Nonuniform code concatenation for universal fault-tolerant quantum computing

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(Received 5 June 2016; revised manuscript received 9 January 2017; published 27 September 2017)

Using transversal gates is a straightforward and efficient technique for fault-tolerant quantum computing. Since transversal gates alone cannot be computationally universal, they must be combined with other approaches such as magic state distillation, code switching, or code concatenation to achieve universality. In this paper we propose an alternative approach for universal fault-tolerant quantum computing, mainly based on the code concatenation approach proposed in [T. Jochym-O'Connor and R. Laflamme, Phys. Rev. Lett. **112**, 010505 (2014)], but in a nonuniform fashion. The proposed approach is described based on nonuniform concatenation of the 7-qubit Steane code with the 15-qubit Reed-Muller code, as well as the 5-qubit code with the 15-qubit Reed-Muller code, which lead to two 49-qubit and 47-qubit codes, respectively. These codes can correct any arbitrary single physical error with the ability to perform a universal set of fault-tolerant gates, without using magic state distillation.

DOI: 10.1103/PhysRevA.96.032337

I. INTRODUCTION

Quantum computers harness physical phenomena unique to quantum mechanics to realize a fundamentally new mode of information processing [1]. They can overcome the limitations of classical computers in efficiently solving hard computational problems for some tasks such as integer factorization [2] and database search [3].

Unfortunately, quantum computers are highly susceptible to noise due to decoherence and imperfect quantum operations that lead to the decay of quantum information [1,4]. Unless we can successfully mitigate the noise problem, maintaining large and coherent quantum states for a long enough time to perform quantum algorithms will not be readily possible. Quantum error correction codes were developed to address this problem [5–7]. To do so, data are encoded into a code and gates are applied directly on the encoded quantum states without a need to decode the states [1]. The encoded gates are applied fault-tolerantly in a way that they do not propagate errors in the circuit. Furthermore, quantum codes can be concatenated recursively to increase their ability to correct errors even further. In this way, almost perfectly reliable quantum computation is possible with polylogarithmic overhead using noisy physical devices as long as the noise level is below a threshold value [8].

A straightforward and efficient technique for fault-tolerant quantum computing is using transversal gates. An encoded gate which can be implemented in a bitwise fashion is known as a transversal gate [1]. No quantum code with a universal set of transversal gates exists [9]. So a common solution for applying nontransversal gates is by using a special state prepared by the magic state distillation (MSD) protocol [10]. However, the overhead of state preparation using MSD remains one of the drawbacks of this approach [11]. The distillation overhead scales as $O(\log^{\gamma}(\epsilon_{\text{in}}/\epsilon_{\text{out}}))$, where γ is determined by the distillation protocol and ϵ_{in} and ϵ_{out} are the input state accuracy and desired output accuracy, respectively [12]. There have been several efforts to reduce the overhead of this scheme such as [12–14].

A work on universal fault-tolerant quantum computing without MSD using only one quantum error correcting code was proposed by Paetznick and Reichardt [15]. In this approach, all of the gates from the considered universal set, e.g., {Pauli gates, H,CCZ} were implemented transversally, where H and CCZ are Hadamard and controlled-controlled-Z, respectively. However, as applying the transversal H gate disturbs the code space, additional error correction and transversal measurements are needed to recover the code space after the application of this gate.

Recently, similar approaches for universal quantum computing without using MSD have been proposed. These approaches are based on combining two different codes, say C_1 and C_2, where each nontransversal gate in C_1 has a transversal implementation on C_2 and vice versa. This approach is pursued in two different ways: (1) by combining C_1 and C_2 based on code switching [16–18] and (2) by combining C_1 and C_2 in a uniform concatenated fashion [19]. We call a concatenated code *uniform* if it uses only one quantum code in each level to encode all of the qubits of that level.

In the code-switching scheme, since the two selected codes have different sets of transversal gates, one can implement a universal set of gates transversally by switching to C_2 for transversal implementation of a gate, which is nontransversal in C_1. However, a fault-tolerant switching circuit is needed, which imposes an additional cost and thus, in some cases, it may incur a higher cost compared to MSD [18]. On the other hand, in the uniform code concatenation method, the logical information is encoded by C_1 where each qubit of C_1 is, in turn, encoded into the code of C_2. Therefore, the number of necessary physical qubits to code the logical information is relatively large (the product of the number of qubits for the two codes). For instance, if 7-qubit Steane and 15-qubit Reed-Muller codes are used a code [[105, 1, 9]] will be produced. However, the code has the ability to correct only one arbitrary single error because of the error propagation in a codeword during T and H implementation, where $T = \text{diag}(1, e^{\frac{i\pi}{4}})$.

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Recently, Yoder *et al.* [20] proposed the pieceable fault-tolerant concept to provide universal fault-tolerance by developing nontransversal, yet still fault-tolerant gates. In this approach a nontransversal circuit is broken into fault-tolerant pieces and rounds of intermediate error correction is applied in between to correct errors before they become uncorrectable by propagating in the circuit.

In this paper, we propose an alternative method for universal fault-tolerant quantum computing mainly based on the code concatenation approach proposed in [19], but in a nonuniform fashion. The proposed method opens a perspective of code concatenation for universal fault-tolerant computation by considering the structural details of nontransversal gates and reduces the overhead of the uniform code concatenation method proposed by the authors of [19].

II. NONUNIFORM CODE CONCATENATION

The proposed approach is based on the nonuniform code concatenation of C_1 and C_2. In this approach, a logical qubit is encoded using C_1 in the first level of the coding hierarchy. However, in the second level, only some of the C_1 qubits are encoded using C_2 depending on the implementation of the nontransversal gates in C_1, as opposed to the method employed in [19] which encoded all of the C_1 qubits using C_2 in the second level of concatenation. The remaining qubits can be encoded using C_1 or kept unchanged. In contrast with uniform concatenated codes, we call such a code *nonuniform*, which uses more than one code in at least one level of its coding hierarchy. The idea of nonuniform code concatenation is motivated by the observation that the application of a nontransversal gate in C_1 does not necessarily involve all of the C_1 qubits. Therefore, it is not necessary to encode all of the C_1 qubits using C_2. The C_1 qubits can be partitioned into two nonoverlapping sets: the set B_1 which contains qubits that are coupled during the application of a nontransversal gate in C_1 and B_2 which contains the uncoupled qubits. Indeed, we only need to encode qubits of B_1 using C_2 and can leave the B_2 qubits unchanged. If there is more than one nontransversal gate in C_1, the set B_1 will contain the union of all involved qubits in the implementation of each nontransversal gate. Figure 1 depicts a schematic overview of the proposed approach.

C_1 and C_2 must have the same properties as stated in [19] for the uniform code concatenation approach: (i) C_1 and C_2 must have at least a distance of three. (ii) For any logical gate in the universal gate set with nontransversal implementation on C_1, there must exist an application of this gate using only gates that are transversal in C_2. (iii) The error correction operations and syndrome measurement on C_1 and C_2 must be globally transversal in the concatenated code space. However, for our method to produce superior results compared to [19], it is necessary to have $|B_2| > 0$. Fortunately, for a stabilizer code $[[n,1,d]]$, there is a useful family of gates which can be implemented by coupling only d qubits of each code block.

Theorem. For a stabilizer code $[[n,1,d]]$, a logical $C^k Z(\theta)$ gate can be implemented nontransversally by coupling only d qubits of each code block as shown in Fig. 2, where $Z(\theta) = \mathrm{diag}(1,e^{i\theta})$ and $k \in \{0,1,2,\ldots\}$.

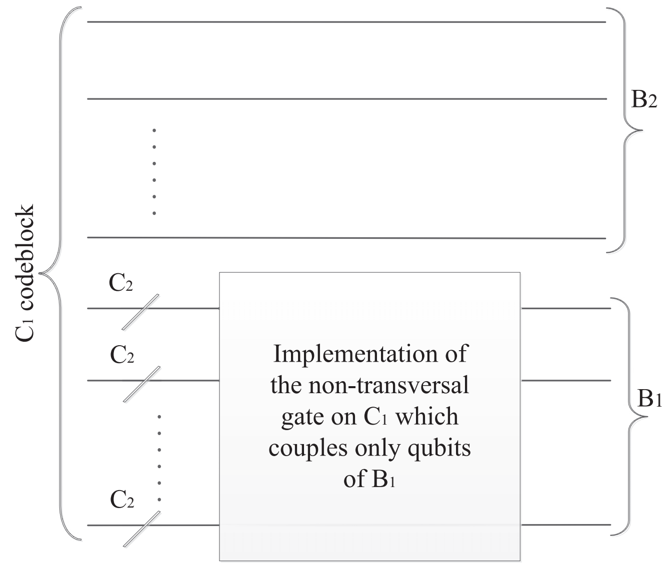

FIG. 1. The schematic overview of the nonuniform code concatenation approach. Logical information is encoded by C_1. In the second level of concatenation each qubit of B_1 is, in turn, encoded into the code of C_2 and the B_2 qubits are left unchanged without encoding.

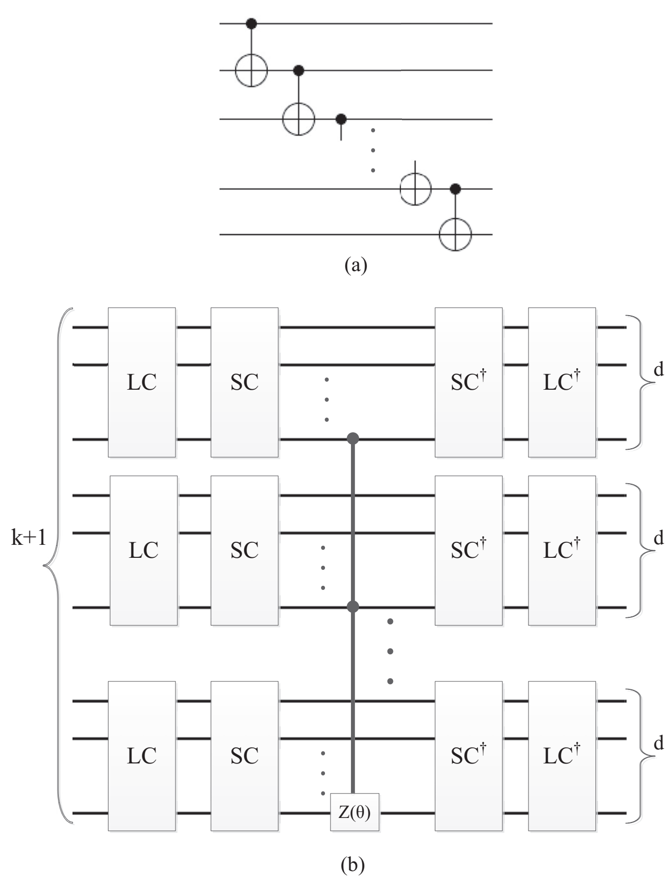

FIG. 2. (a) Staircase of CNOTs. (b) Nontransversal application of $C^k Z(\theta)$ gate for a stabilizer code by involving only d qubits of each code block, where d is the code distance. SC is an acronym for staircase of CNOTs and LC is a circuit containing only local Clifford gates which transform the original logical Z operator into a form consisting of only Pauli Z's and I's. Note that only the qubits of B_1 are shown.

Proof. When the code distance is d, there is a Pauli operator in the normalizer of the stabilizer group (that does not belong to the stabilizer group itself) with weight d, where the weight of an n-qubit Pauli operator is defined as the number of its nonidentity members. The d qubits corresponding to the nonidentity members of this operator are the only qubits that are coupled during the application of $C^k Z(\theta)$ and therefore belong to B_1. For simplicity and without loss of generality, we suppose that the qubits of B_1 are the last qubits of the code block, as shown in Fig. 1.

The mentioned Pauli operator is a logical operator and can be thought of as a logical Z gate. This logical Z operator can be transformed into a form consisting of only Pauli I's and Z's with positive sign ($I^{\otimes n-d} \otimes Z^{\otimes d}$) by applying some single-qubit Clifford gates, namely LC. The application of a staircase of contolled-NOT (CNOT) gates (SC), on these d qubits (as shown in Fig. 2), changes the logical Z gate from $I^{\otimes n-d} \otimes Z^{\otimes d}$ to $I^{\otimes n-1} \otimes Z$ as CNOT maps ZZ to IZ under conjugation. The ability to perform a logical Z gate by applying a single physical Z gate on a qubit (for example, the last qubit) implies that the new logical state $|\bar{0}\rangle$ is an eigenstate of Z_n and $|\bar{1}\rangle$ is an eigenstate of $-Z_n$. Therefore, these new logical states must have the form $|\bar{0}\rangle = |\phi_0\rangle|0\rangle$ and $|\bar{1}\rangle = |\phi_1\rangle|1\rangle$, where $|\phi_0\rangle$ and $|\phi_1\rangle$ are some $(n-1)$ qubit stabilizer states. Consequently, if we apply a phase gate $U = Z(\theta)$ to the last qubit, we get

$$|\bar{0}\rangle \rightarrow |\phi_0\rangle(U|0\rangle) = |\phi_0\rangle|0\rangle = |\bar{0}\rangle,$$
$$|\bar{1}\rangle \rightarrow |\phi_1\rangle(U|1\rangle) = |\phi_1\rangle(e^{i\theta}|1\rangle) = e^{i\theta}|\bar{1}\rangle.$$

Similarly, if we apply a multiqubit phase gate $U = C^k Z(\theta)$ to the last qubit of $k+1$ logical qubits, we have

$$|\bar{t}\rangle \rightarrow \begin{cases} |\bar{t}\rangle & t \in \{0,1,2,\ldots,2^{k+1}-2\} \\ e^{i\theta}|\bar{t}\rangle & t = 2^{k+1}-1 = 11\ldots1, \end{cases}$$

where we use binary notation to denote the $k+1$-qubit state $|\bar{t}\rangle$.

Therefore, the codespace is preserved and a logical U is implemented. Then, one can simply invert the SC and LC circuits to take the new logical states back to the original ones.

It is worth noting that while the theorem holds for any Z rotation, transversal single-qubit Z rotations are restricted to be inside the Clifford hierarchy for any code, specifically for C_2 [21]. The fault-tolerant application of nontransversal gates in C_1, nontransversal gates in C_2, and error correction procedure in the proposed code are described in the following.

A. Fault-tolerant implementation of the nontransversal gates in C_1

A single physical error on one of the qubits of B_1, occurring in the nontransversal application of these gates on C_1 only propagates between the qubits of B_1, which are themselves encoded blocks of C_2. Since implementations of these gates on C_1 consist of only transversal gates in C_2, this single physical error only propagates to a single physical error in each of the B_1 qubits. As these qubits are encoded using C_2, this single physical error can be corrected by error correction procedure on C_2 code blocks.

A single physical error on the B_2 qubits during the application of these nontransversal gates in C_1 does not propagate to any other qubits of C_1 code block and can be corrected using the error correction procedure on C_1.

B. Fault-tolerant implementation of the nontransversal gates in C_2

These gates have transversal implementation on C_1 and therefore, a single physical error on one of the C_1 qubits, does not propagate to any other qubits of C_1, during the application of these gates. However, as they are nontransversal in C_2, a single error on a particular C_2 code block (qubits of B_1) can propagate to a noncorrectable set of errors on that code block which introduces a C_2 logical error. However, this error only leads to a single error on one of the qubits of C_1 which can be corrected using the error correction procedure on C_1.

C. Error correction procedure

Regarding the third necessary condition for code concatenation, the correction procedure is globally transversal and therefore fault-tolerant in the concatenated code space. This feature is essential not only for preventing error propagation during the error correction procedure, as described in [19], it also makes the nonuniform code concatenation possible. Indeed, this feature guarantees that there is no interaction among qubits of the sets B_1 and B_2 which are encoded blocks of different codes during error correction procedure.

Although straight concatenation of the two codes $[[n_1,k,d_1]]$ and $[[n_2,1,d_2]]$ leads to a code $[[n_1 n_2,k,d_1 d_2]]$ [22], our code concatenation scheme reduces the effective distance of the concatenated code to achieve universal fault tolerance. By effective distance we mean the code distance considering the error propagations that occur during application of the nontransversal gates.

While the proposed approach is general and can be applied to any code combination that satisfies the mentioned conditions, in the rest of paper we will focus on the 7-qubit Steane and 5-qubit quantum error correction codes (the smallest quantum codes with a distance of three), as C_1, in combination with the 15-qubit Reed-Muller (RM) code (the smallest known quantum code with a distance of three and transversal T and CCZ gates), as C_2.

D. Nonuniform concatenation of the Steane and 15-qubit Reed-Muller codes

The universal set $\{H, S, T, \text{CNOT}\}$ is chosen as the gate library in this section, where $S = T^2$. S, H, and CNOT and therefore, any gates from the Clifford group have transversal implementation on the Steane code. The T gate remains the only nontransversal gate from the set. As shown in Fig. 3, for this code $B_1 = \{1,2,7\}$ and $B_2 = \{3,4,5,6\}$. On the other hand, the T gate is transversal in the RM code but the Hadammard gate is not [19]. Both of these codes have distances of three and the combination of their set of transversal gates produces a universal gate set. Based on the nonuniform approach, there is no need to encode all of the Steane qubits using the RM code. We need only to encode qubits of B_1 using the RM code and can leave the B_2 qubits unencoded. In doing so, a

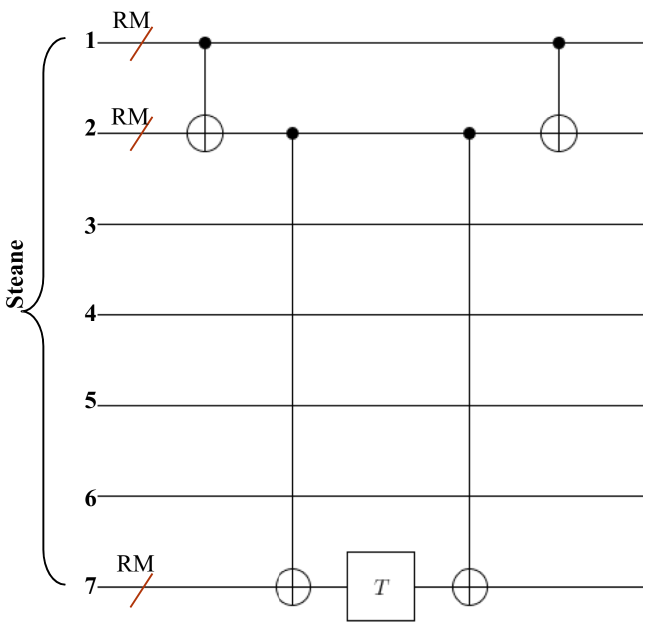

FIG. 3. Fault-tolerant application of the T gate for the proposed 49-qubit nonuniform concatenated code. A logical qubit is encoded by Steane where qubits 1, 2, and 7 are themselves encoded blocks of RM code and the other four qubits are left unchanged.

49-qubit code is constructed which has the ability to correct any single physical error like the 105-qubit code proposed in [19], but with a substantial improvement in resource overhead as the number of qubits and operations are reduced significantly.

As both the Steane and RM quantum codes have the same property that the S and CNOT gates can be implemented transversally, then for the proposed 49-qubit code they have also transversal implementation. Additionally, all syndrome measurements and Pauli corrections will be transversal within both codes [19] and therefore the error correction procedure on the Steane and RM code blocks are globally transversal and fault-tolerant in the 49-qubit code space.

The CCZ $= C^2Z(\pi)$ can also be applied fault-tolerantly for the proposed 49-qubit code, as its implementation on the Steane code has the same structure as T and it is transversal in the RM code.

E. Concatenation of the 5-qubit code with the 15-qubit Reed-Muller code

The stabilizers and logical Pauli operators of the 5-qubit code can be written as follows [20]:

$$S_5 = \left\langle \begin{matrix} ZZXIX \\ XZZXI \\ IXZZX \\ XIXZZ \end{matrix} \right\rangle,$$

$$\overline{Z}_5 = -XIZIX, \quad \overline{X}_5 = -YIXIY.$$

Let $M = \{T = C^0Z(\frac{\pi}{4}), S = C^0Z(\frac{\pi}{2}), CZ = C^1Z(\pi), \text{CCZ} = C^2Z(\pi)\}$. The gates of M are transversal in RM and along with H provide a universal set (but not a minimal) for quantum computation. H is transversal for the 5-qubit code but by permutation [20]. However, this permutation is in contrast with the proposed concept of nonuniform code concatenation, which makes it unusable for our nonuniform construction. The gates of M [generally $C^kZ(\theta)$] can be applied nontransversally on the 5-qubit code block (not the codes constructed by concatenation of the 5-qubit and 15-qubit RM codes) as shown in Fig. 4, where $K = S.H$. Note that K is not transversal in the RM code. Therefore, the nontransversal implementation of the gates of M on the 5-qubit code does not involve only gates that are transversal in RM and thus, violates the second necessary condition for code concatenation.

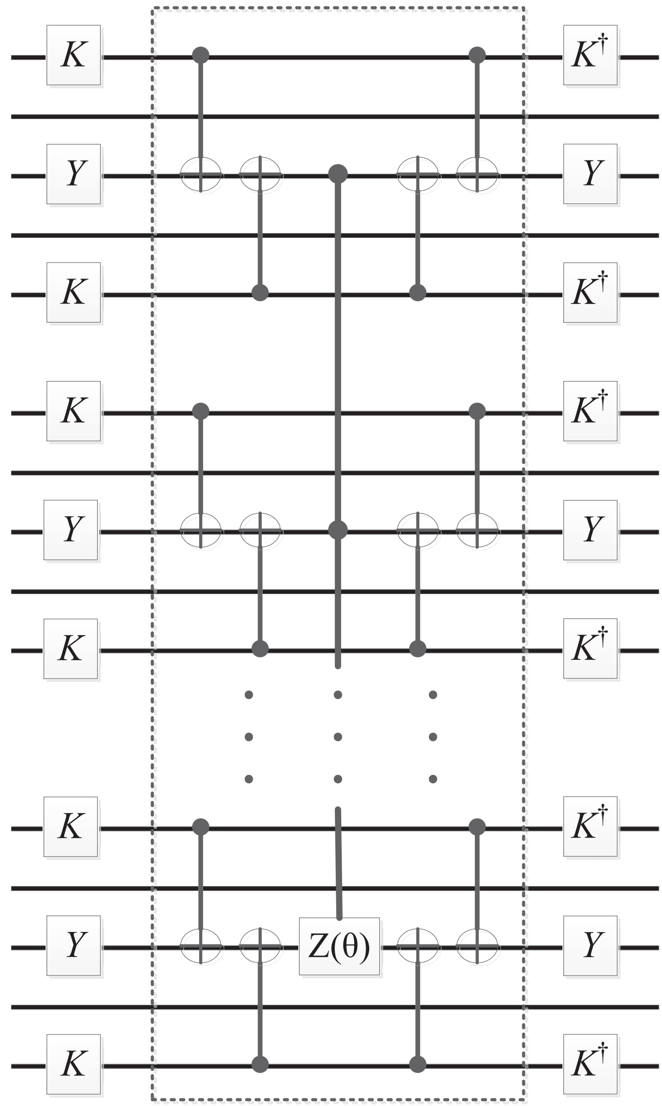

FIG. 4. Nontransversal implementation of the $C^kZ(\theta)$ gate for the 5-qubit code. The circuit specified by the dotted box shows the implementation of this gate for the 5′-qubit code. Note that while this circuit generally holds for the 5-qubit code, not every $C^kZ(\theta)$ can be applied fault-tolerantly for the proposed 75- and 47-qubit concatenated codes, as they may be nontransversal in RM. Indeed, from the $C^kZ(\theta)$ gates, only the gates of $M = \{T,S,CZ,\text{CCZ}\}$, which are transversal in RM, can be applied fault-tolerantly for the proposed 75- and 47-qubit concatenated codes.

Consequently, the 5-qubit code in its standard form is not suitable for the proposed nonuniform code concatenation with RM. However, one can alter this code to an equivalent code, namely the 5′-qubit code, by applying $KIYIK$ on the 5-qubit code block [20]. This new code has the following set of stabilizers:

$$S_{5'} = \left\langle \begin{matrix} -YZXIZ \\ -ZZZXI \\ -IXZZZ \\ -ZIXZY \end{matrix} \right\rangle,$$

where the logical Pauli operators are defined as

$$\overline{Z}_{5'} = ZIZIZ, \quad \overline{X}_{5'} = XIXIX.$$

For this code, the K gate can be applied transversally as $(IIZII).K^{\otimes 5}$. The gates of M can also be implemented as shown in the dotted box of Fig. 4. This implementation only consists of the gates that are transversal in the RM code and therefore satisfies the second condition for code concatenation.

As $H = S^{\dagger}.K$, K along with the gates of M provide a universal set of quantum gates. This set is considered as the gate library for the codes proposed in this section. Considering the 5′-qubit code as C_1 and the RM code as C_2 satisfies the necessary condition for code concatenation regarding this universal set. The concatenation of these codes uniformly leads to a 75-qubit code where all of the C_1 qubits are encoded blocks of RM. Furthermore, for the 5′-qubit code $B_1 = \{1,3,5\}$ and $B_2 = \{2,4\}$. Therefore, the nonuniform concatenation of them produces a 47-qubit code where the B_1 qubits are encoded by the RM code in the second level of concatenation and the qubits of B_2 are left unencoded. Both the 75-qubit and 47-qubit codes have the ability to perform the gates of universal set fault-tolerantly.

III. DISCUSSION

The proposed nonuniform 47 and 49-qubit codes reduce the overall distance of their corresponding uniform codes (e.g., 75 and 105-qubit codes, respectively), as they leave the qubits of B_2 unencoded. Nevertheless, the B_2 qubits can be encoded using the C_1 code in the second level of concatenation, which leads to two 55 and 73-qubit codes, respectively. Doing so will increase the overall distance of the codes to nine, much like the uniform ones. However, in the worst case, the effective distance of these codes remains unchanged with the ability to correct a single physical error. This is because two physical errors on the qubits of B_1 may corrupt all of the B_1 qubits during application of the nontransversal gates, which cannot be corrected using the C_1 error correction procedure and therefore, leads to a logical error. Table I compares the produced concatenated codes based on the Steane and RM codes in terms of number of qubits, overall distance, and effective distance.

Chamberland *et al.* in their recent follow-up work [23] performed extensive numerical analysis into the proposed 49-qubit code and cited the preprint version of this paper. They obtained a depolarizing noise threshold of 9.69×10^{-4} for the 49-qubit code which is comparable to the 105-qubit threshold result of 1.28×10^{-3}, with about one order of magnitude lower resource overhead.

TABLE I. Comparison of the produced concatenated codes based on the Steane and RM codes in terms of number of qubits, overall distance, and effective distance.

Code concatenation method	No. qubits	overall distance	Effective distance
Uniform [19]	105	9	3
Nonuniform (case I)	49	5	3
Nonuniform (case II)	75	9	3

The 7-qubit Steane and 15-qubit Reed-Muller (RM) codes have unique features as follows, which make their concatenation efficient. The Steane code is the smallest CSS code with a distance of three and with the ability to implement a universal set of Clifford gates, transversally. The T gate is a nontransversal gate in the Steane code which can be applied by involving only three qubits (Fig. 3) and along with the Clifford gates provides a universal set of gates. The RM code is the smallest known code with transversal T gate and also a CSS code. Therefore, their concatenation leads to the smallest concatenated CSS code based on the proposed approach with the ability to perform a universal set of fault-tolerant gates. It should be noted that the CSS codes have some useful properties which make them good choices for fault-tolerant quantum computation [22]. Furthermore, the Steane and RM codes have the minimum number of unshared transversal gates, e.g., T and H. While the codes produced using the 5′-qubit code have fewer qubits, they are non-CSS and also the effective distance of the concatenated codes is reduced for all of the gates from the universal set. This is because there are no shared transversal gates between the 5′-qubit and RM codes.

IV. CONCLUSION

In this paper, a nonuniform code concatenation approach is proposed for fault-tolerant quantum computing without using MSD. Four 47, 49, 55, and 73-qubit codes are constructed based on this approach with the ability to correct an arbitrary single physical error which outperforms their counterpart uniform concatenated codes. Introducing the nonuniform code concatenation concept and exploiting it in the design of a new universal fault-tolerant quantum computation method by considering the implementation details of the nontransversal gates in C_1, is the main contribution of the proposed approach. It is worth noting that, in such code concatenation schemes (both uniform and nonuniform), the effective distance of the concatenated code is reduced to make the universal fault-tolerant computation possible. Although the proposed approach is described based on the 5-qubit and Steane codes in concatenation with the 15-qubit Reed-Muller code, one may pursue this work by investigating other code combinations.

ACKNOWLEDGMENT

The authors acknowledge the financial support by the Iranian National Science Foundation (INSF).

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