\right|^2} \\ &\underset{O(|g\,\alpha_p|)}{\overset{\text{high}\varrho}{\longrightarrow}} \frac{\left|G^{(L)*}_{aa}(\omega)\, G^{(L)}_{bb}(\omega)\right|^2}{\left|G^{(L)*}_{aa}(\omega)\, \tilde{H}^{(L)}_{bb}(\omega)\right|^2 + \left|\tilde{H}^{(L)*}_{aa}(\omega)\, G^{(L)}_{bb}(\omega)\right|^2} = \frac{\gamma_a\,\gamma_b}{\gamma_a\,\gamma'_a + \gamma_b\,\gamma'_b}, \end{aligned} \tag{42}$$

and that for the heralding efficiency of, say, an idler photon by the measurement of a signal photon is

$$\begin{aligned} R^{(\text{mrr})}_{\text{herald}}(\omega) &= \frac{\left|\psi^{(2)}_{ab,\,\text{mrr}}(\omega)\right|^2}{\left|\phi^{(1)}_{a,\,\text{mrr}}(\omega)\right|^2 C_{bb}(\omega) + \left|\psi^{(2)}_{ab,\,\text{mrr}}(\omega)\right|^2} \\ &\underset{O(|g\,\alpha_p|)}{\overset{\text{high}\varrho}{\longrightarrow}} \frac{\left|G^{(L)*}_{aa}(\omega)\, G^{(L)}_{bb}(\omega)\right|^2}{\left|G^{(L)*}_{aa}(\omega)\, \tilde{H}^{(L)}_{bb}(\omega)\right|^2 + \left|G^{(L)*}_{aa}(\omega)\, G^{(L)}_{bb}(\omega)\right|^2} = \frac{\gamma_a\,\gamma_b}{\gamma_a\,\gamma'_b + \gamma_a\,\gamma_b} = \frac{\gamma_b}{\Gamma_b}. \end{aligned} \tag{43}$$

Equations (42) and (43) generalize the expressions of Tsang [10], which were computed for a cavity using the standard Langevin input-output formalism [recalling that to lowest order in $|g\,\alpha_p\, T_{ab}|$ we have $C_{kk} \approx \gamma'_k$ for $k \in \{a,b\}$, so that $H^{(L)}_{bb}(\omega)\, C_{kk} \to \tilde{H}^{(L)}_{kk}(\omega)$]. Both of the above expressions suggest that the minimization of internal propagation losses $\gamma'_k \ll \gamma_k$ is desirable for the generation of pure entangled photons.

The expression for the biphoton production rate inside the mrr is given by

$$R^{(\text{mrr})}_{ab} = |r_{ab}(\omega)|^2 \left|\psi^{(2)}_{ab,\text{mrr}}(\omega)\right|^2, \tag{44}$$

where the two-photon wave function inside the mrr is given by

$$\begin{aligned} \psi^{(2)}_{ab,\,\text{mrr}}(\omega) &= e^{i\theta_p(\omega)}\, G^{(L)*}_{aa}(\omega)\, G^{(L)}_{bb}(\omega) + e^{-i\theta_p(\omega)}\, G^{(L)}_{ab}(\omega)\, G^{(L)*}_{ba}(\omega) \\ &= \frac{\alpha_a\alpha_b\, \tau_a\tau_b\, e^{i(\theta_b+\theta_p)}\,[e^{i\theta_b}|r_a|\,|r_b| - (1-e^{i\theta_a}\alpha_a\rho_a)(e^{i\theta_b}-\alpha_b\rho_b)]}{[e^{i(\theta_a+\theta_b)}|r_a|\,|r_b| - (e^{i\theta_a}-\alpha_a\rho_a)\,(e^{i\theta_b}-\alpha_b\rho_b)][(1-e^{i\theta_a}\alpha_a\rho_a)\,(1-e^{i\theta_b}\alpha_b\rho_b) - |r_a|\,|r_b|]}. \end{aligned} \tag{45}$$

In Fig. 2 we plot $\tilde{R}^{(\text{mrr})}_{ab} = |\psi^{(2)}_{ab,\text{mrr}}(\omega)|^2 = R^{(\text{mrr})}_{ab}/|r_{ab}(\omega)|^2$ for a weak driving pump $|r_a| = |r_b| = r = 10^{-5}$ on mrr resonance $\theta = \omega\, T = 0$ and slightly off resonance at $\theta = 0.1$. In this plot (and subsequent ones), we have considered equal mrr round-trip times $T_a = T_b = T$ for both the signal and idler so that $\theta_a = \theta_b \equiv \theta = \omega\, T$, as well as equal coupling $\rho_a = \rho_b \equiv \rho$ and internal loss $\alpha_a = \alpha_b \equiv \alpha$. Here, $\alpha = (0.99, 0.95)$ represents the physically relevant values of 1% and 5% propagation loss within the mrr, respectively. Note that $\tilde{R}^{(\text{mrr})}_{ab}$ is independent of the pump phase θ_p, as can be observed from the overall factor of $e^{i\theta_p(\omega)}$ in Eq. (45).

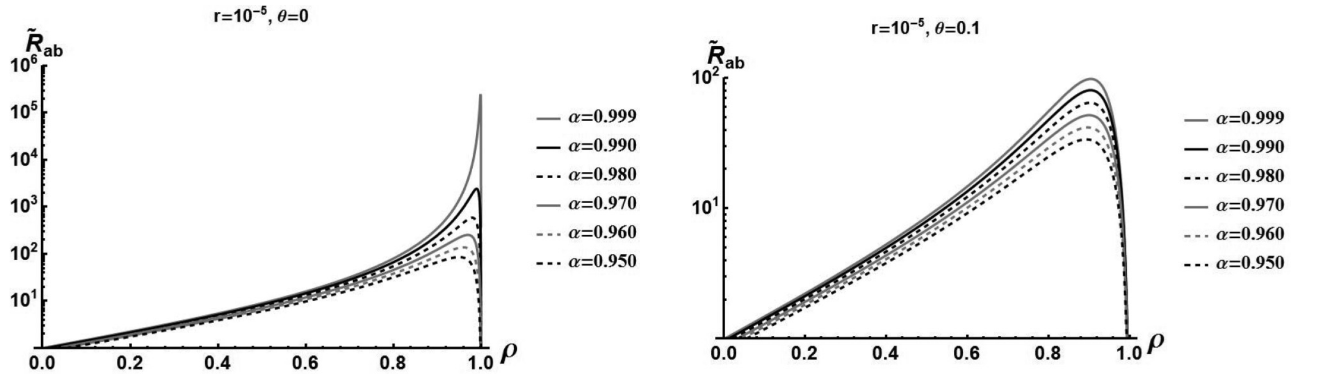

FIG. 2. $\tilde{R}^{(\mathrm{mrr})}_{ab} = |\psi^{(2)}_{ab,\mathrm{mrr}}(\omega)|^2 = R^{(\mathrm{mrr})}_{ab}/|r_{ab}(\omega)|^2$ for $r = 10^{-5}$ and (left) on mrr resonance $\theta = 0$ and (right) slightly off mrr resonance $\theta = 0$ for $\alpha = (0.999,0.99,0.98,0.97,0.95,0.95)$.

The surface of $\tilde{R}^{(\mathrm{mrr})}_{ab} \geqslant 1$ for resonance $\theta = 0$ as a function of coupling ρ and internal propagation loss α is plotted in Fig. 3. This plot indicates that strong biphoton-pair production is favored by a high cavity Q ($\rho \to 1$) and low internal propagation loss ($\alpha \to 1$). In Fig. 4 we plot $\tilde{R}^{(\mathrm{mrr})}_{ab}$ as a function of $\theta = \omega T$ for $\rho = 0.95$ and $\rho = 0.50$, where the effect of the resonance structure of the mrr is manifest. In the Appendix we compare the expression for the biphoton generation rate in the high-cavity-Q limit with other expressions derived in the literature [7,10] using the standard Langevin approach.

The expressions for $R^{(\mathrm{mrr})}_{\mathrm{CAR}}(\omega)$ in Eq. (42) and $R^{(\mathrm{mrr})}_{\mathrm{herald}}(\omega)$ in Eq. (43) take on simple analytic forms given by

$$R^{(\mathrm{mrr})}_{\mathrm{CAR}}(\omega) = \frac{\alpha_a^2\,\alpha_b^2\,\tau_a^2\,\tau_a^2}{(1-|r_a|^2)\,\alpha_b^2\,\alpha_b^2 + \alpha_a^2\big[\big(1-|r_b|^2-\alpha_b^2\big)\tau_a^2 - \alpha_b^2\,\alpha_b^2\big]} \to \frac{\alpha^2\,(1-\rho^2)}{2\,(1-|r|^2-\alpha^2)}, \tag{46}$$

where in the last expression we have again used $T_a = T_b = T$, $\rho_a = \rho_b = \rho$, and $\alpha_a = \alpha_b = \alpha$. Note $R^{(\mathrm{mrr})}_{\mathrm{CAR}}(\omega)$ is independent of θ_a, θ_b. In Fig. 5 we plot $R^{(\mathrm{mrr})}_{\mathrm{CAR}}(\omega)$ with $r = 10^{-5}$ for the operationally relevant (for $\alpha \leqslant 0.99$) internal propagation loss values $\alpha = (0.999,0.99,0.98,0.97,0.95,0.95)$.

The heralding efficiency $R^{(\mathrm{mrr})}_{\mathrm{herald}}(\omega)$ takes an even simpler form, which again is independent of the phase accumulation angle θ_b:

$$R^{(\mathrm{mrr})}_{\mathrm{herald}}(\omega) = \frac{\alpha_b^2\,\big(1-\rho_b^2\big)}{\big(1-|r_b|^2-\alpha_b^2\,\rho_b^2\big)}. \tag{47}$$

In Fig. 6 we plot $R^{(\mathrm{mrr})}_{\mathrm{herald}}(\omega)$ with $r = 10^{-5}$ for the internal propagation loss values (left) $\alpha = (0.99,0.95,0.90,0.85,0.80,0.75)$ and for the operationally relevant (for $\alpha \leqslant 0.99$) internal propagation loss values $\alpha = (0.999,0.99,0.98,0.97,0.95,0.95,0.90)$. Even for high values of loss ($\alpha \leqslant 0.95$), the heralding efficiencies remain relatively high over a broad range of the coupling parameter ρ_b.

V. SUMMARY AND DISCUSSION

In this work we have investigated photon-pair generation via SPDC and SFWM in a single-bus microring resonator using a formalism that explicitly takes into account the round-trip circulation of the fields inside the cavity. We investigated the biphoton generation, coincidence-to-accidental, and heralding efficiency rates as a function of the bus-mrr coupling loss $\rho = e^{-\gamma\,T/2}$ and internal propagation loss $\alpha = e^{-\gamma'\,T/2}$ at rates γ and γ', respectively [with T being the round-trip circulation time of the field(s)]. We showed in Eq. (21) that the signal-idler output fields $\vec{a}_{\mathrm{out}}(\omega)$ can be expressed in terms of the input fields $\vec{a}_{\mathrm{in}}(\omega)$ and quantum noise operators $\vec{f}(\omega)$ as $\vec{a}_{\mathrm{out}}(\omega) = G(\omega)\,\vec{a}_{\mathrm{in}}(\omega) + H(\omega)\,\vec{f}(\omega)$. The matrix $G(\omega)$ encodes the classical phenomenological loss (for $\alpha < 1$) [25,26] of the mrr, while the matrix $H(\omega)$ incorporates the coupling and internal propagation loss due to the quantum Langevin noise fields $\vec{f}(\omega)$ required to preserve the unitarity of the composite system (signal-idler) and environment (noise) structure. While the standard Langevin input-output formalism often used in the literature is valid in the high-cavity-Q limit ($\rho \approx 1 - \gamma\,T/2 \to 1$, $\omega\,T \ll 1$) and near cavity resonances, the formulation developed here is valid

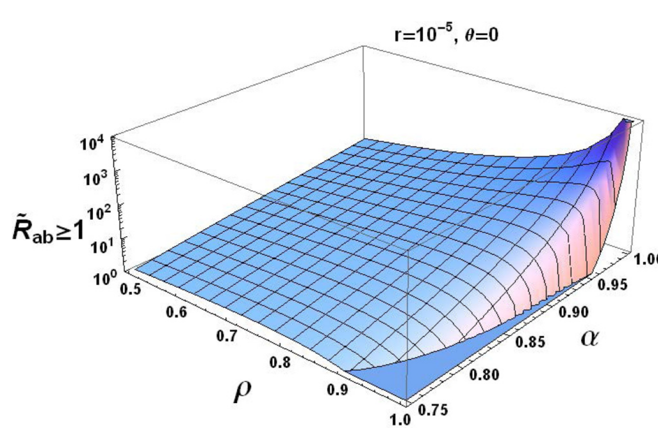

FIG. 3. $\tilde{R}^{(\mathrm{mrr})}_{ab} = |\psi^{(2)}_{ab,\mathrm{mrr}}(\omega)|^2$ for $r = 10^{-5}$ on mrr resonance $\theta = 0$ for $0.5 \leqslant \rho \leqslant 1.0$ and $0.75 \leqslant \alpha \leqslant 1.0$.

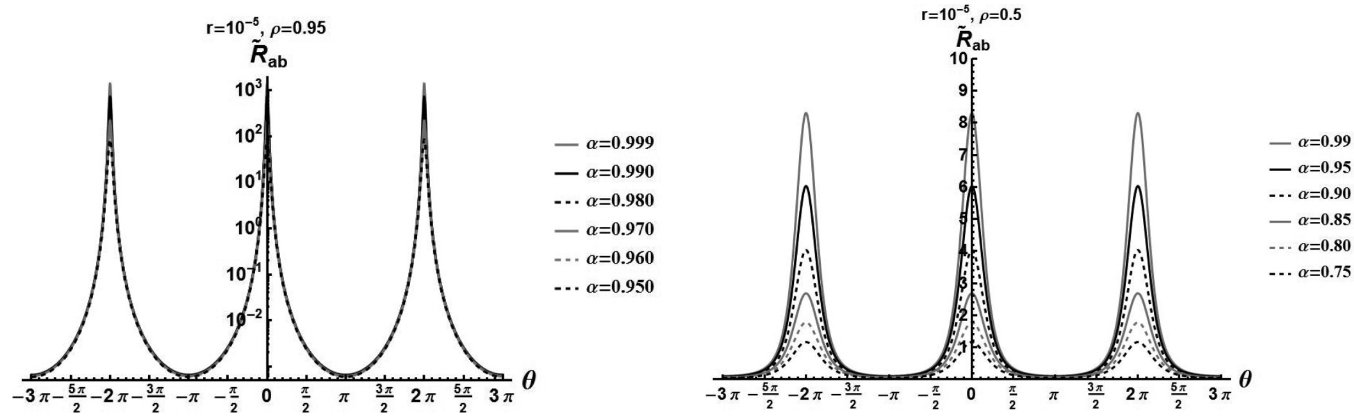

FIG. 4. $\tilde{R}_{ab}^{(\mathrm{mrr})} = |\psi_{ab,\mathrm{mrr}}^{(2)}(\omega)|^2$ for $r = 10^{-5}$ and (left) $\rho = 0.95$ and $\alpha = (0.999,0.99,0.98,0.97,0.95,0.95)$ and (right) $\rho = 0.5$ and $\alpha = (0.99,0.95,0.90,0.85,0.80,0.75)$.

throughout the free spectral range of the mrr. We explored values of the noise-field commutators which were uniquely derived by invoking the unitarity of the input and output fields (which required the latter's commutators to have the canonical form for free fields). For unequal signal and idler group velocities the cross-noise commutators were nonzero, while in general, the noise commutators contained pump-dependent contributions.

This work purposely concentrated on the weak (undepleted) pump limit and perfect phase matching in order to focus on the influence of the mrr coupling ρ and internal propagation loss α parameters. As indicated earlier in this work, nonzero phase matching can be straightforwardly included, which modifies the G and H matrices with multiplicative sinc function contributions. Similarly, this work included only the effects of dispersion through the mrr round-trip times $T_k = L/v_k$ for $k \in \{a,b\}$ for the signal (a) and idler (b) fields with possibly different group velocities v_k. Expansion of the frequency-dependent momentum vectors for the signal and idler fields about a central frequency could also be straightforwardly accommodated. A further logical extension of this work would be to consider the strong pump field regime in the spirit of the recent work by Vernon and Sipe [15] in which effects such as pump depletion and self-phase and cross-phase modulation could be taken into account.

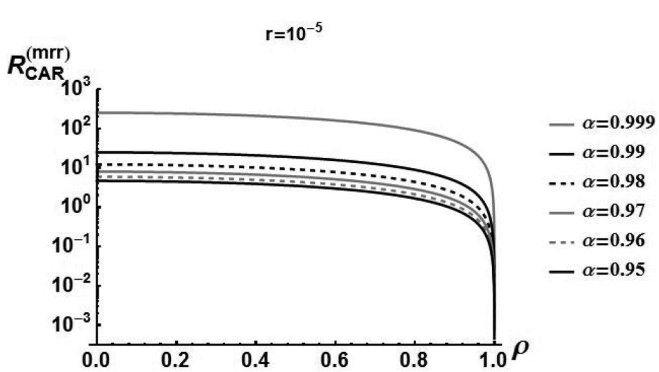

FIG. 5. Coincidence-to-accidental rate $R_{\mathrm{CAR}}^{(\mathrm{mrr})}(\omega)$ with $r = 10^{-5}$ for $\alpha = (0.999,0.99,0.98,0.97,0.96,0.95)$.

ACKNOWLEDGMENTS

P.M.A. would like to acknowledge support for this work from the Office of the Secretary of Defense (OSD) ARAP QSEP program and to thank J. Schneeloch and M. Fanto for helpful discussions. E.E.H. would like to acknowledge support for this work was provided by the Air Force Research Laboratory (AFRL) Visiting Faculty Research Program (VFRP) SUNY-IT Grant No. FA8750-16-3-6003. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of Air Force Research Laboratory.

APPENDIX: THE HIGH-CAVITY-Q LIMIT

1. The high-Q limit and reduction to the standard Langevin input-output formalism for a single mrr field

Both Raymer and McKinstrie [21] and Alsing *et al.* [22] considered the comparison of their formulations to the high-Q limit. Raymer and McKinstrie define the high-Q limit through the physical conditions (see [21], Sec. III): (i) the cross coupling τ_a is very small so that the cavity storage time is long, (ii) the cavity round-trip time T_a is small compared to the duration of the input-field pulse, i.e., $\omega\, T_a \ll 1$, and (iii) the input field is a narrow band and thus well contained within a single free spectral range of the mrr. By defining (now including internal loss)

$$\rho_a \equiv e^{-\gamma_a T_a/2} \approx 1 - \gamma_a T_a/2, \quad \tau_a = \sqrt{1-\rho_a^2} \approx \sqrt{\gamma_a T_a},$$
$$\alpha_a = e^{-\gamma_a' T_a/2} \approx 1 - \gamma_a' T_a/2, \quad e^{i\,\omega\, T_a} \approx 1 + i\,\omega\, T_a, \tag{A1}$$

we have from Eq. (8)

$$G_{\mathrm{out,in}}(\omega) = \left(\frac{\rho_a - \alpha_a\, e^{i\theta_a}}{1-\rho_a\,\alpha_a\, e^{i\theta_a}}\right) \underset{\mathrm{high}\,Q}{\longrightarrow} \frac{i\omega + (\gamma_a - \gamma_a')/2}{-i\omega + (\gamma_a + \gamma_a')/2}, \tag{A2a}$$

$$H_{\mathrm{out,in}}(\omega) \equiv |H_{\mathrm{out,in}}(\omega)| = \frac{|\tau_a|^2\left(1-\alpha_a^2\right)}{|1-\rho_a\,\alpha_a\, e^{i\,\theta_a}|^2} \underset{\mathrm{high}\,Q}{\longrightarrow} \frac{\sqrt{\gamma_a\,\gamma_a'}}{\omega^2 + [(\gamma_a+\gamma_a')/2]^2}, \tag{A2b}$$

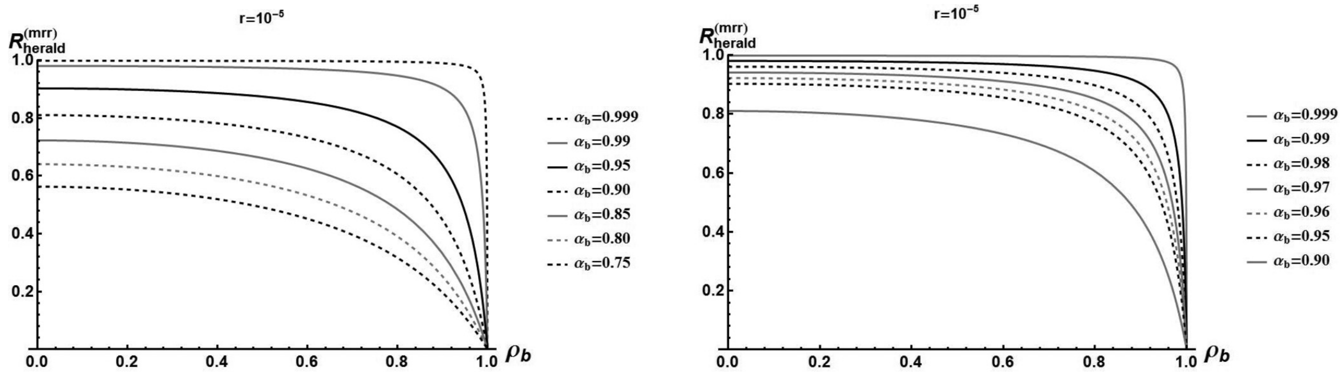

FIG. 6. Heralding efficiency $R^{(\mathrm{mrr})}_{\mathrm{herlad}}(\omega)$ with $r = 10^{-5}$ for (left) $\alpha = (0.999,0.99,0.95,0.90,0.85,0.80,0.75)$ and (right) the operationally relevant values $\alpha = (0.999,0.99,0.98,0.97,0.95,0.95,0.90)$.

where, without loss of generality, we have taken the phase of $H_{\mathrm{out,in}}(\omega)$ to be zero [or, equivalently, absorbed into the definition of the noise operator $f_a(\omega)$].

Under the assumptions of the high-cavity-Q limit one has $a(L_-,t) \approx a(0_+,t)$. Raymer and McKinstrie [21] showed that by defining the rescaled cavity field as $a(t) \equiv \sqrt{T_a}\, a(0_+,t)$ and considering the transfer function $G_{0_+,\mathrm{in}}(t)$ in the time domain, the equation of motion (without noise) becomes $\partial_t\, a(t) = -\frac{1}{2}\gamma'_a\, a(t) + \sqrt{\gamma'_a}\, a_{\mathrm{int}}(t)$. Additionally, the output boundary condition (2b), in the limit $\rho_a \to 1$, $\tau_a \to \sqrt{\gamma'_a\, T_a}$, becomes $a_{\mathrm{out}}(t) = \sqrt{\gamma'_a}\, a(t) - a_{\mathrm{in}}$, which is the standard Langevin boundary condition $a_{\mathrm{in}} + a_{\mathrm{out}} = \sqrt{\gamma_a}\, a$ [17,20].

2. The high-cavity-Q limit of $G(\omega)$ and $H(\omega)$

The high-cavity-Q limit is defined by (see Raymer and McKinstrie [21]) $\rho_k \equiv e^{-\gamma_k T_k/2} \approx 1 - \gamma_k\, T_k/2$ for $k \in \{a,b\}$, which implies $\tau_k^2 \approx \gamma_k\, T_k$, and by taking the limit $\omega\, T_k \ll 1$, $e^{i\theta_k} \approx 1 + i\,\omega\, T_k$. If we further assume that the internal propagation loss is small, we can also take $\alpha_k \approx 1 - \gamma'_k\, T_k/2$. We then have $S_k = (1 - \rho_k\alpha_k\, e^{i\,\theta_k})^{-1} \approx [(s + \Gamma_k/2)\, T_k]^{-1}$, a complex Lorentzian line-shape factor, where for simplicity we have defined $s \equiv -i\omega$ (s can be considered a Laplace-transform-solution variable) and have defined the total decay rate $\Gamma_k = \gamma_k + \gamma'_k$. Let us also further define $\Delta_k = \gamma_k - \gamma'_k$ as the difference between the coupling and internal propagation losses. Then, $e^{i\xi_k L} - \rho_k \to (-s + \Delta_k/2)\, T_k$, and $D(s) \approx [(s + \Gamma_a/2)\,(s + \Gamma_b/2) - |g\alpha_p|^2 t] T_a\, T_b \equiv \tilde{D}(s)\, T_a\, T_b$. We then obtain from Eq. (22b)

$$G(\omega) \underset{\text{high } Q}{\longrightarrow} \frac{1}{\tilde{D}(s)} \begin{pmatrix} (-s + \Delta_a/2)\,(s + \Gamma_b/2) + |g\alpha_p|^2\,[1 - \gamma_a\, T_a/2] & i\, g\, \alpha_p\, \sqrt{\gamma_a\, \gamma_b}\, \sqrt{T_b/T_a}\, [1 - (s + \gamma'_b/2)\, T_b] \\ -i\, g\, \alpha_p^*\, \sqrt{\gamma_a\, \gamma_b}\, \sqrt{T_a/T_b}\, [1 - (s + \gamma'_a/2)\, T_a] & (-s + \Delta_b/2)\,(s + \Gamma_a/2) + |g\alpha_p|^2\,[1 - \gamma_b\, T_b/2] \end{pmatrix}. \tag{A3}$$

For the noise terms, let us redefine the noise operators as $\tilde{f}_k(\omega) \equiv (1 - \alpha_k^2)^{-1/2}\, f_k(\omega)$ for $k \in \{a,b\}$ and, equivalently, the values of the commutators as $[f_k(\omega), f_k^\dagger(\omega)] = C_{kk}(\omega) \equiv (1 - \alpha_k^2)\, \tilde{C}_{kk}(\omega)$ and $D_{ab}(\omega) \equiv (1 - \alpha_a^2)^{1/2}\, (1 - \alpha_b^2)^{1/2}\, \tilde{D}_{ab}(\omega)$, so that $[\tilde{f}_k(\omega), \tilde{f}_k(\omega')] = \tilde{C}_{kk}(\omega)\, \delta(\omega - \omega')$ and $[\tilde{f}_a(\omega), \tilde{f}_b(\omega')] = \tilde{D}_{ab}(\omega)\, \delta(\omega - \omega')$. Then

$$\vec{a}_{\mathrm{out}} \equiv G(\omega)\, \vec{a}_{\mathrm{in}} + \tilde{H}(\omega)\, \vec{\tilde{f}}(\omega), \quad \tilde{H}(\omega) = H(\omega)\, \Lambda_\alpha, \quad \Lambda_\alpha \equiv \begin{pmatrix} \left(1 - \alpha_a^2\right)^{1/2} & 0 \\ 0 & \left(1 - \alpha_b^2\right)^{1/2} \end{pmatrix}. \tag{A4}$$

From Eqs. (24) and (A4) in the high-Q limit, where $(1 - \alpha_k^2)^{1/2} \to \sqrt{\gamma'_k\, T_k}$, we then have

$$\tilde{H}(\omega) \underset{\text{high}Q}{\longrightarrow} \frac{1}{\tilde{D}(s)} \begin{pmatrix} \sqrt{\gamma_a\, \gamma'_a}\, (s + \Gamma_b/2) & i\, g\, \alpha_p\, \sqrt{\gamma_a\, \gamma'_b}\, \sqrt{T_b/T_a} \\ -i\, g\, \alpha_p^*\, \sqrt{\gamma'_a\, \gamma_b}\, \sqrt{T_a/T_b} & \sqrt{\gamma_b\, \gamma'_b}\, (s + \Gamma_a/2) \end{pmatrix}. \tag{A5}$$

Except for the extra correction factors indicated in the square brackets in $G(\omega)$ (which can be safely approximated as unity to lowest order in $|g\, \alpha_p|$), these matrices are the same expressions as obtained by Tsang [see (4.11) in [10]] using the standard Langevin input-output procedure and assuming $T_a = T_b = L/v$.

Note further that to zeroth order in $|g\alpha_p|$ we have $D^{-1} \approx S_a\, S_b/(T_a\, T_b) \to [(s + \Gamma_a/2)^{-1}\, (s + \Gamma_b/2)\, T_a\, T_b]^{-1}$, and thus, $G(\omega)$ reduces in first order in $|g\alpha_p|$ to

$$G(\omega) \underset{\substack{\text{high}Q \\ O(|g\,\alpha_p|)}}{\longrightarrow} \begin{pmatrix} \frac{-s+\Delta_a/2}{s+\Gamma_a/2} & i\, g\, \alpha_p \left(\frac{\sqrt{\gamma_a}}{s+\Gamma_a/2}\right) \left(\frac{\sqrt{\gamma_b}}{s+\Gamma_b/2}\right) \\ -i\, g\, \alpha_p^* \left(\frac{\sqrt{\gamma_a}}{s+\Gamma_a/2}\right) \left(\frac{\sqrt{\gamma_b}}{s+\Gamma_b/2}\right) & \frac{-s+\Delta_b/2}{s+\Gamma_b/2} \end{pmatrix}, \tag{A6}$$

where we have also used $e^{i\xi_k L} \approx 1$. In this limit, the diagonal terms G_{kk}, which directly couple $(\vec{a}_{\text{out}})_k$ to $(\vec{a}_{\text{in}})_k$ for $k \in \{a,b\}$, have same frequency-dependent shifts of the output signal-idler fields relative to the internal signal-idler fields as given by the conventional Langevin approach [17,20,36]. The lower-order (in $|g\,\alpha_p|$) off-diagonal terms G_{ab} and G_{ba} contain the product of Lorentzian line-shape factors $\sqrt{\gamma_k}/(s+\Gamma_k/2)$ relating the output signal-idler fields to the opposite idler-signal fields inside the cavity. Similarly, for $\tilde{H}(\omega)$ we have

$$\tilde{H}(\omega) \underset{\substack{\text{high}Q\\ O(|g\,\alpha_p|)}}{\longrightarrow} \begin{pmatrix} \frac{\sqrt{\gamma_a\,\gamma_a'}}{s+\Gamma_a/2} & \frac{i\,g\,\alpha_p\,\sqrt{\gamma_a\,\gamma_b'}}{(s+\Gamma_a/2)\,(s+\Gamma_b/2)} \\ \frac{i\,g\,\alpha_p^*\,\sqrt{\gamma_a'\,\gamma_b}}{(s+\Gamma_a/2)\,(s+\Gamma_b/2)} & \frac{\sqrt{\gamma_b\,\gamma_b'}}{s+\Gamma_b/2} \end{pmatrix}. \tag{A7}$$

3. Biphoton generation rate $R_{ab}(\omega)$ in the high-Q limit

To make a connection with other works, let us more closely examine the two-photon generation rate given by $R_{ab}(\omega) = |r_{ab}(\omega)|^2\,|\psi^{(2)}_{ab}(\omega)|^2$ in the high-cavity-Q limit. Note that from Eq. (23a) we can write $D(s)$ in Eqs. (39a) and (40a) as $D(s) = (1-\rho_a\,\alpha_a\,e^{-s\,T_a})\,(1-\rho_b\,\alpha_b\,e^{-s\,T_b}) - |g\,\alpha_p|^2\,T_a\,T_b$, where $s=-i\,\omega$. The pole structure of $D(s)$ is obtained by the roots $s_\pm$ of $D(s_\pm)=0$. In general this is a transcendental equation which must be solved numerically. If we approximate $e^{-sT_k}\approx 1-sT_k$, we obtain a quadratic equation in s with poles $s_\pm$, and

$$D(s) \approx (s-s_+)\,(s-s_-),$$

$$s_\pm = \frac{1}{2}\left(\frac{y_a}{x_a}+\frac{y_b}{x_b}\right) \pm \sqrt{\left[\frac{1}{2}\left(\frac{y_a}{x_a}-\frac{y_b}{x_b}\right)\right]^2 + |g\,\alpha_p|^2}, \quad x_k=\rho_k\,\alpha_k, \quad y_k = \frac{(1-\rho_k\,\alpha_k)}{T_k} \tag{A8a}$$

$$\underset{\text{high}Q}{\longrightarrow} \left(\frac{\Gamma_a+\Gamma_b}{4}\right) \pm \sqrt{\left(\frac{\Gamma_a-\Gamma_b}{4}\right)^2 + |g\,\alpha_p|^2} \equiv \pi_\pm \tag{A8b}$$

$$\underset{\substack{\text{high}Q\\ O(|g\,\alpha_p|)}}{\longrightarrow} \begin{cases} -\Gamma_a/2 \\ -\Gamma_b/2 \end{cases} \text{for} \quad \Gamma_b > \Gamma_a, \tag{A8c}$$

where $\pi_\pm$ are the poles of $D(s)$ as computed by Tsang [10] using a standard Langevin input-output calculation. Then, the two-photon generation rate becomes to lowest order in $|g\,\alpha_p|^2$

$$R_{ab}(\omega) = |r_{ab}(\omega)|^2\left|\psi^{(2)}_{ab}(\omega)\right|^2 \approx |r_{ab}(\omega)|^2\left|G^{(L)*}_{aa}(\omega)\,G^{(L)}_{bb}(\omega)\right|^2$$

$$\underset{D(s)\approx(s-s_+)\,(s-s_-)}{\longrightarrow} |g\,\alpha_p|^2\,\frac{\left(\frac{\tau_a}{\sqrt{T_a}}\right)^2\left|\frac{1-\rho_b\,e^{i\xi_b L}}{T_b}\right|^2|e^{i\xi_a L}|^2}{(\omega^2+s_+^2)\,(\omega^2+s_-^2)}\,\frac{\left(\frac{\tau_b}{\sqrt{T_b}}\right)^2\left|\frac{1-\rho_a\,e^{i\xi_b L}}{T_a}\right|^2|e^{i\xi_b L}|^2}{(\omega^2+s_+^2)\,(\omega^2+s_-^2)} \tag{A9a}$$

$$\underset{\text{high }Q}{\longrightarrow} |g\,\alpha_p|^2\,\frac{\gamma_a\,[\omega^2+(\Gamma_b/2)^2]}{(\omega^2+s_+^2)\,(\omega^2+s_-^2)}\,\frac{\gamma_b\,[\omega^2+(\Gamma_a/2)^2]}{(\omega^2+s_+^2)\,(\omega^2+s_-^2)} \tag{A9b}$$

$$\underset{\substack{\text{high }Q\\ O(|g\,\alpha_p|^2)}}{\longrightarrow} |g\,\alpha_p|^2\,\frac{\gamma_a}{[\omega^2+(\Gamma_a/2)^2]}\,\frac{\gamma_b}{[\omega^2+(\Gamma_b/2)^2]}, \tag{A9c}$$

where in the third line we have used $|e^{i\xi_k L}|^2\approx 1$ and in the fourth line we have used Eq. (A8c). The above expressions generalize the two-photon rate $R_{ab}(\omega) = |g\,\alpha_p|^2\,\gamma_a\,\gamma_b/[(\omega^2+s_+^2)\,(\omega^2+s_-^2)]$ computed by Tsang [10], which to $O(|g\,\alpha_p|^2)$ agrees with Eq. (A9b). The last line, Eq. (A9c), is the form computed by Scholz using the (complex) Lorentzian modified form $\sqrt{\gamma_a}\,a(\omega)/(s+\Gamma_a/2)$ and $\sqrt{\gamma_b}\,b^\dagger(\omega)/(-s+\Gamma_b/2)$ for the field operators inside the mrr. Expression (A9a), quadratic in the poles $s_\pm$, more fully takes into account the effect of the the field circulation factors $S_k = 1/[1-\rho_k\,e^{i\xi_b L}]$ on the two-photon generation rate.

[1] J. S. Levy, A. Gondarenko, M. A. Foster, A. C. Turner-Foster, and A. L. Gaeta, Nat. Photonics **4**, 37 (2010).

[2] S. Azzini, D. Grassani, M. J. Strain, M. Sorel, L. G. Helt, J. E. Sipe, M. Liscidini, M. Galli, and D. Bajoni, Opt. Express **20**, 23100 (2012).

[3] C. M. Gentry, J. Shainline, M. Wade, M. Stevens, S. Dyer, X. Zeng, F. Pavanello, T. Gerrits, S. Nam, R. Mirin, and M. Popovic, Optica **2**, 1065 (2015).

[4] S. F. Preble, M. L. Fanto, J. A. Steidle, C. C. Tison, G. A. Howland, Z. Wang, and P. M. Alsing, Phys. Rev. Appl. **4**, 021001 (2015).

[5] C. C. Tison, J. A. Steidle, M. L. Fanto, Z. Wang, N. A. Mogent, A. Rizzo, S. F. Preble, and P. M. Alsing, arXiv:1703.08368.

[6] Z. Vernon, M. Menotti, C. C. Tison, J. A. Steidle, M. L. Fanto, P. M. Thomas, S. F. Preble, A. M. Smith, P. M. Alsing, M. Liscidini, and J. E. Sipe, Opt. Lett. **42**, 3638 (2017).

[7] M. Scholz, L. Koch, and O. Benson, Opt. Commun. **282**, 3518 (2009).
[8] J.-T. Shen and S. Fan, Phys. Rev. A **79**, 023837 (2009).
[9] J.-T. Shen and S. Fan, Phys. Rev. A **79**, 023838 (2009).
[10] M. Tsang, Phys. Rev. A **84**, 043845 (2011).
[11] R. M. Camacho, Opt. Express **20**, 21977 (2012).
[12] L. Helt, M. Liscindi, and J. E. Sipe, J. Opt. Soc. Am. B **29**, 2129 (2012).
[13] Z. Vernon and J. E. Sipe, Phys. Rev. A **91**, 053802 (2015).
[14] Z. Vernon, C. M. Liscidini, and J. E. Sipe, Opt. Lett. **41**, 788 (2015).
[15] Z. Vernon and J. E. Sipe, Phys. Rev. A **92**, 033840 (2015).
[16] M. J. Collett and C. W. Gardiner, Phys. Rev. A **30**, 1386 (1984).
[17] D. F. Walls and G. J. Milburn, *Quantum Optics* (Springer, New York, 1994), Chap. 7.
[18] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, 1995), Chaps. 17.2, 17.4.
[19] M. O. Scully and M. S. Zubairy, *Quantum Optics* (Cambridge University Press, Cambridge, 1997), Chap. 9.
[20] M. Orszag, *Quantum Optics* (Springer, New York, 2000), Chap. 14.3-4.
[21] M. G. Raymer and C. J. McKinstrie, Phys. Rev. A **88**, 043819 (2013).
[22] P. M. Alsing, E. E. Hach III, C. C. Tison, and A. M. Smith, Phys. Rev. A **95**, 053828 (2017).
[23] S. M. Barnett, C. R. Gilson, B. Huttner, and N. Imoto, Phys. Rev. Lett. **77**, 1739 (1996).
[24] S. Huang and G. S. Agarwal, Opt. Express **22**, 020936 (2014).
[25] A. Yariv, Electron. Lett. **36**, 321 (2000).
[26] D. G. Rabus, *Integrated Ring Resonators* (Springer, Berlin, 2007).
[27] E. E. Hach III, S. F. Preble, A. W. Elshaari, P. M. Alsing, and M. L. Fanto, Phys. Rev. A **89**, 043805 (2014).
[28] S. M. Barnett, J. Jeffers, A. Gatti, and R. Loudon, Phys. Rev. A **57**, 2134 (1998).
[29] R. Loudon, *Quantum Theory of Light*, 3rd ed. (Oxford University Press, New York, 2000), Chap. 7.5.
[30] Note that ω is the offset from the central pump frequency ω_p, so that $b^\dagger(\omega) \equiv [b(-\omega)]^\dagger$ (see [20]).
[31] This corresponds to dropping higher-order terms describing self-phase and cross-phase pump modulations terms (see [13,15]).
[32] J. Schneeloch, S. H. Knarr, and P. M. Alsing (unpublished).
[33] S. Barzanjeh, S. Guha, C. Weedbrook, D. Vitali, J. H. Shapiro, and S. Pirandola, Phys. Rev. Lett. **114**, 080503 (2015).
[34] J. Schneeloch and J. C. Howell, J. Opt. **18**, 053501 (2016).
[35] J. Chen, Z. H. Levine, J. Fan, and A. L. Migdall, Opt. Express **19**, 1470 (2011).
[36] H. A. Haus, *Waves and Fields in Optoelectronics* (Prentice-Hall, Englewood Cliffs, NJ, 1984), Chap. 7.