Speeding up adiabatic passage by adding Lyapunov control

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We propose a scheme to speed up adiabatic passage by using Lyapunov control theory. This is a good choice to solve the problem that may emerge in Berry's transitionless quantum driving [M. V. Berry, J. Phys. A **42**, 365303 (2009)]. That is, the extra couplings in the counterdiabatic driving Hamiltonian can be avoided by choosing the available control Hamiltonian in an actual physical system. As examples, we shorten the evolution time of adiabatic population transfer in a three-level system and the entanglement generation in a cavity quantum electrodynamics system. Moreover, the occupation of an intermediate state can be sharply suppressed by properly choosing the control Hamiltonian in the three-level system. The scheme can also be generalized to a complex system where the exact expressions of adiabatic eigenstates are difficult to obtain.

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I. INTRODUCTION

Reliable population transfer and entanglement generation have various applications in quantum information processing (QIP) [1–5]. In the context of QIP, quantum control theory has generated increasing interest in the last few years due to its important role in the development of quantum chemistry and quantum optics as well as potential applications in metrology [6,7], communications [8,9], and other technologies [10–13].

To accurately control a quantum system to evolve as one expects, reliable techniques including optimal control [14,15], adiabatic control [16,17], and measurement-based control [18,19] have been widely used. Among them, adiabatic techniques which are famous for their robustness against the noises of driving fields have been holding an irreplaceable position in the quantum control field. Two of the most important applications of adiabatic techniques are stimulated Raman adiabatic passage (STIRAP) [20] and fractional stimulated Raman adiabatic passage (f-STIRAP) [21]. The STIRAP (f-STIRAP) technique was first demonstrated with sodium dimer molecular beams [22], and applied in many contexts both theoretically and experimentally [23–27] from then on. However, the drawback of STIRAP that the system evolution needs to satisfy the adiabatic condition greatly reduces the evolution speed and causes increased decoherence effect. It has been shown that the fidelities of the desired states are very sensitive to the dephasing due to the long evolution time in STIRAP [28,29]. Therefore, from the view of decoherence, accelerating the dynamics towards the perfect final outcome is a good choice and perhaps is the most reasonable way to fight against the decoherence. In order to speed up adiabatic passage, many methods have been proposed, for instance, the transitionless quantum driving algorithm [30–42].

Here we would like to introduce another reliable quantum control technique, the Lyapunov-based open-loop control, which has been extensively studied recently [43–57] due to its simplicity and intuitive nature in the design of control fields. The key point in Lyapunov control is to construct a Lyapunov function and then to design time-dependent control fields. With these control fields, the problems of driving a quantum system to a target state or realizing some specific operations can be successfully solved. For instance, the Lyapunov control method could be applied to generate entanglement between two distant two-level atoms in cavities connected by an optical fiber [58]. In addition, Wang *et al.* [59] proposed two different designs of the Lyapunov function and applied them to adiabatic quantum computation to improve adiabatic evolution.

In this paper, we propose a scheme to speed up adiabatic passage by using Lyapunov control to drive the nonadiabatic transition state back to the target instantaneous eigenstate. The scheme can effectively avoid the requirement of extra couplings in the transitionless quantum driving algorithm, and it can be easily generalized to complex quantum systems where the exact expression of the counterdiabatic Hamiltonian is difficult to obtain. In our concrete illustrations, to ensure high population such as $P_f \geqslant 0.99$, the laser pulse width T required in STIRAP methods is much larger than that in adding Lyapunov control, which means that the adiabatic condition can be effectively weakened by adding Lyapunov control.

The rest of this paper is organized as follows. In Sec. II, we present the Lyapunov control theory in the adiabatic passage, then elucidate the physical mechanism of this control process. In Sec. III, we use Lyapunov control to speed up the population transfer in a three-level system and the entanglement generation in a cavity QED system. Conclusions are presented in Sec. IV.

II. LYAPUNOV CONTROL THEORY IN ADIABATIC PASSAGE

Consider a N-dimensional nondegenerate quantum system with Hamiltonian $H_0(t)$, where the instantaneous eigenstates $|\phi_n(t)\rangle$ and corresponding eigenvalues $E_n(t)\,(n=1,2,\ldots,N)$ satisfy

$$H_0(t)|\phi_n(t)\rangle = E_n(t)|\phi_n(t)\rangle. \tag{1}$$

In the adiabatic approximation, the system always stays in instantaneous eigenstate $|\phi_T(t)\rangle$ if the initial state is $|\phi_T(0)\rangle$.

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However, when the adiabatic condition cannot be satisfied very well (e.g., shortening the evolution time), nonadiabatic transition between instantaneous eigenstates occurs. As a result, the system gradually deviates from the instantaneous eigenstate $|\phi_T(t)\rangle$. In order to overcome this drawback, Berry proposed the transitionless quantum driving algorithm to eliminate the nonadiabatic couplings by adding a counterdiabatic driving Hamiltonian [31]. Nevertheless, the counterdiabatic driving Hamiltonian often requires extra couplings which are unfeasible or even nonexistent in actual physical systems [42]. Furthermore, this algorithm is very difficult to extend to complex quantum systems. In the following, we demonstrate that the drawbacks existing in the transitionless quantum driving algorithm can be effectively overcome when applying Lyapunov control in adiabatic passage.

To keep the system state evolving along with the instantaneous eigenstate $|\phi_T(t)\rangle$, an additional Hamiltonian $H_c(t) = \sum_{k=1}^K f_k(t)H_k$ needs to be added in the original system, where H_k $(k = 1,\dots,K)$ are control Hamiltonians that are available in physical systems and $f_k(t)$ are time-varying real functions, representing the value of control fields. Then, the evolution of a quantum system is governed by the following Schrödinger equation ($\hbar = 1$):

$$i|\dot{\psi}(t)\rangle = [H_0(t) + H_c(t)]|\psi(t)\rangle, \tag{2}$$

where $|\psi(t)\rangle$ is the system state. In order to design the shape of control fields $f_k(t)$ that impose $|\psi(t)\rangle$ approaching to the instantaneous eigenstate $|\phi_T(t)\rangle$, one first should select a Lyapunov function $V(t)$. Here we consider the following form of the Lyapunov function, which is based on the so-called Hilbert-Schmidt (or, equivalently, on the trace) distance [60]:

$$V(t) = 1 - |\langle\phi_T(t)|\psi(t)\rangle|^2. \tag{3}$$

From this Lyapunov function, we find that the value of $V(t)$ becomes smaller when the system state is closer to the instantaneous eigenstate $|\phi_T(t)\rangle$. In particular, the value of $V(t)$ is minimum when the system state entirely stays in the instantaneous eigenstate $|\phi_T(t)\rangle$. By calculating the time derivative of $V(t)$, we have

$$\begin{aligned}\dot{V}(t) = {} & 2\sum_{k=1}^K f_k(t)\mathrm{Im}[\langle\phi_T(t)|\psi(t)\rangle\langle\psi(t)|H_k|\phi_T(t)\rangle] \\ & - 2\mathrm{Re}[\langle\dot{\phi}_T(t)|\psi(t)\rangle\langle\psi(t)|\phi_T(t)\rangle],\end{aligned} \tag{4}$$

where $\mathrm{Im}[\cdot]$ and $\mathrm{Re}[\cdot]$ represent the imaginary and real part of the argument, respectively. In order to satisfy the condition of $\dot{V}(t) \leqslant 0$, the natural choices of control fields $f_k(t)$ are

$$\begin{aligned} f_k(t) &= -A_k\mathrm{Im}[\langle\phi_T(t)|\psi(t)\rangle\langle\psi(t)|H_k|\phi_T(t)\rangle], \quad k \neq k_0, \\ f_{k_0}(t) &= \frac{\mathrm{Re}[\langle\dot{\phi}_T(t)|\psi(t)\rangle\langle\psi(t)|\phi_T(t)\rangle]}{\mathrm{Im}[\langle\phi_T(t)|\psi(t)\rangle\langle\psi(t)|H_{k_0}|\phi_T(t)\rangle]}, \quad k = k_0, \end{aligned} \tag{5}$$

where the positive number A_k is used to adjust the amplitude of control fields. k_0 is specified to satisfy the condition $\mathrm{Im}[\langle\phi_T(t)|\psi(t)\rangle\langle\psi(t)|H_{k_0}|\phi_T(t)\rangle] \neq 0$, and the control field $f_{k_0}(t)$ is used to eliminate the uncontrollable term $-2\mathrm{Re}[\langle\dot{\phi}_T(t)|\psi(t)\rangle\langle\psi(t)|\phi_T(t)\rangle]$. Under the domination of this designated control field $f_k(t)$, the system state $|\psi(t)\rangle$ will converge to the LaSalle invariant set, for which $\dot{V}$ vanishes [61].

In order to get a more clear insight into the physical process of why the system state can evolve along with the instantaneous eigenstate $|\phi_T(t)\rangle$ in the adiabatic passage by adding Lyapunov control, we turn to the "adiabatic frame." To be specific, by performing a unitary transformation with the unitary operator $U(t) = \sum_{n=1}^N |\phi_n(t)\rangle\langle n|$, the system Hamiltonian $H(t) = H_0(t) + \sum_k f_k(t)H_k$ in the adiabatic frame reads

$$\begin{aligned}\mathcal{H}(t) &= U^\dagger(t)H(t)U(t) - iU^\dagger(t)\dot{U}(t) \\ &= \sum_{n=1}^N E_n(t)|n\rangle\langle n| - i\sum_{n,m=1}^N \langle\phi_m(t)|\dot{\phi}_n(t)\rangle|m\rangle\langle n| \\ &\quad + \sum_{m,n=1}^N\sum_{k=1}^K f_k(t)\langle\phi_m(t)|H_k|\phi_n(t)\rangle|m\rangle\langle n|,\end{aligned} \tag{6}$$

where $\{|n\rangle\}$ are the basis states satisfying $\sum_n |n\rangle\langle n| = 1$ and $\langle m|n\rangle = \delta_{mn}$. Suppose that the system state $|\psi(t)\rangle$ can be written as $|\psi(t)\rangle = \sum_{n=1}^N a_n(t)|\phi_n(t)\rangle$, where $a_n(t)$ are the probability amplitudes of instantaneous eigenstate $|\phi_n(t)\rangle$. According to the Schrödinger equation, we find

$$i\frac{\partial}{\partial t}\begin{pmatrix} a_1(t) \\ a_2(t) \\ \vdots \\ a_N(t)\end{pmatrix} = \begin{pmatrix} \mathcal{H}_{11} & \mathcal{H}_{12} & \cdots & \mathcal{H}_{1N} \\ \mathcal{H}_{21} & \mathcal{H}_{22} & \cdots & \mathcal{H}_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{H}_{N1} & \mathcal{H}_{N2} & \cdots & \mathcal{H}_{NN}\end{pmatrix}\begin{pmatrix} a_1(t) \\ a_2(t) \\ \vdots \\ a_N(t)\end{pmatrix}, \tag{7}$$

where the matrix elements $\mathcal{H}_{nn} = -i\langle\phi_n(t)|\dot{\phi}_n(t)\rangle + E_n(t) + \sum_k f_k(t)E_{nk}(t)$ with $E_{nk}(t) = \langle\phi_n(t)|H_k|\phi_n(t)\rangle$, $\mathcal{H}_{mn} = -i\langle\phi_m(t)|\dot{\phi}_n(t)\rangle + \sum_k f_k(t)\mathcal{H}_k^{mn}(t)$ with $\mathcal{H}_k^{mn}(t) = \langle\phi_m(t)|H_k|\phi_n(t)\rangle$ $(m,n = 1,2,\dots,N; m \neq n)$. Note that $\mathcal{H}_k(t)$ represent the control Hamiltonian in the adiabatic frame, and the control fields of Eq. (5) accordingly read

$$f_k(t) = -A_k\mathrm{Im}\left[\sum_{n=1}^N a_T(t)a_n^*(t)\langle\phi_n(t)|H_k|\phi_T(t)\rangle\right]. \tag{8}$$

One can see that Eqs. (7) and (8) represent a nonlinear autonomous dynamical system. This system will necessarily converge to an invariant set defined by $E = \{|\psi(t)\rangle : \dot{V}(t) = 0\}$ according to LaSalle's invariant principle [61], which is also equivalent to $f_k(t) = 0$ in Eq. (8). In general, this invariant set may contain many instantaneous eigenstates, depending on the choice of control Hamiltonian H_k. If the control Hamiltonian H_k is suitably chosen to satisfy $\langle\phi_m(t)|H_k|\phi_T(t)\rangle \neq 0$, which means that the off-diagonal elements of the control Hamiltonian do not vanish in the adiabatic frame [i.e., there exists direct coupling between the instantaneous eigenstate $|\phi_T(t)\rangle$ and all the other eigenstates $|\phi_m(t)\rangle$], the LaSalle invariant set would only contain the instantaneous eigenstate $|\phi_T(t)\rangle$ [49,62,63]. As a result, the system state would be steered into the instantaneous eigenstate $|\phi_T(t)\rangle$ under Lyapunov control in adiabatic passage. In the following we will show some applications for this scheme.

III. APPLICATIONS

A. Speeding up population transfer in a three-level system

Before studying the problem of speeding up the population transfer in a three-level system, we first briefly review how to achieve population transfer in STIRAP [20]. Consider a three-level system with states $|1\rangle$, $|2\rangle$, and $|3\rangle$, where $|1\rangle$ and $|2\rangle$ are coupled by a pump laser with the Rabi frequency $\Omega_p(t)$, and $|2\rangle$ and $|3\rangle$ are coupled by a Stokes laser with the Rabi frequency $\Omega_s(t)$. Under one-photon resonance condition and the rotating-wave approximation, the system Hamiltonian can be written as ($\hbar = 1$)

$$H_0(t) = \tfrac{1}{2}[\Omega_p(t)|1\rangle\langle 2| + \Omega_s(t)|2\rangle\langle 3| + \text{H.c.}]. \tag{9}$$

The instantaneous eigenstates, with the corresponding eigenvalues $E_1(t) = 0$, $E_2(t) = \Omega(t)/2$, and $E_3(t) = -\Omega(t)/2$ $[\Omega(t) = \sqrt{\Omega_s^2(t) + \Omega_p^2(t)}]$, are

$$\begin{aligned} |\phi_1(t)\rangle &= \cos\theta|1\rangle - \sin\theta|3\rangle,\\ |\phi_2(t)\rangle &= \frac{1}{\sqrt{2}}(\sin\theta|1\rangle + |2\rangle + \cos\theta|3\rangle),\\ |\phi_3(t)\rangle &= \frac{1}{\sqrt{2}}(\sin\theta|1\rangle - |2\rangle + \cos\theta|3\rangle), \end{aligned} \tag{10}$$

where $\tan\theta = \Omega_p(t)/\Omega_s(t)$. To achieve perfect population transfer from $|1\rangle$ to $|3\rangle$ along the dark state $|\phi_1(t)\rangle$, we shall slowly change the value of θ in Eq. (10) from zero to $\frac{\pi}{2}$ to guarantee the system satisfies the adiabatic condition, and the boundary conditions read

$$\lim_{t\to 0}\frac{\Omega_p(t)}{\Omega_s(t)} = 0, \quad \lim_{t\to +\infty}\frac{\Omega_s(t)}{\Omega_p(t)} = 0. \tag{11}$$

This requires that the Stokes laser begins and ends earlier than the pump laser, which can be achieved by appropriate spatial displacement of the axes of cw lasers or a suitable time delay between the pump and Stokes lasers [20]. For instance, we choose the Stokes and pump pulses as follows:

$$\begin{aligned} \Omega_p(t) &= \Omega_p^0 e^{-(t+\tau)^2/T^2},\\ \Omega_s(t) &= \Omega_s^0 e^{-(t-\tau)^2/T^2}, \end{aligned} \tag{12}$$

where T is the pulse width and τ is the time delay. According to the adiabatic theorem [64], the adiabatic condition is satisfied well when $\Omega_{\text{eff}}T \gg 1$, where $\Omega_{\text{eff}} = \Omega(t)/2$ denotes the effective laser intensities. For a given T, the inequality can be satisfied well by increasing Ω_{eff}. This is exactly the point of interest for experiments, since it shows that the adiabatic limit can be achieved for strong enough pulses even if the pulse duration is short.

Now, we need to study the regions where the adiabatic condition is unsatisfied for STIRAP by setting the laser amplitude Ω_n^0 $(n = s,p)$ constant and varying the laser pulse width T of Eq. (12). In Fig. 1(a), we show the population $P_f = |\langle\psi(t)|\phi_1(t)\rangle|^2$ as a function of laser pulse width T by using STIRAP. For the blue dashed line in Fig. 1(a), the laser pulse width $T \leqslant 11$ μs, which cannot keep the population $P_f \geqslant 0.99$, is defined as the adiabatic condition unsatisfied regions. To see the STIRAP more clearly, in Fig. 1(b), we show the time evolution of instantaneous eigenstate $|\phi_1(t)\rangle$

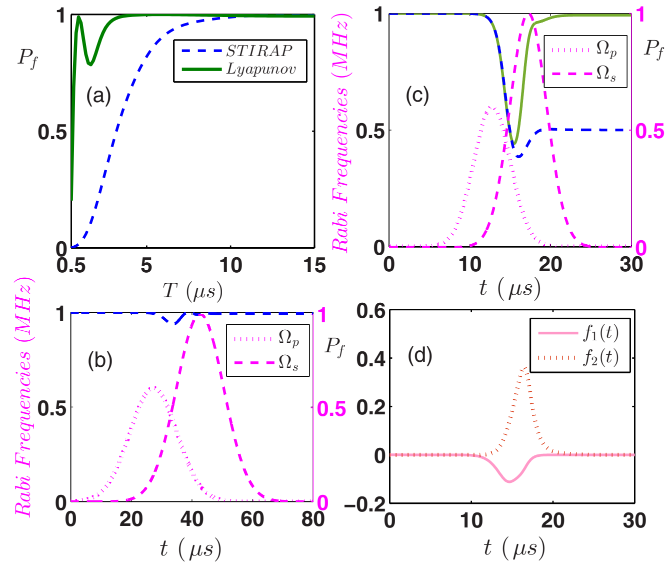

FIG. 1. (a) The population P_f as a function of laser pulse width T in f-STIRAP (blue dashed line) and adding Lyapunov control (green solid line), where $\tau = 0.7T$, $\Omega_s^0 = \frac{5}{3}\Omega_p^0 = 1$ MHz. (b) Time dependence of the Rabi frequencies for STIRAP with laser pulse width $T = 11$ μs (carmine lines, corresponding to the right y coordinate), and the time evolution of population P_f for STIRAP (blue dashed line, corresponding to the left y coordinate). (c) Time dependence of the Rabi frequencies for STIRAP with laser pulse width $T = 3.2$ μs (carmine lines, corresponding to the right y coordinate), and the time evolution of population P_f for STIRAP with the blue dashed line and for Lyapunov control with the green solid line (corresponding to the left y coordinate). (d) The time evolution of corresponding control fields for Lyapunov control with $A_1 = A_2 = 1$.

(the blue dashed line) and the laser pulse sequences of Eq. (12) (the magenta lines) when the pulse width $T = 11$ μs. One can easily find that the population P_f almost keeps unit during the evolution process, meaning that the adiabatic condition is satisfied well in this case. However, when the pulse width $T = 3.2$ μs, as shown in Fig. 1(c), the population P_f can only reach about 0.5 (the blue dashed line), demonstrating that the system cannot satisfy the adiabatic condition very well so that the system state seriously deviates from the instantaneous state $|\phi_1(t)\rangle$. In the following, we show that the adiabatic condition can be weakened by using Lyapunov control.

In order not to add extra interaction in the system, we still take the pump and Stokes lasers as the control Hamiltonians in Lyapunov control, i.e.,

$$H_1 = |1\rangle\langle 2| + |2\rangle\langle 1|, \quad H_2 = |3\rangle\langle 2| + |2\rangle\langle 3|, \tag{13}$$

where the control fields are given by Eq. (5) with $H_{k_0} = H_k$. From Fig. 1(a), the red solid line, we can see that the laser pulse width $T \simeq 3.2$ μs is enough to obtain a high-fidelity population transfer ($P_f \simeq 0.99$) in Lyapunov control strategy, which is greatly shortened as compared to that needed for STIRAP. To be more specific, in Fig. 1(c), the green solid line represents the time evolution of population P_f by adding Lyapunov control in STIRAP with the pulse width $T = 3.2$ μs, and the time varying control fields f_k are plotted in Fig. 1(d),

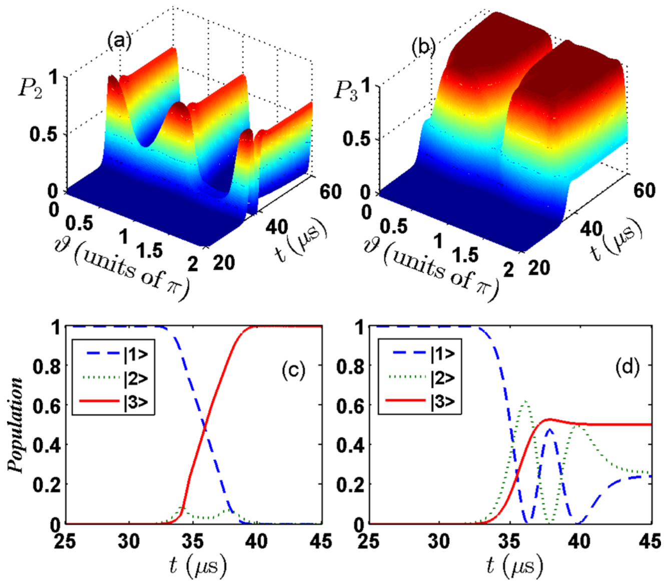

FIG. 2. The population (a) P_2 and (b) P_3 as a function of mixing angle ϑ and evolution time t. The time evolution of population of bare state $|k\rangle$ ($k = 1,2,3$) in STIRAP, (c) with Lyapunov control at mixing angle $\vartheta = 3\pi/2$ and (d) without Lyapunov control, where the parameters are $T = 3.2$ μs, $\tau = 0.7T$, $\Omega_s^0 = \frac{5}{3}\Omega_p^0 = 1$ MHz.

demonstrating that the system state is basically transferred to the instantaneous target state in this case.

In STIRAP, the occupation of intermediate state $|2\rangle$ is usually small. Naturally, one may care about the population of the intermediate state in the Lyapunov control case. Generally, the choice of control Hamiltonian dominates the effectiveness of the control. Note that the population P_f cannot always keep unit [see Fig. 1(c) green solid line] in the whole evolution process, which means the eigenstate $|\phi_1(t)\rangle$ is transferred to other eigenstates at some moment, leading to the nonzero population of intermediate state $|2\rangle$. Nevertheless, the intermediate state $|2\rangle$ can be suppressed by optimizing the control Hamiltonian. For instance, one can choose the common and feasible modulation function to optimize the control Hamiltonian. In Figs. 2(a) and 2(b), the trigonometric modulation functions are used to optimize the control Hamiltonian, i.e., $H_1' = \cos\vartheta(|1\rangle\langle 2| + |2\rangle\langle 1|)$ and $H_2' = \sin\vartheta(|3\rangle\langle 2| + |2\rangle\langle 3|)$. We can see from Figs. 2(a) and 2(b) that the mixing angle ϑ dominates the values of population P_l ($P_l = |\langle l|\psi(t)\rangle|^2, l = 2,3$). In most cases, ϑ can ensure the perfect population transfer from $|1\rangle$ to $|3\rangle$ [see Fig. 2(b)], and the optimal ϑ can be easily obtained according to Fig. 2(a). For instance, for $\vartheta = 3\pi/2$, a relatively small population of the intermediate state can be found when achieving the perfect population transfer from $|1\rangle$ to $|3\rangle$, which is plotted in Fig. 2(c). This not only shows that the single control Hamiltonian $H_1' = |1\rangle\langle 2| + |2\rangle\langle 1|$ can realize the perfect population transfer but also offers us the direction to suppress the population of the intermediate state. In addition, the single control Hamiltonian is also beneficial to experimental operations, indicating the flexibility choice of the control Hamiltonian in Lyapunov control. That is, it is not completely necessary to choose the control Hamiltonian as the same form as the counterdiabatic driving Hamiltonian in the transitionless quantum driving algorithm. For comparison, in Fig. 2(d) we plot the population transfer from $|1\rangle$ to $|3\rangle$ by using the STIRAP method with the same parameters, demonstrating the STIRAP method is invalid when the laser pulse width is shortened.

The physical mechanism to accelerate the adiabatic population transfer in the three-level system can be specified as follows. With the given control Hamiltonian in Eq. (13), the matrix elements in Eq. (7) read

$$\begin{aligned}
\mathcal{H}_{11} &= E_1(t),\\
\mathcal{H}_{22} &= E_2(t) + \sin\theta f_1(t) + \cos\theta f_2(t),\\
\mathcal{H}_{33} &= E_3(t) - \sin\theta f_1(t) - \cos\theta f_2(t),\\
\mathcal{H}_{12} &= (\mathcal{H}_{21})^* = \frac{-i\dot{\theta}}{\sqrt{2}} + \frac{\cos\theta f_1(t)}{\sqrt{2}} - \frac{\sin\theta f_2(t)}{\sqrt{2}},\\
\mathcal{H}_{13} &= (\mathcal{H}_{31})^* = \frac{-i\dot{\theta}}{\sqrt{2}} + \frac{\cos\theta f_1(t)}{\sqrt{2}} - \frac{\sin\theta f_2(t)}{\sqrt{2}},\\
\mathcal{H}_{23} &= (\mathcal{H}_{32})^* = 0.
\end{aligned} \tag{14}$$

From Eq. (14), we can see that the condition $\mathcal{H}_{11} \neq \mathcal{H}_{22} \neq \mathcal{H}_{33}$ is satisfied. Furthermore, the elements of the adiabatic control Hamiltonian read $\mathcal{H}_1^{12}(t) = \cos\theta/\sqrt{2} \neq 0$, $\mathcal{H}_1^{13}(t) = -\cos\theta/\sqrt{2} \neq 0$, $\mathcal{H}_2^{12}(t) = -\sin\theta/\sqrt{2} \neq 0$, and $\mathcal{H}_2^{13}(t) = \sin\theta/\sqrt{2} \neq 0$, which means that the direct couplings between the target eigenstate $[|\phi_1(t)\rangle]$ and other eigenstates $[|\phi_2(t)\rangle$ and $|\phi_3(t)\rangle]$ exist. With the help of control fields $f_1(t)$ and $f_2(t)$, the system will be driven to the invariant set that only contains the target eigenstate $|\phi_1(t)\rangle$ according to Lyapunov control theory [49,62,63]. Thus even if the nonadiabatic transitions occur due to the unsatisfied adiabatic condition during system evolution (for instance, the laser pulse width $T = 3.2$ μs cannot satisfy the adiabatic condition well), they will be driven back to the target eigenstate $|\phi_1(t)\rangle$ by control fields, leading to the system evolution always evolving along with target eigenstate $|\phi_1(t)\rangle$.

Consider a special case in Lyapunov control, that is, Lyapunov function $V(t) = 0$ all the time, which means the system state $|\psi(t)\rangle$ always evolves along with the instantaneous eigenstate $|\phi_T(t)\rangle$. In the three-level system, if the control Hamiltonian is chosen as $H_1 = |1\rangle\langle 3| - |3\rangle\langle 1|$, the matrix elements of Eq. (7) read as below:

$$\begin{aligned}
&\mathcal{H}_{11} = E_1(t), \quad \mathcal{H}_{22} = E_2(t), \quad \mathcal{H}_{33} = E_3(t),\\
&\mathcal{H}_{12} = (\mathcal{H}_{21})^* = -\frac{1}{\sqrt{2}}[f_1(t) - i\dot{\theta}],\\
&\mathcal{H}_{13} = (\mathcal{H}_{31})^* = -\frac{1}{\sqrt{2}}[f_1(t) - i\dot{\theta}],\\
&\mathcal{H}_{23} = (\mathcal{H}_{32})^* = 0.
\end{aligned} \tag{15}$$

In order to satisfy $V(t) = 0$, we must ensure that all the off-diagonal elements of Eq. (15) are zero, i.e., the control field $f_1(t) = i\dot{\theta}$. This is exactly the counterdiabatic driving Hamiltonian $H_{CD} = f_1(t)H_1$ by using Berry's approach [42]. Therefore the transitionless quantum driving algorithm can be regarded as a particular case of Lyapunov control. However, in our method, we can avoid the difficulty of the realization of the counterdiabatic driving Hamiltonian H_{CD} by choosing some available control Hamiltonians.

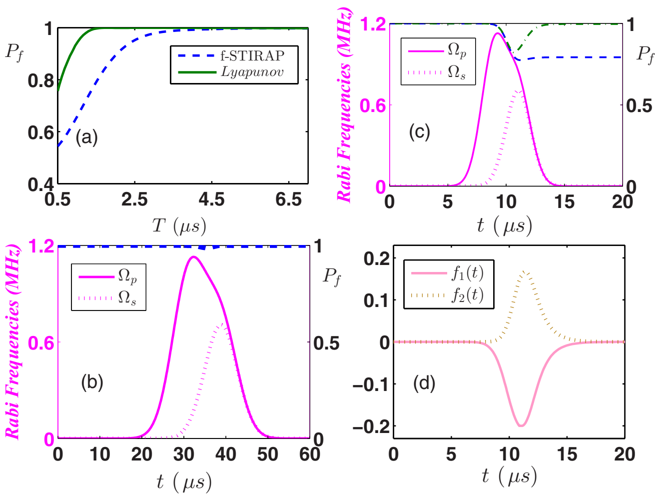

FIG. 3. (a) The population P_f as a function of laser pulse width T in f-STIRAP (blue dashed line) and adding Lyapunov control (green solid line), where $\tau = 0.7T$, $\Omega_s^0 = \Omega_p^0 = 1$ MHz. (b) Time dependence of the Rabi frequencies for f-STIRAP with laser pulse width $T = 5.5$ μs (carmine lines, corresponding to the left y coordinate), and the time evolution of population P_f for f-STIRAP (blue dashed line, corresponding to the right y coordinate). (c) Time dependence of the Rabi frequencies for f-STIRAP with laser pulse width $T = 1.5$ μs (carmine lines, corresponding to the left y coordinate), and the time evolution of population P_f for STIRAP with the blue dashed line and for Lyapunov control with the green dot-dashed line (corresponding to the right y coordinate). (d) The time evolution of corresponding control fields for Lyapunov control with $A_1 = A_2 = 1$.

Note that by suitably choosing the pump and Stokes lasers one can achieve superposition coherent state $|\phi_1(t)\rangle = \cos\theta|1\rangle - \sin\theta|3\rangle$ under f-STIRAP [21], where the pump and Stokes pulses satisfy the following condition:

$$\lim_{t\to 0}\frac{\Omega_p(t)}{\Omega_s(t)} = 0, \quad \lim_{t\to+\infty}\frac{\Omega_p(t)}{\Omega_s(t)} = \tan\theta. \tag{16}$$

In this case, the Stokes pulse still comes first and is followed after a certain time delay by the pump pulse, but the two pulses vanish simultaneously eventually, i.e., the two pulses can be chosen as

$$\begin{aligned}\Omega_p(t) &= \Omega_p^0 \sin\theta e^{-(t-\tau)^2/T^2},\\ \Omega_s(t) &= \Omega_s^0 e^{-(t+\tau)^2/T^2} + \Omega_s^0\cos\theta e^{-(t-\tau)^2/T^2}.\end{aligned} \tag{17}$$

Following the same routine to the case of perfect population transfer in STIRAP, Fig. 3(a) shows the evolution of population $P_f = |\langle\psi(t)|\phi_1(t)\rangle|^2$ versus laser pulse width T under f-STIRAP and adding Lyapunov control. We can see that ensuring high population transfer such as $P_f \geqslant 0.99$ requires the laser pulse width $T \geqslant 5.5$ μs for f-STIRAP but only $T \geqslant 1.5$ μs by adding Lyapunov control strategy, which means it needs shorter interaction time between lasers and system for Lyapunov control. In Fig. 3(b), we show the time evolution of instantaneous eigenstate $|\phi_1(t)\rangle$ and the laser pulse sequences of Eq. (17) for laser pulse width $T = 5.5$ μs. One can find that the system evolution always follows the instantaneous eigenstate $|\phi_1(t)\rangle$, which indicates the adiabatic condition is satisfied very well with $T = 5.5$ μs for f-STIRAP, while if the laser pulse width T reduces to 1.5 μs the system evolution will deviate away from the instantaneous eigenstate, which is shown in Fig. 3(c) with the blue dashed line. However, when we add Lyapunov control to this case, as we can see from the green solid line in Fig. 3(c), the system evolution will follow the instantaneous eigenstate $|\phi_1(t)\rangle$ again. Figure 3(d) shows the time evolution of control fields for which we added Lyapunov control to f-STIRAP.

B. Speeding up entanglement generation in a cavity QED system

Another application is to accelerate adiabatic generation of atom-atom entanglement in a cavity QED system [65], where two Λ-type atoms are trapped in two distant single-mode optical cavities connected by an optical fiber. The kth atomic transition $|0\rangle_k \to |e\rangle_k$ $(k = 1,2)$ is resonantly coupled to the kth cavity with coupling coefficient $g_k(t)$. The kth classical field drives the atomic transition $|1\rangle_k \to |e\rangle_k$ resonantly with coupling coefficient $\Omega_k(t)$. For simplicity we assume both $g_k(t)$ and $\Omega_k(t)$ are real. In the rotating wave approximation, the system Hamiltonian can be written as ($\hbar = 1$)

$$H_0 = \sum_{k=1}^{2}[g_i(t)|e\rangle_k\langle 0| + \Omega_k(t)|e\rangle_k\langle 1|] + \nu b(a_1^\dagger + a_2^\dagger) + \text{H.c.}, \tag{18}$$

where a_k and b are the annihilation operator for the kth cavity mode and fiber mode, respectively, and ν is the coupling strength between fiber and cavities. By defining the excitation number operator $N_e = \sum_{k=1}^{2}(|e\rangle_k\langle e| + |1\rangle_k\langle 1| + a^\dagger a) + b^\dagger b$, due to $[N_e, H_0] = 0$, the subspace with $N_e = 1$ can be spanned by the state vectors

$$\begin{aligned}|\varphi_1\rangle &= |10\rangle_a|00\rangle_c|0\rangle_f, \quad |\varphi_2\rangle = |e0\rangle_a|00\rangle_c|0\rangle_f,\\ |\varphi_3\rangle &= |00\rangle_a|10\rangle_c|0\rangle_f, \quad |\varphi_4\rangle = |00\rangle_a|00\rangle_c|1\rangle_f,\\ |\varphi_5\rangle &= |00\rangle_a|01\rangle_c|0\rangle_f, \quad |\varphi_6\rangle = |0e\rangle_a|00\rangle_c|0\rangle_f,\\ |\varphi_7\rangle &= |01\rangle_a|00\rangle_c|0\rangle_f,\end{aligned} \tag{19}$$

where $|m_1m_2\rangle_a|n_1n_2\rangle_c|n_f\rangle_f$ denotes the atomic state $|m_k\rangle$ $(m_k = 0,1,e; k = 1,2)$, n_k $(k = 1,2)$ photons in the kth cavity, and n_f photons in the fiber. The Hamiltonian H_0 has the following dark state:

$$\begin{aligned}|\phi_T(t)\rangle = K_{12}[&g_1(t)\Omega_2(t)|\varphi_1\rangle - \Omega_1(t)\Omega_2(t)|\varphi_3\rangle\\ &+\Omega_1(t)\Omega_2(t)|\varphi_5\rangle - g_2(t)\Omega_1(t)|\varphi_7\rangle],\end{aligned} \tag{20}$$

where the normalization constant $K_{12} = (g_1^2\Omega_2^2 + 2\Omega_1^2\Omega_2^2 + g_2^2\Omega_1^2)^{-1/2}$. Suppose the initial state of the system is $|\varphi_1\rangle$. If $g_1(t), g_2(t) \gg \Omega_1(t), \Omega_2(t)$, and the pulse shapes satisfy

$$\lim_{t\to 0}\frac{g_2(t)\Omega_1(t)}{g_1(t)\Omega_2(t)} = 0, \quad \lim_{t\to+\infty}\frac{g_2(t)\Omega_1(t)}{g_1(t)\Omega_2(t)} = \tan\beta, \tag{21}$$

the initial state can be adiabatically transferred to $|\phi_T(t\to\infty)\rangle = (\cos\beta|10\rangle_a - \sin\beta|01\rangle_a)|00\rangle_c|0\rangle_f$, which is the entangled state for two atoms in fact. The time-dependent

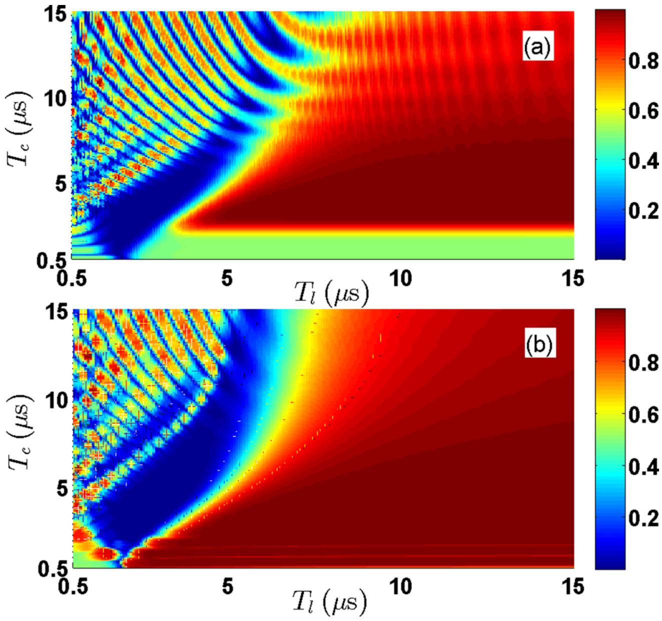

FIG. 4. The population P_f as a function of T_c and T_l in (a) the f-STIRAP process and (b) the f-STIRAP process with Lyapunov control. The pulse parameters are chosen as $\Gamma/2\pi = 10$ MHz, $\Omega_0 = 5\Gamma$, $g_0 = 25\Gamma$, $\nu = 30\Gamma$, $d = 3/\Gamma$. We take $t = 16/\gamma$ for the simulation where the system has been a steady state.

coupling coefficients are given as below:

$$g_1(t) = g_2(t) = g_0 e^{-t^2/T_c^2},$$
$$\Omega_1(t) = \Omega_0 \sin\beta e^{-(t-d)^2/T_l^2},$$
$$\Omega_2(t) = \Omega_0 e^{-(t+d)^2/T_l^2} + \Omega_0 \cos\beta e^{-(t-d)^2/T_l^2}, \quad (22)$$

where T_c and T_l are the Gaussian pulse widths of the cavity and the laser fields, respectively. d is the distance between the center of the cavity and the laser axis. The goal here is to speed up the adiabatic transfer process from the initial state $|\psi(0)\rangle = |\varphi_1\rangle$ to the target state $|\phi_T(t)\rangle = (\cos\beta|10\rangle_a - \sin\beta|01\rangle_a)|00\rangle_c|0\rangle_f$ by adding Lyapunov control. In order not to add extra interaction in the system, the control Hamiltonians are also chosen as

$$H_1 = |e\rangle_1\langle 1| + |1\rangle_1\langle e|,$$
$$H_2 = |e\rangle_2\langle 1| + |1\rangle_2\langle e|. \quad (23)$$

In Figs. 4(a) and 4(b), we simulate the evolution of population $P_f = |\langle\psi(t)|\phi_T(t)\rangle|^2$ as a function of T_c and T_l for the conventional adiabatic passage and Lyapunov control, respectively. The control fields are designed by Eq. (5) with $A_1 = 1$ and $H_{k_0} = H_1'$. We see that the regions of high populated target state are enlarged for Lyapunov control as compared with the conventional adiabatic passage. Especially, high population of the target state can also be realized for small T_c under Lyapunov control while it is invalid for conventional adiabatic passage. Moreover, Lyapunov control removes population oscillations that emerge in the conventional adiabatic passage when $T_l > 6$ μs.

IV. CONCLUSIONS

We have presented an effective scheme to speed up adiabatic evolution by using Lyapunov control. Specifically, in a three-level system associating with STIRAP and f-STIRAP, we have shown that the requirement of laser pulse width for perfect population transfer can be greatly shortened by using Lyapunov control. Additionally, we have also explored the application of the scheme to speed up entanglement generation in a cavity QED system. The advantages of the scheme are as follows. First, no extra couplings are needed for the control Hamiltonian to perform adiabatic quantum state transfer. Second, the occupation of the intermediate state can be effectively suppressed by properly choosing the control Hamiltonian. Third, the proposed scheme may be applied to accelerate adiabatic evolution in complex quantum systems where the exact expressions of adiabatic eigenstates cannot be reachable. We expect that the scheme may offer insight on adiabatic theory and quantum information processing.

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