

Entanglement enhancement through multirail noise reduction for continuous-variable measurement-based quantum-information processing

Yung-Chao Su and Shin-Tza Wu*

Department of Physics, National Chung Cheng University, Chiayi 621, Taiwan

(Received 7 November 2016; published 18 September 2017)

We study theoretically the teleportation of a controlled-PHASE (CZ) gate through measurement-based quantum-information processing for continuous-variable systems. We examine the degree of entanglement in the output modes of the teleported CZ-gate for two classes of resource states: the canonical cluster states that are constructed via direct implementations of two-mode squeezing operations and the linear-optical version of cluster states which are built from linear-optical networks of beam splitters and phase shifters. In order to reduce the excess noise arising from finite-squeezed resource states, teleportation through resource states with different multirail designs will be considered and the enhancement of entanglement in the teleported CZ gates will be analyzed. For multirail cluster with an arbitrary number of rails, we obtain analytical expressions for the entanglement in the output modes and analyze in detail the results for both classes of resource states. At the same time, we also show that for uniformly squeezed clusters the multirail noise reduction can be optimized when the excess noise is allocated uniformly to the rails. To facilitate the analysis, we develop a trick with manipulations of quadrature operators that can reveal rather efficiently the measurement sequence and corrective operations needed for the measurement-based gate teleportation, which will also be explained in detail.

DOI: [10.1103/PhysRevA.96.032327](https://doi.org/10.1103/PhysRevA.96.032327)

I. INTRODUCTION

Measurement has been the foundation of physical science since Galileo Galilei's time [1]. It has also played a pivotal role in the development of quantum mechanics. Studies of quantum measurements have had great impacts on our understanding of the foundation of quantum mechanics [2]. At the same time, manipulations of quantum systems through measurements have been exploited in many aspects of modern quantum technologies [3,4].

In quantum-information sciences, quantum-information processing typically involves sequence of transformations that are represented by unitary operators acting on the information carriers [5]. Experimental realizations for these processes, however, have been challenging due to the highly demanding control over coherence [6,7]. Measurement-based quantum-information processing offers a partial solution to such difficulties [8]. The quantum-information processing in this approach is effected by appropriately designed sequence of local measurements over highly entangled resource states. By feeding forward the measurement outcomes, gate operation over the input state can be achieved through appropriate corrective operations upon the output. In this way, the intended gate operation can be “teleported” to the output of the entangled resource state [9–11].

Originally, measurement-based quantum information processing was proposed for discrete-variable systems, i.e., systems with finite-dimensional state space [8]. It was later extended to systems with continuous degrees of freedom, which are often referred to as “continuous-variable” (CV) systems [12,13]. Since then, optical systems have been providing an appealing platform for CV quantum information processing through the measurement-based scheme due to their advantages in not only state preparation but also the

detection and manipulation of states [14–17]. Nevertheless, the archetypal resource state for measurement-based CV quantum-information processing requires two-mode squeezing operations over a set of quantum modes prepared in the zero-momentum eigenstate to form a highly entangled state commonly known as a “cluster state” (or a “graph state”) [13] over which definite correlations among the “nodes” of the cluster are imposed by the entanglement. Since an ideal momentum eigenstate demands infinite squeezing (hence infinite energy), only approximated, finite-squeezed momentum states are available in the laboratories. In practice, therefore, CV cluster states always have nonideal correlations among their nodes due to finite squeezing, which in turn deteriorate the quality of the gate teleportation with such resource states. Moreover, although two-mode squeezing operation can be implemented through inline squeezers and beam splitters [18,19], the rapidly growing demand for its implementation along with the size of the canonical CV cluster renders the scheme experimentally challenging for practical applications. In order to alleviate the demand for inline squeezers, van Loock and colleagues proposed in Ref. [20] a linear-optical approach to the construction of CV cluster states, which consists of finding appropriate linear-optical network that can implement the desired cluster correlations over a collection of offline squeezed modes through combinations of beam splitters and phase shifters. Making use of this approach, experimental demonstrations for measurement-based single-mode operations [21], two-mode operation [22], and their sequential operation [23] have been achieved through linear-optical CV cluster states. However, the linear-optical approach suffers from difficulties with scalability, since the number of optical elements in the optical network increases rapidly with the size of the cluster. Subsequent endeavors to improve the scalability of optical CV cluster states via encoding in the frequency domain [24] or in the time domain [25,26] have made a lot of progress in recent years. It has also been shown that fault-tolerant quantum computing in the measurement-based

*phystw@gmail.com

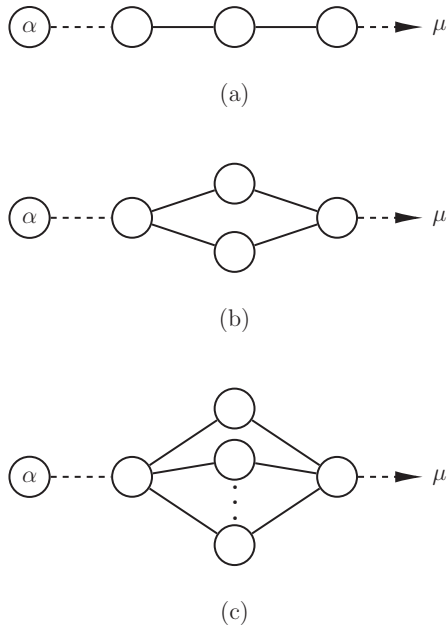


FIG. 1. Excess-noise reduction for measurement-based teleportation through multirail designs for CV cluster states proposed by van Loock *et al.* [20] with (a) single-rail, (b) two-rail, and (c) general multirail structures. Here each circle represents a qumode, with the one labeled α the input mode and those unlabeled the cluster modes. The solid lines connecting the cluster nodes represent two-mode correlations which can be established through two-mode squeezing operations or linear-optical networks (see the text). In each figure, the coupling between the input mode α and the cluster is represented by a dashed line. To teleport the state of α toward the right of the cluster, a sequence of quadrature measurements must be applied to the cluster modes. Upon feeding forward the measurement outcomes and applying the corresponding corrective operations to the rightmost node of the cluster, as indicated by the arrow, one would get the output mode μ which holds the teleported state.

CV scheme can be achieved with finite squeezing, although at a level that is still experimentally demanding [27]. These developments have made measurement-based CV approach a promising direction for realistic quantum computing.

As with canonical cluster states, the linear-optical cluster states (the qualifier “CV” will henceforth be omitted when it is clear in context) are also nonideal in practice, since the initial offline squeezed modes for the cluster are always squeezed finitely. This results in imperfect cluster correlations among the cluster nodes and causes “excess noise” in measurement-based teleportation through such resource states. As a remedy, van Loock and coworkers proposed in Ref. [20] a multirail design (see Fig. 1) for the CV cluster state that is capable of reducing excess noise in teleportation. They showed that, with increasing number of rails in the design, the excess noise becomes progressively smaller in one of the quadratures of the teleported mode. A comparison for the noise reduction was drawn in Ref. [20] for single-mode teleportation through multirail canonical cluster states and linear-optical cluster states. Here we extend the consideration by examining the teleportation of a two-mode entangling gate, the controlled-PHASE (or controlled-Z, abbreviated CZ) gate, through these two

classes of CV cluster states. In particular, with the reduction of excess noise through the multirail design, we expect to find improvements in the quality of the teleported CZ gate, which should be revealed in the enhanced entanglement in the output modes of the cluster. Since the multirail design can increase the size of the cluster massively (see Fig. 1), it is necessary to have a way to find out of the teleportation scenario efficiently. To this end, we devise an intuitive Heisenberg approach that can reveal the measurement sequence and the corresponding corrective operations for the teleportation through simple manipulations of the quadrature operators [28]. We are able to find out analytical results for the entanglement of the teleported CZ gates for both canonical and linear-optical cluster states with general multirail configurations. In the process of this analysis, we will also show that noise reduction in the multirail scheme can be optimized when the excess noise is distributed uniformly to each of the multirails.

We will begin in the following section by providing background information for CV cluster states necessary to our calculations and setting up the notations. In particular, we will explain the CZ-gate teleportation in measurement-based CV quantum information processing using a Heisenberg approach that involves certain tricks for manipulating the quadrature operators which will be essential to our calculations. We will then study in Sec. III the CZ-gate teleportation through canonical cluster states and linear-optical cluster states in two different subsections, where the noise-reduction mechanism in the multirail scheme will also be analyzed. Comparison of the results for the two classes of resource states will be presented at the later part of Sec. III. Finally, we close in Sec. IV with brief comments on the results and their possible extensions. For the sake of clarity of our presentation, a number of details of our results have been relegated to the appendices.

II. FORMULATION

In CV systems, each quantum mode (or “qumode,” in analogy with “qubit” for quantum bit) is described as a quantized harmonic oscillator [14]. For a qumode k , if the annihilation operator is \hat{a}_k , the corresponding quadrature amplitude (or “position”) and quadrature phase (or “momentum”) operators are then given, respectively, by [29]

$$\hat{q}_k \equiv \frac{\hat{a}_k + \hat{a}_k^\dagger}{2} \quad \text{and} \quad \hat{p}_k \equiv \frac{\hat{a}_k - \hat{a}_k^\dagger}{2i}, \quad (1)$$

where \dagger indicates Hermitian conjugation. These quadrature operators possess continuous eigenvalues and the corresponding eigenstates provide bases for encoding CV quantum information [14]. For any two qumodes k and l , it follows from (1) and the commutation relation $[\hat{a}_k, \hat{a}_l^\dagger] = \delta_{kl}$ that

$$[\hat{q}_k, \hat{p}_l] = \frac{i}{2} \delta_{kl}, \quad (2)$$

which corresponds to the canonical commutation relations for position and momentum operators in mechanical systems with $\hbar = 1/2$.

Canonical cluster states for CV systems are constructed by applying CZ gates over a set of momentum-squeezed vacuum modes that constitute the “nodes” of the cluster. Explicitly, the

CZ gate that acts on modes k and l is given by

$$CZ_{kl} = e^{2i \hat{q}_k \hat{q}_l}, \quad (3)$$

which is clearly symmetrical between modes k and l . By virtue of the commutation relation (2), it follows that the CZ-gate operation on the quadrature operators for mode l yields

$$CZ_{kl}^\dagger \hat{q}_l CZ_{kl} = \hat{q}_l, \quad CZ_{kl}^\dagger \hat{p}_l CZ_{kl} = \hat{p}_l + \hat{q}_k, \quad (4)$$

and likewise for mode k . Therefore, we see that the CZ gate has established correlations between the quadrature operators of the two modes. Experimentally, the transformation (4) corresponds to a quantum nondemolition (QND) process, where the position operator is unchanged, while the momentum operator picks up a shift from the other mode [29]. The CZ gate is thus also often referred to as a ‘‘QND gate’’ [12]. For definiteness, hereafter we will refer to the operation that connects the nodes of a canonical cluster the QND gate, while the gate (4) to be teleported using measurement-based schemes the CZ-gate, although the two indeed function identically.

According to (4), if a collection of modes with annihilator operators $\hat{a}_k = \hat{q}_k + i \hat{p}_k$ are coupled through QND gates, one then has for the resultant mode k

$$\hat{q}_k = \hat{q}_k, \quad \hat{p}_k = \hat{p}_k + \sum_{l \in N_k} \hat{q}_l, \quad (5)$$

where N_k denotes the set of modes that are coupled with mode k through QND gates. For initial modes that are momentum-squeezed vacuum states, one can define

$$\begin{aligned} \hat{\delta}_k &\equiv \hat{p}_k - \sum_{l \in N_k} \hat{q}_l \\ &= \hat{p}_k = e^{-r_k} \hat{p}_k^{(0)}, \end{aligned} \quad (6)$$

where r_k is the squeezing parameter for mode k , $\hat{p}_k^{(0)}$ is the momentum operator for the respective vacuum mode, and we have used (5) in arriving at the second line. It is then clear that in the ideal, infinite-squeezing limit $r_k \rightarrow \infty$ for all k , the $\hat{\delta}_k$'s would vanish identically, and the state would approach an ideal cluster state, which has perfect quadrature correlations among its nodes [12,17]. These operators $\hat{\delta}_k$ thus represent the noise in the cluster correlations due to finite-squeezed initial modes, and are often called the ‘‘excess-noise’’ operators, or the ‘‘nullifiers’’ of the cluster state, since the ideal cluster state is an eigenstate of these operators with eigenvalue zero [17]. Depending on the context, we will use both terms for $\hat{\delta}_k$ interchangeably and at times, for brevity, also refer to them simply as the ‘‘noise operators.’’

For the teleportation of CZ gate using different types of CV cluster states that we will discuss in the following, rather than the Schrödinger approach [13], we will resort to a Heisenberg approach that keeps track of the teleportation process through the time evolution of the quadrature operators [28,30]. In particular, by manipulating the quadrature operators judiciously, we will provide an intuitive and efficient way for establishing the measurement sequence and the corrective operations for the teleportation. This will simplify the calculation significantly and make the analysis for teleportation involving large clusters manageable tasks.

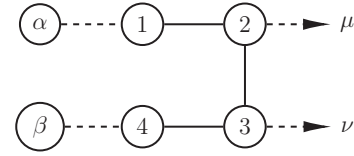


FIG. 2. Teleportation of a CZ gate through a linear four-mode cluster. Here and in all figures below, the input modes α, β are coupled to the cluster nodes via either a QND coupling or a beam-splitter coupling (see the text), which is represented by a dashed line here. To the right of the cluster, the output modes μ and ν are generated from nodes 2 and 3, respectively, where the arrows indicate appropriate corrective operations in accordance with measurement outcomes in the measurement sequence. For an ideal cluster state, the teleported state registered by μ, ν would be identical to that of the input modes α, β operated with a CZ gate. Depending on the context, the cluster state here (and also in other figures of this paper) can be a canonical one or a linear-optical one.

As an orientation, let us consider a simple case with a linear four-mode cluster [13] as depicted in Fig. 2. In view of the CZ-gate transformation (4), the teleportation that we wish to accomplish here amounts to implementing the mapping between the quadrature operators of the output modes μ, ν and the input modes α, β

$$\begin{pmatrix} \hat{q}_\mu \\ \hat{p}_\mu \end{pmatrix} \rightarrow \begin{pmatrix} \hat{q}_\alpha \\ \hat{p}_\alpha + \hat{q}_\beta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \hat{q}_\nu \\ \hat{p}_\nu \end{pmatrix} \rightarrow \begin{pmatrix} \hat{q}_\beta \\ \hat{p}_\beta + \hat{q}_\alpha \end{pmatrix}. \quad (7)$$

Since the input modes α, β are coupled to the cluster nodes 1, 4, respectively, by way of QND gates, it follows from (4) that after these couplings, we have

$$\begin{pmatrix} \hat{q}'_\rho \\ \hat{p}'_\rho \end{pmatrix} = \begin{pmatrix} \hat{q}_\rho \\ \hat{p}_\rho + \hat{q}_k \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \hat{q}'_k \\ \hat{p}'_k \end{pmatrix} = \begin{pmatrix} \hat{q}_k \\ \hat{p}_k + \hat{q}_\rho \end{pmatrix}, \quad (8)$$

where the subscripts $(\rho, k) = (\alpha, 1)$ and $(\beta, 4)$. To accomplish the mapping (7), we note that the output mode μ is obtained by correcting the final state of node 2 in the teleportation. Thus, in anticipation of $\hat{q}_2 \rightarrow \hat{q}_\alpha$ after corrective operations, we write the following trivial identity making use of the \hat{p}'_1 entry of (8) (i.e., setting $(\rho, k) = (\alpha, 1)$ there and using the entry involving \hat{p}'_1)

$$\begin{aligned} \hat{q}_2 &= \hat{q}_2 + (\hat{p}'_1 - \hat{p}_1 - \hat{q}_\alpha) \\ &= -\hat{q}_\alpha + \hat{p}'_1 + (\hat{q}_2 - \hat{p}_1), \end{aligned} \quad (9)$$

where we have grouped the terms that would constitute a noise operator of (6) in reaching the second line. Rearranging terms in the final identity above, one can write accordingly

$$\begin{aligned} \hat{q}_\mu &\equiv -\hat{q}_2 + \hat{p}'_1 \\ &= \hat{q}_\alpha + (\hat{p}_1 - \hat{q}_2). \end{aligned} \quad (10)$$

Here we have written from (9) by collecting the image \hat{q}_α of the intended mapping $\hat{q}_2 \rightarrow \hat{q}_\alpha$ and the noise operator $\hat{\delta}_1 = (\hat{p}_1 - \hat{q}_2)$ to the same side of the equation and relegating the rest to the other side, which is redefined as \hat{q}_μ . Immediately, we see from this result that by flipping the phase of the position operator \hat{q}_2 for node 2 and then displacing in accordance with the measurement outcome for \hat{p}'_1 , one would

be able to produce an output mode μ with \hat{q}_μ that differs from the input quadrature \hat{q}_α by just the excess noise $\hat{\delta}_1$.

Similarly, anticipating $\hat{p}_2 \rightarrow \hat{p}_\alpha + \hat{q}_\beta$ subject to corrections, one can make use of the \hat{p}'_α and the \hat{p}'_4 entries of (8) and write down the following trivial identity for the momentum operator of node 2

$$\begin{aligned}\hat{p}_2 &= \hat{p}_2 + (\hat{p}'_\alpha - \hat{p}_\alpha - \hat{q}_1) + (\hat{p}'_4 - \hat{p}_4 - \hat{q}_\beta) \\ &= -(\hat{p}_\alpha + \hat{q}_\beta) + \hat{p}'_\alpha + \hat{p}'_4 + (\hat{p}_2 - \hat{q}_1 - \hat{q}_3) - (\hat{p}_4 - \hat{q}_3).\end{aligned}\quad (11)$$

Rearranging terms in the same way as in obtaining (10) from (9), we get from the final identity in the last equation

$$\begin{aligned}\hat{p}_\mu &\equiv -\hat{p}_2 + \hat{p}'_\alpha + \hat{p}'_4 \\ &= \hat{p}_\alpha + \hat{q}_\beta - (\hat{p}_2 - \hat{q}_1 - \hat{q}_3) + (\hat{p}_4 - \hat{q}_3),\end{aligned}\quad (12)$$

where the terms in the parentheses in the last expression are the noise operators $\hat{\delta}_2$ and $\hat{\delta}_4$ of (6). For the other output mode ν , due to the symmetry between the modes in Fig. 2, one can obtain similar equations by exchanging the indices $1 \leftrightarrow 4$, $2 \leftrightarrow 3$, and $\alpha \leftrightarrow \beta$ in (10) and (12), and arrive at

$$\begin{aligned}\hat{q}_\nu &\equiv -\hat{q}_3 + \hat{p}'_4 \\ &= \hat{q}_\beta + (\hat{p}_4 - \hat{q}_3), \\ \hat{p}_\nu &\equiv -\hat{p}_3 + \hat{p}'_\beta + \hat{p}'_1 \\ &= \hat{p}_\beta + \hat{q}_\alpha - (\hat{p}_3 - \hat{q}_2 - \hat{q}_4) + (\hat{p}_1 - \hat{q}_2).\end{aligned}\quad (13)$$

As the first line in each of the results (10), (12), and (13) suggests, the CZ-gate teleportation here requires measurements for the quadrature operators \hat{p}'_α , \hat{p}'_β , \hat{p}'_1 , and \hat{p}'_4 . Supposing that the measurement results are, respectively, s_α , s_β , s_1 , and s_4 , the first lines of (10), (12), and (13) indicate that the corrective operations necessary for nodes 2 and 3 are

$$\hat{X}_2(s_1) \hat{Z}_2(s_\alpha + s_4) \hat{F}_2^2 \hat{X}_3(s_4) \hat{Z}_3(s_\beta + s_1) \hat{F}_3^2. \quad (14)$$

Here $\hat{X}_k \equiv e^{-2i\hat{p}_k s}$ and $\hat{Z}_k \equiv e^{2i\hat{q}_k s}$ are the Weyl-Heisenberg operators, and $\hat{F}_k \equiv e^{i\frac{\pi}{2}\hat{a}_k^2}$ is the Fourier transform operator for mode k that rotates its quadrature operators clockwise by an angle $(\pi/2)$ over the corresponding (quantum mechanical) phase space [31,32]. It should be noted that due to the finitely squeezed cluster state, the teleported CZ gate is imperfect. As one can read from the second lines of (10), (12), and (13), here the quadrature operators for the output modes differ from the ideal results [i.e., the image of the mapping in (7)] by excess-noise contributions.

The example considered above is a relatively simple one, which, nevertheless, already shows the simplicity of our Heisenberg approach. To prepare for more elaborate tasks, let us explain in greater details the tricks behind the manipulations of quadrature operators using the case of \hat{q}_μ above. First, as pointed out earlier, one must notice that in the desired mapping (7), \hat{q}_μ is constructed out of \hat{q}_2 . Thus, we start by writing

$$\hat{q}_2 = \hat{q}_2 + \dots, \quad (15)$$

where the dots represent identities added from the entries of the input-coupling equation (8) which serve two purposes: (i) They must include the image of \hat{q}_μ for the mapping (7),

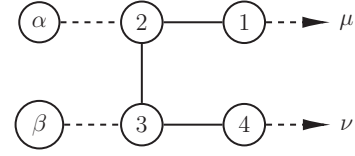


FIG. 3. CZ-gate teleportation using a linear four-mode cluster with a node arrangement different from that of Fig. 2.

which is \hat{q}_α here, and (ii) they must help “eliminate” \hat{q}_2 on the right-hand side of (15) by forming combination of terms that constitutes a nullifier of (6). Clearly, of the \hat{q}'_α and \hat{p}'_1 entries of (8) that contain \hat{q}_α , it is the latter that shall meet both demands (i) and (ii) here. It is then easy to arrive at (9) and then (10) accordingly. Similarly, the construction for \hat{p}_μ in (12) can be achieved in the same manner, except that two identities from the input-coupling equation (8) and two nullifiers of (6) are invoked in this case.

As it turns out, when applying these tricks to different cluster structures and input-mode couplings, there exist four major categories:

(a) trivial cases: for which the quadrature manipulations can be carried out easily;

(b) FT-input cases: for which the quadrature manipulations require Fourier transformations (FT) over the input modes prior to input couplings;

(c) FT-output cases: for which the quadrature manipulations require Fourier transformations over the output cluster nodes at the output stage; and

(d) FT-cluster cases: for which the quadrature manipulations require Fourier transformations over some or all of the cluster nodes prior to input couplings.

It should be noted that these four categories are not mutually exclusive, and the association of any gate teleportation to these categories is also not unique. For instance, we shall now illustrate with a case which can be treated as a mixture of categories (b) and (c), or purely that of (d). Let us consider again CZ-gate teleportation using a canonical linear four-mode cluster, but now with different input-mode connections as depicted in Fig. 3. As one can check through the prescriptions above, the quadrature manipulations in this case are no longer trivial [33], and one must consider possible scenarios with categories (b), (c), and (d) above.

After trial and error, one can find that the CZ-gate teleportation in the arrangement of Fig. 3 can be achieved by first applying inverse Fourier transforms to the input modes α , β before they are coupled to the cluster nodes:

$$\begin{pmatrix} \hat{q}_\rho \\ \hat{p}_\rho \end{pmatrix} \xrightarrow{\hat{F}_\rho^\dagger} \begin{pmatrix} \hat{q}'_\rho \\ \hat{p}'_\rho \end{pmatrix} = \begin{pmatrix} \hat{p}_\rho \\ -\hat{q}_\rho \end{pmatrix} \quad (16)$$

with $\rho = \alpha, \beta$. Subsequently, the Fourier-transformed input modes are coupled to the cluster nodes 2 and 3 via QND gates and yield [cf. (8)]

$$\begin{aligned}\begin{pmatrix} \hat{q}''_\rho \\ \hat{p}''_\rho \end{pmatrix} &= \begin{pmatrix} \hat{q}'_\rho \\ \hat{p}'_\rho + \hat{q}_k \end{pmatrix} = \begin{pmatrix} \hat{p}_\rho \\ -\hat{q}_\rho + \hat{q}_k \end{pmatrix}, \\ \begin{pmatrix} \hat{q}'_k \\ \hat{p}'_k \end{pmatrix} &= \begin{pmatrix} \hat{q}_k \\ \hat{p}_k + \hat{q}'_\rho \end{pmatrix} = \begin{pmatrix} \hat{q}_k \\ \hat{p}_k + \hat{p}_\rho \end{pmatrix},\end{aligned}\quad (17)$$

where the subscripts are $(\rho, k) = (\alpha, 2)$ and $(\beta, 3)$. Like previously, in view of the CZ-gate mapping (7) and because the output mode μ corresponds to the corrected cluster mode 1, one can proceed similarly to what was done in (9). However, things are a little more intricate here. It transpires that Fourier transformations over the output cluster nodes are also necessary when producing the output modes here. Therefore, for node 1, in place of $\hat{q}_1 \rightarrow \hat{q}_\alpha$, we have here $\hat{p}_1 \rightarrow \hat{q}_\alpha$ when supplemented with corrections. According to the prescriptions (i) and (ii) described after (15), we can make use of the \hat{p}_α'' entry of (17) and write

$$\hat{p}_1 = \hat{p}_1 + (\hat{p}_\alpha'' + \hat{q}_\alpha - \hat{q}_2) = \hat{q}_\alpha + \hat{p}_\alpha'' + (\hat{p}_1 - \hat{q}_2). \quad (18)$$

The last equality thus suggests

$$\begin{aligned} \hat{q}_\mu &\equiv \hat{p}_1 - \hat{p}_\alpha'' \\ &= \hat{q}_\alpha + (\hat{p}_1 - \hat{q}_2) \end{aligned} \quad (19)$$

with the terms in the parentheses being the noise operator $\hat{\delta}_1$. Similarly, due to the Fourier transformation, instead of $\hat{p}_1 \rightarrow (\hat{p}_\alpha + \hat{q}_\beta)$, we expect here $-\hat{q}_1 \rightarrow (\hat{p}_\alpha + \hat{q}_\beta)$ when corrections are applied. We can thus write by way of the \hat{p}_2' and the \hat{p}_β'' entries of (17)

$$\begin{aligned} \hat{q}_1 &= \hat{q}_1 + (\hat{p}_2' - \hat{p}_2 - \hat{p}_\alpha) - (\hat{p}_\beta'' + \hat{q}_\beta - \hat{q}_3) \\ &= -(\hat{p}_\alpha + \hat{q}_\beta) - \hat{p}_\beta'' + \hat{p}_2' - (\hat{p}_2 - \hat{q}_1 - \hat{q}_3). \end{aligned} \quad (20)$$

Rearranging terms in the last identity above thus yields

$$\begin{aligned} \hat{p}_\mu &\equiv -\hat{q}_1 - \hat{p}_\beta'' + \hat{p}_2' \\ &= \hat{p}_\alpha + \hat{q}_\beta + (\hat{p}_2 - \hat{q}_1 - \hat{q}_3), \end{aligned} \quad (21)$$

where the term in the parentheses in the last expression is the noise operator $\hat{\delta}_2$.

As previously, the symmetry in the arrangement of Fig. 3 allows us to write down the formulas for the output mode ν easily from (19) and (21). We find

$$\begin{aligned} \hat{q}_\nu &\equiv \hat{p}_4 - \hat{p}_\beta'' \\ &= \hat{q}_\beta + (\hat{p}_4 - \hat{q}_3), \\ \hat{p}_\nu &\equiv -\hat{q}_4 - \hat{p}_\alpha'' + \hat{p}_3' \\ &= \hat{p}_\beta + \hat{q}_\alpha + (\hat{p}_3 - \hat{q}_2 - \hat{q}_4). \end{aligned} \quad (22)$$

Therefore, according to the first lines of the results in (19), (21), and (22), the CZ-gate teleportation in the setting of Fig. 3 requires measurements for the quadrature operators \hat{p}_α'' , \hat{p}_β'' , \hat{p}_2' , and \hat{p}_3' . If the corresponding measurement outcomes are, respectively, s_α , s_β , s_2 , and s_3 , the first lines of (19), (21), and (22) indicate that the corrective operations needed for the cluster modes 1 and 4 are

$$\hat{X}_1^\dagger(s_\alpha) \hat{Z}_1^\dagger(s_\beta - s_2) \hat{F}_1^\dagger \hat{X}_4^\dagger(s_\beta) \hat{Z}_4^\dagger(s_\alpha - s_3) \hat{F}_4^\dagger. \quad (23)$$

This furnishes the scenario for the CZ-gate teleportation using a canonical linear four-mode cluster in the arrangement of Fig. 3. We see that the quadrature manipulations here have invoked the Fourier transformations stipulated for categories (b) and (c). As pointed out earlier, one can also treat this case following the scheme of category (d); this is illustrated in Appendix A.

Although the Heisenberg approach presented above does seem less systematic than the Schrödinger approach [13].

However, with the limited number of categories listed above, usually trial and error can quickly bring forward the correct teleportation scenarios, and the calculation requires much less algebra than the Schrödinger approach does. This is especially appealing when large clusters are involved in the teleportation process, such as the cases that we will study in the following section.

We have so far been focusing on establishing the procedures for CZ-gate teleportation through given cluster designs. An important issue next is then how one can quantify the quality of the teleported CZ gate. Since the CZ gate is capable of entangling its two input modes, we will quantify the quality of the teleported gate through the entanglement in the output modes when nonentangled input modes are supplied. Although entanglement quantification for general CV states remains a major challenge [15,34], the teleported states that will concern us in this work belong entirely to the class of two-mode Gaussian states. Their entanglement properties can be fully quantified through partial transposition of their covariance matrices [35,36]. This is based on the fact that for any Gaussian state all symplectic eigenvalues of its covariance matrix cannot be less than $1/4$ [15,37]. For a two-mode Gaussian state, suppose the partial transposition of its covariance matrix has symplectic eigenvalues λ_\pm with $\lambda_- \leq \lambda_+$. Since $\lambda_+ \geq 1/4$ always [38], here the physicality of the partially transposed covariance matrix is determined entirely by the smaller symplectic eigenvalue λ_- , and a measure for the degree of entanglement for the state is provided by the logarithmic negativity (or log-negativity, for short) [39]

$$E_N = -\ln(\min\{1, 4\lambda_-\}), \quad (24)$$

where the function $\min(x, y)$ yields the smaller of x and y . For entangled two-mode Gaussian states, violation of physicality upon partial transposition is then signaled by $\lambda_- < 1/4$, which results in nonzero, positive E_N according to (24).

If we denote the output quadratures of the teleported CZ gate in the form of a column vector $\hat{\xi} \equiv (\hat{q}_\mu, \hat{p}_\mu, \hat{q}_\nu, \hat{p}_\nu)^T$, the covariance matrix for the output modes μ, ν is then a real, symmetric 4×4 matrix with elements [29]

$$V_{kl} \equiv \frac{1}{2} \langle \{\Delta \hat{\xi}_k, \Delta \hat{\xi}_l\} \rangle, \quad (25)$$

where $k, l = 1 \sim 4$, $\Delta \hat{\xi}_k \equiv \hat{\xi}_k - \langle \hat{\xi}_k \rangle$, and $\{A, B\} \equiv AB + BA$. Since we are using the Heisenberg picture for the time evolution, the expectation values here are evaluated with respect to the initial state of the system. As it emerges, the covariance matrices for the output modes μ, ν for all cases considered in this work share an ‘‘X form’’

$$V = \begin{pmatrix} a & 0 & 0 & c \\ 0 & b & c & 0 \\ 0 & c & a & 0 \\ c & 0 & 0 & b \end{pmatrix} \quad (26)$$

with, according to (25),

$$a = \langle (\Delta \hat{q}_\sigma)^2 \rangle, \quad b = \langle (\Delta \hat{p}_\sigma)^2 \rangle, \quad c = \langle \{\Delta \hat{q}_\sigma, \Delta \hat{p}_{\sigma'}\} \rangle / 2, \quad (27)$$

where $\sigma, \sigma' = \mu, \nu$, and $\sigma' \neq \sigma$ in the expression for c . The partial transposition for the covariance matrix (26) with respect to, say, the output mode ν is effected through replacing in (27) $\hat{q}_\nu \rightarrow +\hat{q}_\nu$ and $\hat{p}_\nu \rightarrow -\hat{p}_\nu$ (that is, ‘‘time reversal’’ in mode

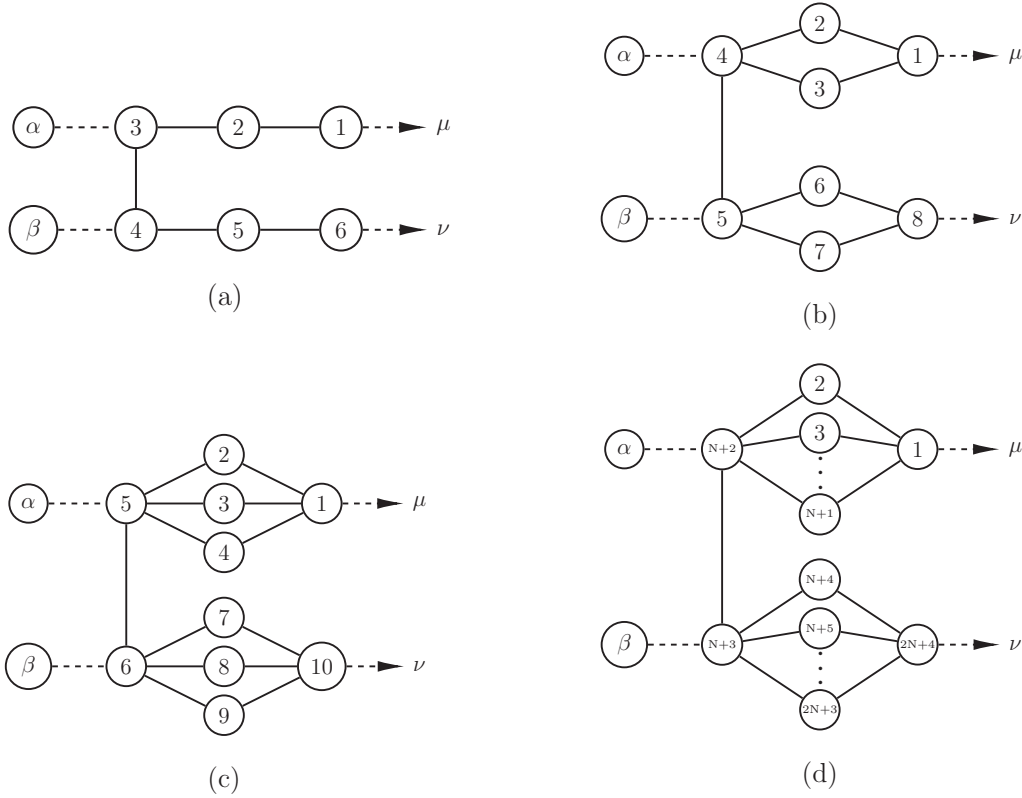


FIG. 4. CZ-gate teleportation using clusters with multirail designs of (a) single-rail, (b) two-rail, (c) three-rail, and (d) N -rail structures. Namely, the “teleporting arms” (i.e., the segments of nodes 3–2–1 and 4–5–6) of the single-rail cluster in panel (a) are replaced with multirail structures in panels (b)–(d) according to the designs of Fig. 1. Here the dashed lines and the arrows have the same meanings as those in Figs. 2 and 3.

ν [35]). The symplectic eigenvalues of the partially transposed covariance matrix are then found to be [38]

$$\lambda_{\pm} = |\sqrt{ab} \pm c|. \quad (28)$$

Substituting the expression for λ_- into (24), we can thus write the log-negativity

$$E_N = \max\{0, -\ln(4|\sqrt{ab} - c|)\} \quad (29)$$

with $\max\{x, y\}$ being the larger of x and y .

As an illustration, let us look back at the CZ-gate teleported in the arrangement of Fig. 3 with a canonical linear four-mode cluster. Suppose the input modes α, β are independent coherent states; one then has $\langle(\Delta\hat{q}_\rho)^2\rangle = \langle(\Delta\hat{p}_\rho)^2\rangle = 1/4$ and $\langle\{\Delta\hat{q}_\rho, \Delta\hat{p}_\rho\}\rangle = 0$ for both $\rho = \alpha, \beta$. If the cluster state had been prepared from momentum-squeezed vacuum states with identical squeezing parameter r , so that $\langle\hat{\delta}_k^2\rangle = e^{-2r}/4$ for all $k = 1 \sim 4$ in (6). Using the results (19), (21), and (22), one can obtain the quadrature correlators (27)

$$a = \frac{(1 + e^{-2r})}{4}, \quad b = \frac{(2 + e^{-2r})}{4}, \quad c = \frac{1}{4}. \quad (30)$$

It then follows from (29) that the log-negativity here reads

$$E_{N,L4} = \max\{0, -\ln(\sqrt{(1 + e^{-2r})(2 + e^{-2r})} - 1)\}, \quad (31)$$

where the additional subscript “L4” indicates the “linear four-mode.” One can check that for $r < \ln(\frac{\sqrt{17}+3}{2}) \simeq 0.29$ the log-negativity $E_{N,L4}$ vanishes identically. In other words, the presence of the excess noise in the teleported CZ gate here

has entirely corrupted the entanglement in the output modes for finite range of squeezing levels. We will examine in the following section how the multirail design shown in Fig. 1 can help reduce the excess noise and hence improve entanglement in the output modes of the teleported CZ gate.

III. CONTROLLED-PHASE (CZ) GATE TELEPORTATION USING MULTIRAIL CLUSTERS

The noise-reduction scheme through multirail designs shown in Fig. 1 was originally proposed in Ref. [20] for the teleportation of single-qumode gates. For the CZ gate, since there are two input qumodes, it is necessary to implement the multirail structure for each of the “teleporting arms,” such as the segments of nodes 2–1 and nodes 3–4 in the case of Fig. 3. As is clear from Fig. 1, the minimal cluster for the multirail design requires three nodes for each arm. Therefore, for CZ-gate teleportation, we will start with a linear six-mode cluster and then increase the complexity of the cluster by building the multirail structure into the arms as shown in Fig. 4.

In what follows, we will consider two classes of resource states for the CZ-gate teleportation. The first is the canonical cluster states that are fabricated through QND gates, as we have already considered in the previous section. The second will be a class of cluster states proposed by van Loock and coworkers [20] which can be constructed using linear-optical networks and thus will be termed the linear-optical cluster states in the following. As we will see, these two classes of cluster states

possess different noise structures and hence are not identical at finite squeezing. However, in the ideal, infinite-squeezing limit, they become completely identical since all excess noise tend to zero in this limit. Therefore, at finite squeezing, we expect to see different entanglement properties in the teleported CZ gates when different classes of cluster states are employed for the teleportation, as we shall now investigate.

A. Canonical cluster states

Let us start by considering CZ-gate teleportation using a canonical linear six-mode cluster as illustrated in Fig. 4(a), which corresponds to a “single-rail” design. The quadrature manipulations in this case are very similar to those for the linear four-mode cluster of Fig. 3 demonstrated in the previous section. Again, it is necessary to subject the input modes α, β to the Fourier transformations (16) before they are coupled with the cluster nodes. Subsequent QND couplings between the Fourier-transformed input modes α', β' and the cluster nodes 3, 4, respectively, thus yield again (17) with here the sets of subscripts $(\rho, k) = (\alpha, 3)$ and $(\beta, 4)$. By making use of these coupling equations and the nullifiers for the linear six-mode cluster, one can obtain in the same manner as before

$$\begin{aligned}\hat{q}_\mu &\equiv -\hat{q}_1 - \hat{p}_\alpha'' + \hat{p}_2 \\ &= \hat{q}_\alpha + (\hat{p}_2 - \hat{q}_1 - \hat{q}_3), \\ \hat{p}_\mu &\equiv -\hat{p}_1 - \hat{p}_\beta'' + \hat{p}_3' \\ &= \hat{p}_\alpha + \hat{q}_\beta - (\hat{p}_1 - \hat{q}_2) + (\hat{p}_3 - \hat{q}_2 - \hat{q}_4),\end{aligned}\quad (32)$$

and

$$\begin{aligned}\hat{q}_\nu &\equiv -\hat{q}_6 - \hat{p}_\beta'' + \hat{p}_5 \\ &= \hat{q}_\beta + (\hat{p}_5 - \hat{q}_4 - \hat{q}_6), \\ \hat{p}_\nu &\equiv -\hat{p}_6 - \hat{p}_\alpha'' + \hat{p}_4' \\ &= \hat{p}_\beta + \hat{q}_\alpha - (\hat{p}_6 - \hat{q}_5) + (\hat{p}_4 - \hat{q}_3 - \hat{q}_5).\end{aligned}\quad (33)$$

Accordingly, from the first line in each of the results for the output quadratures above, we see that here the CZ-gate teleportation calls for measurements over the momentum operators $\hat{p}_\alpha'', \hat{p}_\beta'', \hat{p}_2, \hat{p}_3', \hat{p}_4'$, and \hat{p}_5 . Suppose the corresponding measurement outcomes are, respectively, $s_\alpha, s_\beta, s_2, s_3, s_4$, and s_5 ; then the gate teleportation can then be achieved upon correcting the cluster nodes 1 and 6 with the operations

$$\hat{X}_1(s_2 - s_\alpha) \hat{Z}_1(s_3 - s_\beta) \hat{F}_1^2 \hat{X}_6(s_5 - s_\beta) \hat{Z}_6(s_4 - s_\alpha) \hat{F}_6^2, \quad (34)$$

which have been read off from (32) and (33) in the same way as detailed previously for the cases of linear four-mode clusters.

We next turn to the multirail variants of the linear six-mode cluster. As explained in the beginning of this section, in order to implement the multirail design of Fig. 1 into the linear six-mode cluster in Fig. 4(a), we replace each of the teleporting arms (i.e., the segments 3–2–1 and 4–5–6) of the cluster with multi-rail structures as in Figs. 4(b)–4(d). For the two-rail cluster of Fig. 4(b), we find that the CZ-gate teleportation can again be achieved by first applying the Fourier transformations (16) over the input modes α, β before they are coupled to the cluster. After that, QND couplings between the Fourier-transformed input modes α', β' and the cluster nodes 4, 5, respectively, again lead to (17) with the subscripts now $(\rho, k) = (\alpha, 4)$ and $(\beta, 5)$. With the help of these

coupling equations and the nullifiers for the present cluster, it is straightforward to establish the following through the same tricks as before:

$$\begin{aligned}\hat{q}_\mu &\equiv -\hat{q}_1 - \hat{p}_\alpha'' + \frac{\hat{p}_2 + \hat{p}_3}{2} \\ &= \hat{q}_\alpha + \frac{1}{2}[(\hat{p}_2 - \hat{q}_1 - \hat{q}_4) + (\hat{p}_3 - \hat{q}_1 - \hat{q}_4)], \\ \hat{p}_\mu &\equiv -\hat{p}_1 - \hat{p}_\beta'' + \hat{p}_4' \\ &= \hat{p}_\alpha + \hat{q}_\beta - (\hat{p}_1 - \hat{q}_2 - \hat{q}_3) + (\hat{p}_4 - \hat{q}_2 - \hat{q}_3 - \hat{q}_5),\end{aligned}\quad (35)$$

and

$$\begin{aligned}\hat{q}_\nu &\equiv -\hat{q}_8 - \hat{p}_\beta'' + \frac{\hat{p}_6 + \hat{p}_7}{2} \\ &= \hat{q}_\beta + \frac{1}{2}[(\hat{p}_6 - \hat{q}_5 - \hat{q}_8) + (\hat{p}_7 - \hat{q}_5 - \hat{q}_8)], \\ \hat{p}_\nu &\equiv -\hat{p}_8 - \hat{p}_\alpha'' + \hat{p}_5' \\ &= \hat{p}_\beta + \hat{q}_\alpha - (\hat{p}_8 - \hat{q}_6 - \hat{q}_7) + (\hat{p}_5 - \hat{q}_4 - \hat{q}_6 - \hat{q}_7).\end{aligned}\quad (36)$$

Here, however, it should be noted that in (35) and (36), one can in general set for \hat{q}_μ and \hat{q}_ν

$$\begin{aligned}\hat{q}_\mu &\equiv -\hat{q}_1 - \hat{p}_\alpha'' + (\eta_1 \hat{p}_2 + \eta_2 \hat{p}_3) \\ &= \hat{q}_\alpha + [\eta_1 (\hat{p}_2 - \hat{q}_1 - \hat{q}_4) + \eta_2 (\hat{p}_3 - \hat{q}_1 - \hat{q}_4)], \\ \hat{q}_\nu &\equiv -\hat{q}_8 - \hat{p}_\beta'' + (\eta_3 \hat{p}_6 + \eta_4 \hat{p}_7) \\ &= \hat{q}_\beta + [\eta_3 (\hat{p}_6 - \hat{q}_5 - \hat{q}_8) + \eta_4 (\hat{p}_7 - \hat{q}_5 - \hat{q}_8)],\end{aligned}\quad (37)$$

where $\eta_1 + \eta_2 = 1$ and $\eta_3 + \eta_4 = 1$, so that in each equation the first line would always be identical to the second. Clearly, since both $(\hat{q}_\mu - \hat{q}_\alpha)$ and $(\hat{q}_\nu - \hat{q}_\beta)$ in (37) remain to yield linear combinations of excess-noise operators, these choices of \hat{q}_μ and \hat{q}_ν should also serve well for the CZ-gate teleportation. However, as one can prove easily, for equally squeezed initial cluster modes, the symmetrical arrangement $\eta_k = 1/2$ for all $k = 1 \sim 4$ in (37) [i.e., corresponding to (35) and (36)] would minimize the excess noise in the teleported \hat{q}_μ and \hat{q}_ν . This is also the case for the three-rail design that we shall discuss shortly, and is in fact the reason behind the noise reduction in this multirail approach. For instance, the choice $\eta_{1,3} = 1$ and $\eta_{2,4} = 0$ in (37) would yield \hat{q}_μ and \hat{q}_ν identical to those for the single-rail cluster in (32) and (33), respectively, except for change of indices in the nodes. In this case, there would thus be no any noise reduction in the teleportation despite the implemented multirail structure.

For the case of the three-rail cluster illustrated in Fig. 4(c), the calculation proceeds almost identically to that for the two-rail case above, apart from modifications in the node indices. Instead of repeating similar details, here we list only the results that we find:

$$\begin{aligned}\hat{q}_\mu &\equiv -\hat{q}_1 - \hat{p}_\alpha'' + \frac{\hat{p}_2 + \hat{p}_3 + \hat{p}_4}{3} \\ &= \hat{q}_\alpha + \frac{1}{3}[(\hat{p}_2 - \hat{q}_1 - \hat{q}_5) + (\hat{p}_3 - \hat{q}_1 - \hat{q}_5) \\ &\quad + (\hat{p}_4 - \hat{q}_1 - \hat{q}_5)],\end{aligned}$$

$$\begin{aligned}\hat{p}_\mu &\equiv -\hat{p}_1 - \hat{p}_\beta'' + \hat{p}_5' \\ &= \hat{p}_\alpha + \hat{q}_\beta - (\hat{p}_1 - \hat{q}_2 - \hat{q}_3 - \hat{q}_4) \\ &\quad + (\hat{p}_5 - \hat{q}_2 - \hat{q}_3 - \hat{q}_4 - \hat{q}_6),\end{aligned}\quad (38)$$

and

$$\begin{aligned}\hat{q}_v &\equiv -\hat{q}_{10} - \hat{p}_\beta'' + \frac{\hat{p}_7 + \hat{p}_8 + \hat{p}_9}{3} \\ &= \hat{q}_\beta + \frac{1}{3}[(\hat{p}_7 - \hat{q}_6 - \hat{q}_{10}) + (\hat{p}_8 - \hat{q}_6 - \hat{q}_{10}) \\ &\quad + (\hat{p}_9 - \hat{q}_6 - \hat{q}_{10})], \\ \hat{p}_v &\equiv -\hat{p}_{10} - \hat{p}_\alpha'' + \hat{p}_6' \\ &= \hat{p}_\beta + \hat{q}_\alpha - (\hat{p}_{10} - \hat{q}_7 - \hat{q}_8 - \hat{q}_9) \\ &\quad + (\hat{p}_6 - \hat{q}_5 - \hat{q}_7 - \hat{q}_8 - \hat{q}_9).\end{aligned}\quad (39)$$

As for the two-rail case above, here one could in general put \hat{q}_μ and \hat{q}_v in the form

$$\begin{aligned}\hat{q}_\mu &\equiv -\hat{q}_1 - \hat{p}_\alpha'' + (\eta_1 \hat{p}_2 + \eta_2 \hat{p}_3 + \eta_3 \hat{p}_4) \\ &= \hat{q}_\alpha + [\eta_1 (\hat{p}_2 - \hat{q}_1 - \hat{q}_5) + \eta_2 (\hat{p}_3 - \hat{q}_1 - \hat{q}_5) \\ &\quad + \eta_3 (\hat{p}_4 - \hat{q}_1 - \hat{q}_5)], \\ \hat{q}_v &\equiv -\hat{q}_{10} - \hat{p}_\beta'' + (\eta_4 \hat{p}_7 + \eta_5 \hat{p}_8 + \eta_6 \hat{p}_9) \\ &= \hat{q}_\beta + [\eta_4 (\hat{p}_7 - \hat{q}_6 - \hat{q}_{10}) + \eta_5 (\hat{p}_8 - \hat{q}_6 - \hat{q}_{10}) \\ &\quad + \eta_6 (\hat{p}_9 - \hat{q}_6 - \hat{q}_{10})],\end{aligned}\quad (40)$$

where $\eta_1 + \eta_2 + \eta_3 = 1$ and $\eta_4 + \eta_5 + \eta_6 = 1$, so that each identity would always hold. Nevertheless, one can again show easily that it is the symmetrical choice $\eta_k = 1/3$ for all $k = 1 \sim 6$ of (38) and (39) that would minimize the excess noise in the teleported \hat{q}_μ and \hat{q}_v had the cluster been prepared from uniformly squeezed vacuum states. We will prove below that this result can indeed be generalized to arbitrary N -rail design for the canonical cluster.

With the foregoing results, it is straightforward to extend the consideration to a design with an arbitrary number of rail. For the N -rail canonical cluster in Fig. 4(d), from Eqs. (32), (33), (35), (36), (38), and (39), one can write down inductively for the output modes of the teleported CZ gate

$$\begin{aligned}\hat{q}_\mu &= \hat{q}_\alpha + \frac{1}{N} \sum_{k=2}^{N+1} \hat{\delta}_k, \\ \hat{p}_\mu &= \hat{p}_\alpha + \hat{q}_\beta - \hat{\delta}_1 + \hat{\delta}_{N+2}, \\ \hat{q}_v &= \hat{q}_\beta + \frac{1}{N} \sum_{k=N+4}^{2N+3} \hat{\delta}_k, \\ \hat{p}_v &= \hat{p}_\beta + \hat{q}_\alpha - \hat{\delta}_{2N+4} + \hat{\delta}_{N+3}.\end{aligned}\quad (41)$$

Note that here we have omitted the parts of the equations that would reveal the measurement sequence and corrective operations for the teleportation [namely, corresponding to the first lines of the results (32), (33), and etc.]. This is because here we are concerned primarily with the entanglement properties of the output modes, and thus need only the parts of the equations listed in (41). Also note that we have expressed the formulas here in terms of the noise operators $\hat{\delta}_k$ instead of

the quadrature operators, as this will be more convenient for evaluating the quadrature correlators (27).

Like previously for the case of Fig. 3 with a canonical linear four-mode cluster, we shall suppose that the input modes α, β here are independent coherent states and that the cluster nodes are uniformly squeezed with squeezing parameter r . Moreover, according to (6), the noise operators $\hat{\delta}_k$ for canonical cluster states are independent from each other. We can thus find from (41) the quadrature correlators (27) for the output modes of the teleported CZ gate

$$a = \frac{(1 + \frac{1}{N}e^{-2r})}{4}, \quad b = \frac{(1 + e^{-2r})}{2}, \quad c = \frac{1}{4}. \quad (42)$$

It should be noted that the correlators b and c are both N -independent here. One can now obtain immediately from (29) the log-negativity for the output modes

$$E_{N,NR} = \max \left\{ 0, -\ln \left(\sqrt{2 \left(1 + \frac{e^{-2r}}{N} \right) (1 + e^{-2r})} - 1 \right) \right\} \quad (43)$$

with the subscript NR standing for “ N rail.” Note that here the result for the two-rail ($N = 2$) case is identical to that for the linear four-mode cluster (31) in the previous section. We will defer further analysis for the result (43) until the end of the next subsection, where comparisons between the results from two classes of resource states will be made. Before closing this subsection, let us look back at the expressions for \hat{q}_μ and \hat{q}_v in (41). Like previously for the two-rail and the three-rail cases, the coefficient $1/N$ in front of the summation over the noise operators $\hat{\delta}_k$ in fact corresponds to the optimal choice for reducing the excess noise in the teleported \hat{q}_μ and \hat{q}_v when the cluster is uniformly squeezed. A generic expression for \hat{q}_μ and \hat{q}_v in the present N -rail teleportation takes the form

$$\hat{q}_\sigma = \hat{q}_\rho + \sum_k \eta_k \hat{\delta}_k, \quad (44)$$

where $(\sigma, \rho) = (\mu, \alpha)$ or (v, β) , and the summation has the same range as in (41). For the same reason as in (37) and (40), here we must constrain the coefficients η_k such that $\sum_k \eta_k = 1$. Since the noise operators $\hat{\delta}_k$ are independent from each other, the excess noise in the teleported \hat{q}_σ is simply $\langle (\hat{q}_\sigma - \hat{q}_\rho)^2 \rangle = \sum_k \eta_k^2 \langle \hat{\delta}_k^2 \rangle$. For uniformly squeezed clusters $\langle \hat{\delta}_k^2 \rangle$ has the same value for all k ; thus the minimization for the excess noise is the same as that for the sum $\sum_k \eta_k^2$. Geometrically, this amounts to finding the point on the N -dimensional hyperplane $\sum_k \eta_k = 1$ which has the shortest (Euclidean) distance to the origin. The answer is clearly the “symmetric point” with $\eta_k = 1/N$ for all k . A rigorous derivation for this result is elementary, for instance, based on the method of Lagrange multipliers. We note that for \hat{q}_σ in (44) the summation over k covers all midrail nodes in each set of the multirails, i.e., nodes $k = 2 \sim (N + 1)$ for the set with $\sigma = \mu$ and $k = (N + 4) \sim (2N + 3)$ for the set with $\sigma = v$ in Fig. 4(d). Thus, the symmetric arrangement $\eta_k = 1/N$ in (44) corresponds to distributing the excess noise to each of the multirail with equal weight.

B. Linear-optical cluster states

The construction of canonical cluster states requires implementation of QND gates, which can be experimentally challenging especially when the cluster is large and the qumodes are encoded through spatially separated optical modes. In this context, van Loock and coworkers proposed in Ref. [20] an alternative route to preparing optical cluster states through offline squeezers and linear-optical networks of beam splitters and phase shifters. In essence, one has a set of spatially separated optical modes that are squeezed locally and sent through an optical network of passive elements, which is capable of entangling these modes and setting up correlations among their quadrature operators that are akin to those in canonical clusters. In particular, despite the more complicated noise operators $\hat{\delta}_k$ in this case (see below), in the limit of infinite squeezing, just like canonical cluster states, all $\hat{\delta}_k$ would tend to zero and the state would approach an ideal cluster state. These states have thus been dubbed “linear-optical cluster states” in the present paper.

In a linear-optical cluster state, the cluster correlations among its nodes are established through a unitary transformation associated with the action of a linear-optical network [18,40] over a set of offline squeezed initial modes. If the annihilation operators for these initial modes are \hat{a}_l , the linear-optical network induces the transformation

$$\hat{a}_k = \sum_l U_{kl} \hat{a}_l, \quad (45)$$

where U_{kl} is the k,l element of the corresponding unitary matrix U , and \hat{a}_k is the annihilation operator for the resultant cluster mode k . In pursuance of furnishing the cluster correlations, the k th row \vec{u}_k of U must take the particular form [20]

$$\vec{u}_k = \vec{\alpha}_k + i \sum_{l \in N_k} \vec{\alpha}_l, \quad (46)$$

where $\vec{\alpha}_k$ are real row vectors and N_k indicates the set of nodes in the cluster that are connected to node k . Unitarity of the matrix U entails orthonormality of the row vectors \vec{u}_k , which yields

$$\begin{aligned} \vec{\alpha}_k \vec{\alpha}_l^T + \sum_{m \in N_k, n \in N_l} \vec{\alpha}_m \vec{\alpha}_n^T &= \delta_{kl}, \\ \vec{\alpha}_k \sum_{n \in N_l} \vec{\alpha}_n^T - \sum_{m \in N_k} \vec{\alpha}_m \vec{\alpha}_l^T &= 0, \end{aligned} \quad (47)$$

where T signifies matrix transposition and δ_{kl} is Kronecker's δ . For any given geometry of the cluster, one can obtain the conditions $\vec{\alpha}_k$ (or, in fact, their inner products) must satisfy in accordance with (47). Writing

$$G_{kl} \equiv \vec{\alpha}_k \vec{\alpha}_l^T, \quad (48)$$

one can solve easily (though often tediously) from (47) the values of G_{kl} for all k,l . Since G_{kl} is symmetric in k and l , a cluster with M nodes would have $M(M+1)/2$ such “geometric constraints” for the $\vec{\alpha}_k$'s. Following these constraints, one can construct $\vec{\alpha}_k$ accordingly, and hence the row vectors \vec{u}_k of U through (46). Clearly, the choices for $\vec{\alpha}_k$ and thus U are not unique. As long as (47) is satisfied, the matrix U would always lead to the desired cluster

correlations for the linear-optical cluster state [20]. Details of these calculations for the linear-optical cluster states that will concern us are summarized in Appendix B. Here we just quote the expression for the noise operators [20]

$$\hat{\delta}_k = \sum_l \left(\alpha_{k,l} + \sum_{m \in N_k} \sum_{n \in N_m} \alpha_{n,l} \right) \hat{p}_l, \quad (49)$$

where \hat{q}_k, \hat{p}_k are quadrature operators for node k , $\alpha_{k,l}$ denotes the l th component of the vector $\vec{\alpha}_k$, and, as in (6), $\hat{p}_l = e^{-r_l} \hat{p}_l^{(0)}$ is the momentum operator for the offline squeezed initial mode l . It is clear that here the noise operators have much more complicated dependence on the initial mode operators than that in (6) for canonical cluster states. Nevertheless, in the ideal limit of infinite squeezing (i.e., $r_l \rightarrow \infty$ for every l), one would still recover the limit of ideal cluster states as previously for canonical cluster states. Therefore, we see that the unitary transformation (45) with row vectors of the form (46) does implement successfully the intended cluster correlations among the nodes of the cluster.

Let us turn now to the study of CZ-gate teleportation through linear-optical cluster states. As previously, we shall first illustrate the calculation utilizing the simple case of Fig. 3 with a linear four-mode cluster of the linear-optical type. In order to reduce the demand for squeezing resources [18], here the coupling between the input modes α, β and the cluster nodes will be achieved through beam-splitter couplings (via “Bell measurements” [30]), instead of the QND couplings used previously. Specifically, for the input mode $\rho = \alpha, \beta$ and the cluster node $k = 2, 3$ of Fig. 3, the coupling is effected through a 50:50 beam splitter, so that the annihilation operators for the modes transform as [31,32]

$$\begin{pmatrix} \hat{a}_{\rho_1} \\ \hat{a}_{\rho_2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \hat{a}_\rho \\ \hat{a}_k \end{pmatrix}, \quad (50)$$

where the subscripts are $(\rho, k) = (\alpha, 2)$ and $(\beta, 3)$, and hence $(\rho_1, \rho_2) = (\alpha_1, \alpha_2)$ and (β_1, β_2) , correspondingly. Namely, the beam splitter has coupled the input mode ρ and the cluster mode k in producing the modes ρ_1 and ρ_2 , which have the quadrature operators, according to (50),

$$\begin{pmatrix} \hat{q}_{\rho_1} \\ \hat{p}_{\rho_1} \end{pmatrix} = \begin{pmatrix} \frac{\hat{q}_\rho + \hat{q}_k}{\sqrt{2}} \\ \frac{\hat{p}_\rho + \hat{p}_k}{\sqrt{2}} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \hat{q}_{\rho_2} \\ \hat{p}_{\rho_2} \end{pmatrix} = \begin{pmatrix} \frac{\hat{q}_\rho - \hat{q}_k}{\sqrt{2}} \\ \frac{\hat{p}_\rho - \hat{p}_k}{\sqrt{2}} \end{pmatrix}. \quad (51)$$

For CZ-gate teleportation, like before for canonical clusters, again we wish to achieve the mapping (7) through quadrature-manipulation tricks by utilizing the input-coupling equation (51) and the cluster correlations (49). However, it should be noted that previously the input coupling (8) through QND gate had led to trivial transformations in both \hat{q}'_ρ and \hat{q}'_k entries [as they belong to the “controlled” part of the QND operation (4)]. It was thus quite obvious that one must employ the \hat{p}'_ρ and \hat{p}'_k entries there for quadrature manipulations, and consequently the measurement sequence involves invariably the \hat{p}' 's and never the \hat{q}' 's [41]. Here, the situation is more complicated since all entries of (51) are nontrivial and extra care is needed in determining the entries to be used (and hence the quadratures to be measured for the teleportation).

For instance, in the case of Fig. 3 with a linear-optical cluster, anticipating that $\hat{q}_1 \rightarrow \hat{q}_\alpha$ after corrective operations, one would write according to points (i) and (ii) listed below (15)

$$\begin{aligned}\hat{q}_1 &= \hat{q}_1 + \hat{q}_\alpha + \dots \\ &= \hat{q}_\alpha + \hat{q}_1 - \hat{p}_2 + \hat{q}_3 + \dots,\end{aligned}\quad (52)$$

where the dots in the second line remain to be fixed through entries of the input-coupling equation (51). However, it is clear from (51) that the added \hat{p}_2 must appear together with \hat{p}_α , no matter whether it is \hat{p}_{α_1} or \hat{p}_{α_2} that will be measured. Similarly, according to (51), the added term \hat{q}_3 in (52) must bring along \hat{q}_β to the equation. It is thus evident that this scheme will not work and one must attempt with different scenarios. For instance, if one attempt instead with $\hat{p}_1 \rightarrow \hat{q}_\alpha$ up to corrective operations, one can write using the \hat{q}_{α_2} entry of (51)

$$\begin{aligned}\hat{p}_1 &= \hat{p}_1 - (\sqrt{2}\hat{q}_{\alpha_2} - \hat{q}_\alpha + \hat{q}_2) \\ &= \hat{q}_\alpha - \sqrt{2}\hat{q}_{\alpha_2} + (\hat{p}_1 - \hat{q}_2).\end{aligned}\quad (53)$$

Recognizing the nullifier $(\hat{p}_1 - \hat{q}_2) = \hat{\delta}_1$ in the last line, one can thus assign

$$\begin{aligned}\hat{q}_\mu &\equiv \hat{p}_1 + \sqrt{2}\hat{q}_{\alpha_2} \\ &= \hat{q}_\alpha + (\hat{p}_1 - \hat{q}_2).\end{aligned}\quad (54)$$

This result indicates that Fourier transformation is needed here and thus suggests for the output quadrature \hat{p}_μ the mapping $\hat{q}_1 \rightarrow -(\hat{p}_\alpha + \hat{q}_\beta)$, subject to corrections. With this observation, we write, utilizing the \hat{p}_{α_1} and \hat{q}_{β_2} entries of (51),

$$\begin{aligned}\hat{q}_1 &= \hat{q}_1 + (\sqrt{2}\hat{p}_{\alpha_1} - \hat{p}_\alpha - \hat{p}_2) + (\sqrt{2}\hat{q}_{\beta_2} - \hat{q}_\beta + \hat{q}_3) \\ &= -(\hat{p}_\alpha + \hat{q}_\beta) + \sqrt{2}(\hat{p}_{\alpha_1} + \hat{q}_{\beta_2}) - (\hat{p}_2 - \hat{q}_1 - \hat{q}_3).\end{aligned}\quad (55)$$

Rearranging terms in the last expression yields immediately the desired equation

$$\begin{aligned}\hat{p}_\mu &\equiv -\hat{q}_1 + \sqrt{2}(\hat{p}_{\alpha_1} + \hat{q}_{\beta_2}) \\ &= \hat{p}_\alpha + \hat{q}_\beta + (\hat{p}_2 - \hat{q}_1 - \hat{q}_3)\end{aligned}\quad (56)$$

with the terms in the parentheses in the last line the nullifier $\hat{\delta}_2$. Exploiting the symmetry among the modes in Fig. 3, based on the results (54) and (56), one can write immediately for the other output mode ν

$$\begin{aligned}\hat{q}_\nu &\equiv \hat{p}_4 + \sqrt{2}\hat{q}_{\beta_2} \\ &= \hat{q}_\beta + (\hat{p}_4 - \hat{q}_3), \\ \hat{p}_\nu &\equiv -\hat{q}_4 + \sqrt{2}(\hat{p}_{\beta_1} + \hat{q}_{\alpha_2}) \\ &= \hat{p}_\beta + \hat{q}_\alpha + (\hat{p}_3 - \hat{q}_2 - \hat{q}_4).\end{aligned}\quad (57)$$

From the second line in each of the equations in (54), (56), and (57), we see that we have achieved the intended mapping (7) for the CZ gate, except for the presence of excess-noise terms $\hat{\delta}_k$ due to finite squeezing. One can also read off from the first lines of these equations that the CZ-gate teleportation here requires measurements over the quadrature operators \hat{p}_{α_1} , \hat{q}_{α_2} , \hat{p}_{β_1} , and \hat{q}_{β_2} . For measurement outcomes with, respectively, s_{α_1} , s_{α_2} , s_{β_1} , and s_{β_2} , one can find from the first lines of (54), (56), and (57) that the corrective operations

are here

$$\begin{aligned}&\hat{X}_1(\sqrt{2}s_{\alpha_2}) \hat{Z}_1[\sqrt{2}(s_{\alpha_1} + s_{\beta_2})] \\ &\times \hat{F}_1^\dagger \hat{X}_4(\sqrt{2}s_{\beta_2}) \hat{Z}_4[\sqrt{2}(s_{\beta_1} + s_{\alpha_2})] \hat{F}_4^\dagger.\end{aligned}\quad (58)$$

To quantify the quality of the teleported CZ gate, we calculate as ever the log-negativity (29) for the output modes μ and ν . For this purpose, it is necessary to find first the explicit forms for the noise operators $\hat{\delta}_k$, so that one can calculate the quadrature correlators (27) using (54), (56), and (57). Using the matrix U found for linear four-mode clusters in Appendix B, we get

$$\begin{aligned}\hat{\delta}_1 &= \sqrt{2}\hat{p}_1, \quad \hat{\delta}_2 = \frac{5}{\sqrt{10}}\hat{p}_2 + \frac{1}{\sqrt{2}}\hat{p}_4, \\ \hat{\delta}_3 &= \frac{1}{\sqrt{2}}\hat{p}_1 + \frac{5}{\sqrt{10}}\hat{p}_3, \quad \hat{\delta}_4 = \sqrt{2}\hat{p}_4.\end{aligned}\quad (59)$$

As noted following (49), the noise operators here have much more complicated forms than their canonical-cluster counterparts (6). This will also be seen for other linear-optical cluster states that we will consider later.

For the sake of calculating the quadrature correlators (27) explicitly, as before, we shall consider identically squeezed cluster modes with squeezing parameter r and independent coherent-state input modes. For the noise operators, we note that previously for canonical cluster states only the autocorrelators $\langle \hat{\delta}_k^2 \rangle$ are nonvanishing, while here, according to (59), there can be nonzero cross correlators $\langle \hat{\delta}_k \hat{\delta}_l \rangle$ with $k \neq l$. With this precaution, it is then not difficult to find the quadrature correlators (27) by means of (54), (56), and (57), together with (59) and get

$$a = \frac{(1 + 2e^{-2r})}{4}, \quad b = \frac{(2 + 3e^{-2r})}{4}, \quad c = \frac{(1 + e^{-2r})}{4}.\quad (60)$$

Substituting these results into (29), we find the log-negativity for the output modes

$$\begin{aligned}E_{N,L4} &= \max\{0, -\ln(\sqrt{(1 + 2e^{-2r})(2 + 3e^{-2r})} \\ &\quad - (1 + e^{-2r}))\},\end{aligned}\quad (61)$$

where the prime over the subscript $L4$ has been added to tell the cluster from its canonical counterpart earlier [cf. (31)]. As with canonical clusters, here excess noise due to finite squeezing has impaired the entanglement between the output modes μ and ν . In particular, one can check easily that the entanglement (61) in the output modes attained here is worse than that of (31) obtained previously for canonical linear four-mode cluster. It is therefore of interest to examine also the performance of the multirail noise reduction for teleportation with linear-optical clusters. Before delving into this analysis, on the grounds of the foregoing discussions, we note that the calculations for CZ-gate teleportation using linear-optical clusters are indeed very similar to those for canonical cluster states, except for the differences in the input coupling and the correlations in the noise operators. We will therefore be very brief with the calculations for each individual case in the following and take them merely as intermediate steps leading to the general results for CZ-gate teleportation through an arbitrary N -rail linear-optical cluster.

Let us start with the linear six-mode cluster of Fig. 4(a), where the input modes are now coupled to the cluster through two 50:50 beam splitters as in (50). The input coupling between the input modes α, β and the cluster nodes 3, 4, respectively, yields the consequential modes given by (51) with $(\rho, k) = (\alpha, 3)$ and $(\beta, 4)$. Making use of these coupling equations, the same tricks as those illustrated earlier for the linear four-mode cluster produce the results

$$\begin{aligned}\hat{q}_\mu &\equiv \hat{q}_1 - \hat{p}_2 + \sqrt{2}\hat{q}_{\alpha_1} \\ &= \hat{q}_\alpha - (\hat{p}_2 - \hat{q}_1 - \hat{q}_3), \\ \hat{p}_\mu &\equiv \hat{p}_1 + \sqrt{2}(\hat{p}_{\alpha_2} + \hat{q}_{\beta_1}) \\ &= \hat{p}_\alpha + \hat{q}_\beta + (\hat{p}_1 - \hat{q}_2) - (\hat{p}_3 - \hat{q}_2 - \hat{q}_4),\end{aligned}\quad (62)$$

and

$$\begin{aligned}\hat{q}_\nu &\equiv \hat{q}_6 - \hat{p}_5 + \sqrt{2}\hat{q}_{\beta_1} \\ &= \hat{q}_\beta - (\hat{p}_5 - \hat{q}_4 - \hat{q}_6), \\ \hat{p}_\nu &\equiv \hat{p}_6 + \sqrt{2}(\hat{q}_{\alpha_1} + \hat{p}_{\beta_2}) \\ &= \hat{p}_\beta + \hat{q}_\alpha + (\hat{p}_6 - \hat{q}_5) - (\hat{p}_4 - \hat{q}_3 - \hat{q}_5).\end{aligned}\quad (63)$$

By means of the matrix U found for linear six-mode clusters in Appendix B, one can arrive at the noise operators from (49)

$$\begin{aligned}\hat{\delta}_1 &= \sqrt{2}\hat{p}_1, \quad \hat{\delta}_2 = \sqrt{3}\hat{p}_2, \\ \hat{\delta}_3 &= \frac{1}{\sqrt{2}}\hat{p}_1 + \sqrt{\frac{13}{6}}\hat{p}_3 + \frac{1}{\sqrt{3}}\hat{p}_5, \\ \hat{\delta}_4 &= \frac{1}{\sqrt{3}}\hat{p}_2 + \sqrt{\frac{13}{6}}\hat{p}_4 + \frac{1}{\sqrt{2}}\hat{p}_6, \\ \hat{\delta}_5 &= \sqrt{3}\hat{p}_5, \quad \hat{\delta}_6 = \sqrt{2}\hat{p}_6.\end{aligned}\quad (64)$$

For the two-rail cluster of Fig. 4(b), again beam-splitter coupling (50) between the input modes α, β and the cluster nodes 4, 5, respectively, generates the consequential modes given by (51) with $(\rho, k) = (\alpha, 4)$ and $(\beta, 5)$. These results and the quadrature-manipulation tricks lead to the output quadratures for the teleported CZ gate

$$\begin{aligned}\hat{q}_\mu &\equiv \hat{q}_1 - \frac{\hat{p}_2 + \hat{p}_3}{2} + \sqrt{2}\hat{q}_{\alpha_1} \\ &= \hat{q}_\alpha - \frac{1}{2}[(\hat{p}_2 - \hat{q}_1 - \hat{q}_4) + (\hat{p}_3 - \hat{q}_1 - \hat{q}_4)], \\ \hat{p}_\mu &\equiv \hat{p}_1 + \sqrt{2}(\hat{p}_{\alpha_2} + \hat{q}_{\beta_1}) \\ &= \hat{p}_\alpha + \hat{q}_\beta + (\hat{p}_1 - \hat{q}_2 - \hat{q}_3) - (\hat{p}_4 - \hat{q}_2 - \hat{q}_3 - \hat{q}_5),\end{aligned}\quad (65)$$

and

$$\begin{aligned}\hat{q}_\nu &\equiv \hat{q}_8 - \frac{\hat{p}_6 + \hat{p}_7}{2} + \sqrt{2}\hat{q}_{\beta_1} \\ &= \hat{q}_\beta - \frac{1}{2}[(\hat{p}_6 - \hat{q}_5 - \hat{q}_8) + (\hat{p}_7 - \hat{q}_5 - \hat{q}_8)], \\ \hat{p}_\nu &\equiv \hat{p}_8 + \sqrt{2}(\hat{q}_{\alpha_1} + \hat{p}_{\beta_2}) \\ &= \hat{p}_\beta + \hat{q}_\alpha + (\hat{p}_8 - \hat{q}_6 - \hat{q}_7) - (\hat{p}_5 - \hat{q}_4 - \hat{q}_6 - \hat{q}_7).\end{aligned}\quad (66)$$

The noise operators (49) for the two-rail linear-optical cluster can again be obtained via the corresponding unitary matrix U

listed in Appendix B. We find

$$\begin{aligned}\hat{\delta}_1 &= \sqrt{3}\hat{p}_1, \quad \hat{\delta}_2 = \sqrt{3}\hat{p}_2, \quad \hat{\delta}_3 = \frac{2}{\sqrt{3}}\hat{p}_2 + \sqrt{\frac{5}{3}}\hat{p}_3, \\ \hat{\delta}_4 &= \frac{2}{\sqrt{3}}\hat{p}_1 + \sqrt{\frac{34}{15}}\hat{p}_4 + \frac{1}{\sqrt{15}}\hat{p}_6 + \frac{1}{\sqrt{3}}\hat{p}_7, \\ \hat{\delta}_5 &= \frac{1}{\sqrt{3}}\hat{p}_2 + \frac{1}{\sqrt{15}}\hat{p}_3 + \sqrt{\frac{34}{15}}\hat{p}_5 + \frac{2}{\sqrt{3}}\hat{p}_8, \\ \hat{\delta}_6 &= \sqrt{\frac{5}{3}}\hat{p}_6 + \frac{2}{\sqrt{3}}\hat{p}_7, \quad \hat{\delta}_7 = \sqrt{3}\hat{p}_7, \quad \hat{\delta}_8 = \sqrt{3}\hat{p}_8.\end{aligned}\quad (67)$$

Finally, for the three-rail cluster of Fig. 4(c), the beam-splitter coupling (50) between the input modes α, β and, respectively, the cluster nodes 5, 6 yields the postcoupling modes of (51) with $(\rho, k) = (\alpha, 5)$ and $(\beta, 6)$. Immediately, the same tricks as before bring forth

$$\begin{aligned}\hat{q}_\mu &\equiv \hat{q}_1 - \frac{\hat{p}_2 + \hat{p}_3 + \hat{p}_4}{3} + \sqrt{2}\hat{q}_{\alpha_1} \\ &= \hat{q}_\alpha - \frac{1}{3}[(\hat{p}_2 - \hat{q}_1 - \hat{q}_5) + (\hat{p}_3 - \hat{q}_1 - \hat{q}_5) \\ &\quad + (\hat{p}_4 - \hat{q}_1 - \hat{q}_5)], \\ \hat{p}_\mu &\equiv \hat{p}_1 + \sqrt{2}(\hat{p}_{\alpha_2} + \hat{q}_{\beta_1}) \\ &= \hat{p}_\alpha + \hat{q}_\beta + (\hat{p}_1 - \hat{q}_2 - \hat{q}_3 - \hat{q}_4) \\ &\quad - (\hat{p}_5 - \hat{q}_2 - \hat{q}_3 - \hat{q}_4 - \hat{q}_6),\end{aligned}\quad (68)$$

and

$$\begin{aligned}\hat{q}_\nu &\equiv \hat{q}_{10} - \frac{\hat{p}_7 + \hat{p}_8 + \hat{p}_9}{3} + \sqrt{2}\hat{q}_{\beta_1} \\ &= \hat{q}_\beta - \frac{1}{3}[(\hat{p}_7 - \hat{q}_6 - \hat{q}_{10}) + (\hat{p}_8 - \hat{q}_6 - \hat{q}_{10}) \\ &\quad + (\hat{p}_9 - \hat{q}_6 - \hat{q}_{10})], \\ \hat{p}_\nu &\equiv \hat{p}_{10} + \sqrt{2}(\hat{q}_{\alpha_1} + \hat{p}_{\beta_2}) \\ &= \hat{p}_\beta + \hat{q}_\alpha + (\hat{p}_{10} - \hat{q}_7 - \hat{q}_8 - \hat{q}_9) \\ &\quad - (\hat{p}_6 - \hat{q}_5 - \hat{q}_7 - \hat{q}_8 - \hat{q}_9).\end{aligned}\quad (69)$$

The U matrix obtained in Appendix B for the three-rail cluster enables us to find the noise operators (49) here explicitly:

$$\begin{aligned}\hat{\delta}_1 &= 2\hat{p}_1, \quad \hat{\delta}_2 = \sqrt{3}\hat{p}_2, \quad \hat{\delta}_3 = \frac{2}{\sqrt{3}}\hat{p}_2 + \sqrt{\frac{5}{3}}\hat{p}_3, \\ \hat{\delta}_4 &= \frac{2}{\sqrt{3}}\hat{p}_2 + \frac{2}{\sqrt{15}}\hat{p}_3 + \sqrt{\frac{7}{5}}\hat{p}_4, \\ \hat{\delta}_5 &= \frac{3}{2}\hat{p}_1 + \frac{1}{2}\sqrt{\frac{65}{7}}\hat{p}_5 + \frac{1}{\sqrt{35}}\hat{p}_7 + \frac{1}{\sqrt{15}}\hat{p}_8 + \frac{1}{\sqrt{3}}\hat{p}_9, \\ \hat{\delta}_6 &= \frac{1}{\sqrt{3}}\hat{p}_2 + \frac{1}{\sqrt{15}}\hat{p}_3 + \frac{1}{\sqrt{35}}\hat{p}_4 + \frac{1}{2}\sqrt{\frac{65}{7}}\hat{p}_6 + \frac{3}{2}\hat{p}_{10}, \\ \hat{\delta}_7 &= \sqrt{\frac{7}{5}}\hat{p}_7 + \frac{2}{\sqrt{15}}\hat{p}_8 + \frac{2}{\sqrt{3}}\hat{p}_9, \\ \hat{\delta}_8 &= \sqrt{\frac{5}{3}}\hat{p}_8 + \frac{2}{\sqrt{3}}\hat{p}_9, \quad \hat{\delta}_9 = \sqrt{3}\hat{p}_9, \quad \hat{\delta}_{10} = 2\hat{p}_{10}.\end{aligned}\quad (70)$$

Before proceeding to the discussion for general N -rail configurations, we would like to point out here that, as before for canonical clusters, the two-rail results (65), (66) and the three-rail results (68), (69) correspond to the minimum excess noises in the quadratures \hat{q}_μ and \hat{q}_ν when the nodes are uniformly squeezed. For example, for the two-rail results (65) and (66), one can have instead for \hat{q}_μ and \hat{q}_ν the general expressions

$$\begin{aligned}\hat{q}_\mu &\equiv \hat{q}_1 - (\eta_1 \hat{p}_2 + \eta_2 \hat{p}_3) + \sqrt{2}\hat{q}_{\alpha_1} \\ &= \hat{q}_\alpha - [\eta_1 (\hat{p}_2 - \hat{q}_1 - \hat{q}_4) + \eta_2 (\hat{p}_3 - \hat{q}_1 - \hat{q}_4)], \\ \hat{q}_\nu &\equiv \hat{q}_8 - (\eta_3 \hat{p}_6 + \eta_4 \hat{p}_7) + \sqrt{2}\hat{q}_{\beta_1} \\ &= \hat{q}_\beta - [\eta_3 (\hat{p}_6 - \hat{q}_5 - \hat{q}_8) + \eta_4 (\hat{p}_7 - \hat{q}_5 - \hat{q}_8)],\end{aligned}\quad (71)$$

where $\eta_1 + \eta_2 = 1$ and $\eta_3 + \eta_4 = 1$. Despite the more complicated noise operators (67), for identically squeezed cluster nodes, one can again show that it is the symmetrical arrangement $\eta_k = 1/2$ for all $k = 1 \sim 4$ in (65) and (66) that would minimize the excess noise in the teleported \hat{q}_μ and \hat{q}_ν . Similarly, this is also the case with the three-rail results in (68) and (69). As we had noted when studying multirail canonical clusters, optimization through the choice of coefficients such as in (71) is the underlying mechanism for excess-noise reduction through multirail designs. We will come back to this issue shortly in our discussion for general N -rail clusters in the following.

We are now in a position to extend the results above to general linear-optical cluster states with an arbitrary N -rail design. Upon inspecting the results (62), (63), (65), (66), (68), and (69), for the CZ-gate teleportation in Fig. 4(d) with an N -rail linear-optical cluster, one can obtain by induction the output quadratures:

$$\begin{aligned}\hat{q}_\mu &= \hat{q}_\alpha - \frac{1}{N} \sum_{k=2}^{N+1} \hat{\delta}_k, \\ \hat{p}_\mu &= \hat{p}_\alpha + \hat{q}_\beta + \hat{\delta}_1 - \hat{\delta}_{N+2}, \\ \hat{q}_\nu &= \hat{q}_\beta - \frac{1}{N} \sum_{k=N+4}^{2N+3} \hat{\delta}_k, \\ \hat{p}_\nu &= \hat{p}_\beta + \hat{q}_\alpha + \hat{\delta}_{2N+4} - \hat{\delta}_{N+3}.\end{aligned}\quad (72)$$

As in (41), here we have expressed the formulas in favor of the noise operators (49) both for compactness and for later convenience in calculating the quadrature correlators. Although (72) closely resembles (41), one must be alert to the more complicated correlations among the noise operators $\hat{\delta}_k$ here than those of canonical clusters. Remarkably, for any linear-optical cluster with uniformly squeezed nodes, one can derive an analytical expression for its noise correlators (see Appendix C):

$$\langle \hat{\delta}_k \hat{\delta}_l \rangle = (M_{kl} + \delta_{kl}) \times \frac{e^{-2r}}{4},\quad (73)$$

where r is the squeezing parameter for the nodes and

$$M_{kl} \equiv \begin{cases} \text{(number of nodes connected to node } k), & \text{if } k = l; \\ \text{(number of nodes connected to both nodes } k \text{ and } l), & \text{if } k \neq l. \end{cases}\quad (74)$$

This result will allow us to find an analytical formula for the entanglement in the output modes μ, ν of the teleported CZ gate for general N -rail linear-optical cluster. In addition, we can make use of (73) to understand the noise-reduction mechanism behind the multirail scheme for teleportation with linear-optical clusters. That is, as with canonical clusters [see below (44)], in our expressions for \hat{q}_μ and \hat{q}_ν in (72), the coefficient $\frac{1}{N}$ in the sum over noise operators $\hat{\delta}_k$ corresponds to allocating the excess noise to each of the multirails with equal weight. As shown in Appendix C, this symmetric arrangement would minimize the excess noise in the teleported \hat{q}_μ and \hat{q}_ν .

Equipped with (73), we shall now examine the quality of the teleported CZ gate through an N -rail cluster by calculating the log-negativity for the output modes. As always, we shall assume that we have independent coherent-state input modes α, β , and that the cluster nodes are uniformly squeezed with squeezing parameter r . With the help of (73), the quadrature correlators (27) can then be found readily through (72), which yield

$$a = \frac{(1 + \frac{2N+1}{N}e^{-2r})}{4}, \quad b = \frac{(2 + 3e^{-2r})}{4}, \quad c = \frac{(1 + e^{-2r})}{4}.\quad (75)$$

As with their counterparts for canonical clusters, here b and c are again N independent. Using (75) in (29), one finds that the log-negativity for the output modes in this case reads

$$E_{N, NR'} = \max \left\{ 0, -\ln \left(\sqrt{\left[1 + \frac{(2N+1)e^{-2r}}{N} \right] (2 + 3e^{-2r})} - (1 + e^{-2r}) \right) \right\}.\quad (76)$$

With the entanglement of the output modes available for CZ-gate teleportation through canonical cluster states and linear-optical cluster states, we are now ready for comparisons between them. We plot in Fig. 5 the log-negativities (43) and (76) with respect to the squeezing parameter r for different numbers of rails N , where E_N is normalized relative to the ideal (infinite squeezing) value $E_N(\infty) = -\ln(\sqrt{2} - 1) \simeq 0.88$. We note that for the linear-optical result (76), the log-negativity approaches the result (61) for a linear four-mode cluster in the limit of $N \rightarrow \infty$, which remains to have regimes with zero entanglement for $r \lesssim 0.45$ [visible from the plot for $N = 100$ in Fig. 5(b)]. In contrast, for canonical cluster states, the log-negativity (43) would be identical to the corresponding linear four-mode result (31) when $N = 2$, and for large N the

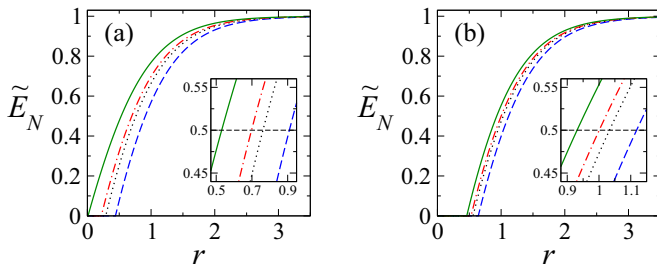


FIG. 5. The log-negativity \tilde{E}_N for the output modes μ, ν of the teleported CZ gate normalized with respect to the ideal (infinite squeezing) value plotted as a function of the squeezing parameter r . Here the resource states are (a) canonical and (b) linear-optical multirail clusters with the number of rails $N = 1$ (dashed lines), 2 (dotted lines), 3 (dot-dashed lines), and 100 (solid lines). In each panel, the inset shows the vicinity of the squeezing level at which the log-negativity reaches 50% of the ideal value.

output modes can have nonzero entanglement even at very low squeezing levels [e.g., see the plot for $N = 100$ in Fig. 5(a)].

As a figure of merit for the CZ-gate teleportation, let us consider the squeezing parameter \bar{r} that would enable the teleportation to achieve one half of the ideal value for the log-negativity (see the insets in Fig. 5). Analytic expressions for \bar{r} can be obtained by solving the equation

$$\sqrt{ab} - c = \frac{\sqrt{\sqrt{2} - 1}}{4}, \quad (77)$$

where the correlators a, b, c are given by (42) for canonical clusters and (75) for linear-optical clusters. We list in Table I the values of \bar{r} for both types of multirail clusters with selected number of rails N . It is seen that for linear-optical clusters, \bar{r} starts with the single-rail value 1.12 (corresponding to -9.73 dB) and, with the implementation of multirail structures, reduces to 1.00 (-8.69 dB) for $N = 3$ and to 0.93 (-8.08 dB) for $N = 100$. Judging from the reduction of \bar{r} achieved by increasing N in the multirail, we see that the improvement in the CZ-gate teleportation is rather limited in this case. In the instance of the canonical clusters, \bar{r} takes the value 0.91 (-7.90 dB) for single-rail and decreases appreciably when multirail design is incorporated: \bar{r} becomes 0.70 (-6.08 dB) when $N = 3$ and 0.53 (-4.60 dB) when $N = 100$. If this trend could persist for even larger values of N , so that \bar{r} would tend to zero for sufficiently large N , one would then be able to achieve ideal log-negativity for the CZ-gate teleportation via multirail canonical clusters with vanishing squeezing. Nonetheless, as one can show analytically, for

TABLE I. Squeezing parameter \bar{r} for achieving 50% of the ideal log-negativity with N -rail clusters.

	N				
	1	2	3	100	∞
\bar{r} (can.) ^a	0.91	0.76	0.70	0.53	$0.52 + O(1/N)$
\bar{r} (l.o.) ^b	1.12	1.03	1.00	0.93	$0.93 + O(1/N)$

^aCanonical clusters.

^bLinear-optical clusters.

both linear-optical and canonical clusters \bar{r} would approach nonzero steady values in the large N limit in the manner of $1/N$ (see Table I). Therefore, despite the impressive reduction of \bar{r} for multirail canonical clusters, it is not possible to reduce \bar{r} without bound toward zero. In other words, for either class of resource states considered here, multirail noise reduction is insufficient to enable ideal CZ-gate teleportation if the resource state had not been sufficiently squeezed.

To understand the reason for the difference between the results for the two classes of resource states, let us recall that the entanglement between the output modes is determined solely from the smaller symplectic eigenvalue λ_- in (28) for the partially transposed covariance matrix of the output modes, which, in turn, is governed by the quadrature correlators (27). From the results (42) and (75), we find that for both canonical and linear-optical clusters, the correlators b and c for the teleported states are entirely independent of the number of rails N . Therefore, the N dependence of λ_- (and hence, of the log-negativity E_N) is completely due to the correlator a . Since we have $\sqrt{ab} > c$ always for both (42) and (75), according to (28) and (29), the entanglement thus changes monotonically with a when b and c stay fixed. In the case of canonical clusters, we see from (42) that the multirail design can eliminate the excess noise in a entirely in the large N limit, so that λ_- can drop below the critical value $1/4$ for entanglement. Thus, with increasing number of rails, the log-negativity (43) would increase steadily and tend to a limit with nonvanishing entanglement for the whole range of squeezing parameter r . In contrast, for linear-optical clusters, the multirail noise reduction can bring down a at best to the linear four-mode expression (60) even with $N \rightarrow \infty$. The entanglement (76) is thus always bounded above by $E_{N,L'}$ of (61), which vanishes for a finite range of squeezing levels. Likewise, for the figure of merit \bar{r} , one can also understand why the multirail reduction works better for canonical clusters than for linear-optical ones. This is because in the large N limit, only the correlator b would be left with an excess-noise term in the case of canonical clusters, while all three correlators a, b , and c would still have excess-noise terms for linear-optical clusters. In fact, this is also why one can never succeed in reducing \bar{r} to zero even with infinite N , since there are always excess-noise terms left in the quadrature correlator(s). If one could design cluster-type resource states for which the output mode quadrature correlators would depend on some parameters that could erase *all* excess-noise terms under appropriate limits, it would then be possible to achieve perfect CZ-gate teleportation with resource states of *arbitrarily low* squeezing.

In order to verify our results experimentally, one can construct the linear-optical clusters, as usual, by passing momentum-squeezed vacuum modes through appropriate networks of beam splitters and phase shifters [20–23,32]. The number of optical elements required, nevertheless, would increase rapidly with the number of multirails implemented. For the canonical clusters, one may achieve the multirail structure by means of cluster shaping over universal two-dimensional cluster states [42]. This is, however, rather wasteful for the resource states and the number of rails achievable would also be limited by the connectivity of the nodes in the original cluster. Alternatively, one may resort to

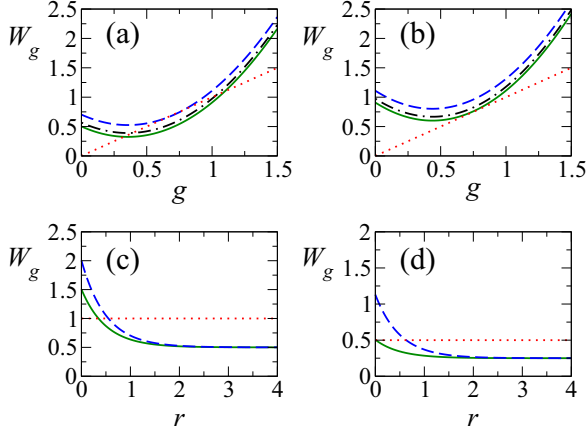


FIG. 6. Panels (a) and (b): W_g as a function of the signal gain g at -3.91 dB ($r = 0.45$) squeezing for teleportations with (a) canonical and (b) linear-optical clusters with multirails $N = 1$ (dashed lines), 3 (dot-dashed lines), and 100 (solid lines). Panels (c) and (d): W_g plotted vs the squeezing parameter r at fixed gain values (c) $g = 1.0$ and (d) $g = 0.5$ for teleportations with canonical (solid lines) and linear-optical (dashed lines) clusters with the number of rails $N = 100$. In each panel, the dotted line depicts the corresponding entanglement bound.

the temporal-encoding-based single-QND approach [25]. By routing and controlling the qumodes exquisitely, it is possible to fabricate the canonical multirail cluster employing a single QND gate (see Appendix D).

For the entanglement detection, rather than the full covariance matrix, experimentally it is favorable to check through the variance sum of suitable quadrature combinations, such as [22]

$$W_g \equiv \langle [\Delta(g\hat{p}_\mu - \hat{q}_\nu)]^2 \rangle + \langle [\Delta(g\hat{p}_\nu - \hat{q}_\mu)]^2 \rangle, \quad (78)$$

where g is a signal gain introduced to sharpen the entanglement bound (see below). Measurement with $W_g < g$ indicates entanglement between the output modes μ and ν . The variance function W_g can be readily expressed in terms of the correlators (27), which yields $W_g = 2(a + g^2b - 2gc)$. Making use of our results (42) for canonical clusters, and (75) for linear-optical clusters, we plot in Figs. 6(a) and 6(b) the dependence of W_g on the gain factor g for canonical and linear clusters with different multirail structures at the squeezing level $r = 0.45$ (-3.91 dB). We see that for canonical clusters, the teleported modes become entangled with the implementation of the multirails, while for linear-optical clusters, the teleported modes remains to have no detectable entanglement even with large number of multirails $N = 100$. This is consistent with our finding through log-negativities shown in Fig. 5.

It should be noted that the gain factor g is crucial here for the entanglement detection. As illustrated in Fig. 6(c) for the canonical cluster with multirails of $N = 100$, with unit gain ($g = 1$), W_g would fail to detect entanglement for $r \lesssim 0.35$, despite the presence of entanglement even at $r \simeq 0.01$ according to the log-negativity [see Fig. 5(a)]. In contrast, as depicted in Fig. 6(d), the choice of $g = 0.5$ changes the situation entirely: W_g would then be able to detect entanglement even at very low squeezing levels. In fact, for the optimal gain $g = (c + 1/4)/b$ that minimizes $(W_g - g)$ (thus providing the most

stringent entanglement bound for a given configuration), one can show easily that the condition $W_g < g$ for entanglement would become exactly identical to the symplectic-eigenvalue based criterion $\lambda_- = (\sqrt{ab} - c) < 1/4$.

IV. CONCLUSION AND DISCUSSION

In summary, we have examined in this paper the teleportation of a CZ gate through a measurement-based scheme in the CV regime, where two classes of resource states have been considered: the canonical cluster states fabricated through QND gates and the linear-optical cluster states from linear-optical networks. We study the entanglement properties of the teleported output modes and examine the performance of a multirail design that aims to reduce the excess noise in the teleported gate. For both classes of resource states, we find analytical expressions for the entanglement and obtain its scaling with the number of rails in the multirail design. It is found that the multirail design can help improve the CZ-gate teleportation significantly when canonical clusters are adopted. In the process of this analysis, we also discuss in detail the noise-reduction mechanism underlying the multirail approach to measurement-based teleportation. And last but not least, in order to facilitate the analysis for teleportation with multirail clusters, we have developed an operator-manipulation trick that can help establish efficiently the measurement sequence and the corrective operations for the CZ-gate teleportation.

Although the quadrature-manipulation trick has been applied in this work only to CZ-gate teleportation, it is equally applicable to teleportation of other gates in measurement-based quantum information processing. For instance, for single-qumode gates that effect translation, rotation, and squeezing of a single qumode, one can apply the same trick and obtain the corresponding measurement sequence and corrective operations in the measurement-based teleportation [28].

It must be noted that the entanglement enhancement in the teleported CZ gate in our result should not be interpreted as a scheme for building an “improved” CZ gate, which can be used, for instance, in the single-QND approach to cluster construction [25]. Instead, the CZ-gate teleportation here is used as a task for testing the performance of a given resource state in the measurement-based scheme. Our result demonstrates that by implementing multirail designs into the resource states, due to suppression of the excess noise, it is possible to entangle two input modes more effectively than the original single-rail clusters. In the light of this result, therefore, it is then interesting to enquire whether this multirail design could help lower the squeezing threshold for fault-tolerant measurement-based CV quantum computing when appropriate error-correcting code is incorporated [27].

Finally, in closing we note that despite the lower quality of the teleported CZ gate through linear-optical cluster states in comparison with that via canonical cluster states in our results, it can be possible to change this situation by exploiting an additional degree of freedom in constructing the U matrix for the linear-optical cluster. Since the geometric constraints resulting from (47) consist of inner products of the $\vec{\alpha}_k$'s [i.e., the G_{kl} of (48)], they are invariant under orthogonal transformations. In other words, for any choice of the $\vec{\alpha}_k$'s

one can always apply orthogonal transformations over these vectors without violating the geometric constraints. This extra degree of freedom thus provides means of optimization for constructing the U matrix according to the cost function one choose to consider [43]. Since the number of nodes can be quite large in our calculation, such optimization can be challenging. Also, it is possible to improve the quality of the teleported gates by introducing appropriate gain factors in the feedback signals [12,32]. We plan to investigate these issues of optimization in a separate work.

ACKNOWLEDGMENTS

We thank Prof. Dian-Jiun Han for useful discussions and Dr. Yi-Chan Lee for valuable suggestions and help in parts of this work. This research is supported by the Ministry of Science and Technology Taiwan through Grant No. MOST 104-2112-M-194-006. It is also supported partly by the Center for Theoretical Sciences, Taiwan.

APPENDIX A: CZ-GATE TELEPORTATION USING FOURIER-TRANSFORMED CLUSTER STATES

In this Appendix, we demonstrate that for the CZ-gate teleportation of Fig. 3 with a canonical linear four-mode cluster, one can perform the quadrature manipulations following the scheme of category (d), rather than those of categories (b) and (c), and arrive at results equivalent to those of Sec. II. For this purpose, it is necessary to apply Fourier transforms over the cluster modes prior to their coupling with the input modes. The quadrature operators for the Fourier-transformed cluster nodes $k = 1 \sim 4$ read

$$\begin{pmatrix} \hat{q}'_k \\ \hat{p}'_k \end{pmatrix} = \begin{pmatrix} -\hat{p}_k \\ +\hat{q}_k \end{pmatrix} = \begin{pmatrix} -\hat{p}_k - \sum_{l \in N_k} \hat{q}_l \\ \hat{q}_k \end{pmatrix}, \quad (\text{A1})$$

where, as always, \hat{q}_k and \hat{p}_k are quadrature operators for the offline-squeezed initial cluster mode k , and we have used (5) in reaching the last expression. It follows immediately from (A1) that, instead of (6), the excess-noise operators (or nullifiers) here become

$$\hat{\delta}_k \equiv \hat{q}'_k + \sum_{l \in N_k} \hat{p}'_l = -\hat{p}_k. \quad (\text{A2})$$

QND couplings between the input modes and the cluster nodes thus yield [cf. (17)]

$$\begin{pmatrix} \hat{q}''_\rho \\ \hat{p}''_\rho \end{pmatrix} = \begin{pmatrix} \hat{q}_\rho \\ \hat{p}_\rho + \hat{q}'_k \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \hat{q}''_k \\ \hat{p}''_k \end{pmatrix} = \begin{pmatrix} \hat{q}'_k \\ \hat{p}'_k + \hat{q}_\rho \end{pmatrix}, \quad (\text{A3})$$

where $(\rho, k) = (\alpha, 2)$ and $(\beta, 3)$, and we have denoted all resultant modes with double-primed notations for clarity.

In anticipation of the mapping

$$\begin{pmatrix} \hat{q}'_1 \\ \hat{p}'_1 \end{pmatrix} \rightarrow \begin{pmatrix} \hat{q}_\alpha \\ \hat{p}_\alpha + \hat{q}_\beta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \hat{q}'_4 \\ \hat{p}'_4 \end{pmatrix} \rightarrow \begin{pmatrix} \hat{q}_\beta \\ \hat{p}_\beta + \hat{q}_\alpha \end{pmatrix} \quad (\text{A4})$$

subject to proper corrective operations, one can write, employing the \hat{p}''_2 entry of (A3),

$$\begin{aligned} \hat{q}'_1 &= \hat{q}'_1 - (\hat{p}''_2 - \hat{p}'_2 - \hat{q}_\alpha) \\ &= \hat{q}_\alpha - \hat{p}'_2 + (\hat{q}'_1 + \hat{p}'_2) \end{aligned} \quad (\text{A5})$$

with $(\hat{q}'_1 + \hat{p}'_2)$ in the last line being the nullifier $\hat{\delta}_1$ of (A2). The last expression above immediately suggests that

$$\begin{aligned} \hat{q}_\mu &\equiv \hat{q}'_1 + \hat{p}''_2 \\ &= \hat{q}_\alpha + (\hat{q}'_1 + \hat{p}'_2). \end{aligned} \quad (\text{A6})$$

Similarly, in view of (A4), we can write, with the help of (A3),

$$\begin{aligned} \hat{p}'_1 &= \hat{p}'_1 - (\hat{p}''_2 - \hat{p}_\alpha - \hat{q}'_2) - (\hat{p}''_3 - \hat{p}'_3 - \hat{q}_\beta) \\ &= \hat{p}_\alpha + \hat{q}_\beta - \hat{p}''_2 - \hat{p}''_3 + (\hat{q}'_2 + \hat{p}'_1 + \hat{p}'_3). \end{aligned} \quad (\text{A7})$$

It thus follows that

$$\begin{aligned} \hat{p}_\mu &\equiv \hat{p}'_1 + \hat{p}''_2 + \hat{p}''_3 \\ &= \hat{p}_\alpha + \hat{q}_\beta + (\hat{q}'_2 + \hat{p}'_1 + \hat{p}'_3). \end{aligned} \quad (\text{A8})$$

As before, by virtue of the symmetry among the modes, for the output mode ν one can obtain from (A6) and (A8) by changing the indices suitably

$$\begin{aligned} \hat{q}_\nu &\equiv \hat{q}'_4 + \hat{p}''_3 \\ &= \hat{q}_\beta + (\hat{q}'_4 + \hat{p}'_3), \\ \hat{p}_\nu &\equiv \hat{p}'_4 + \hat{p}''_2 + \hat{p}''_3 \\ &= \hat{p}_\beta + \hat{q}_\alpha + (\hat{q}'_3 + \hat{p}'_2 + \hat{p}'_4). \end{aligned} \quad (\text{A9})$$

Comparing the second line of each expression for the output quadratures above with its counterpart in (19), (21), and (22), we see that the output modes here differ from those earlier only in sign changes in the excess-noise terms. As far as the entanglement of the output modes is concerned, which depends only on $\langle \hat{\delta}_k^2 \rangle$, the results above are therefore entirely equivalent to those found in Sec. II.

APPENDIX B: CONSTRUCTION OF THE U MATRICES FOR LINEAR-OPTICAL CLUSTER STATES

We explain in this appendix the procedure for constructing the unitary matrix U in (45) which represents the effects of a linear-optical network that implements the desired cluster correlations for a linear-optical cluster state. The results of these calculations related to the linear-optical cluster states considered in the text will also be supplied in the following.

In constructing the U matrix for a given cluster geometry, one starts by solving from (47) the geometric constraints over the row vectors $\vec{\alpha}_k$, which consist of definite values for the inner products $G_{kl} \equiv \vec{\alpha}_k \vec{\alpha}_l^T$. For a cluster with M nodes, there are $M(M+1)/2$ such constraints that can be solved from (47) and the result can be conveniently summarized in terms of an $M \times M$ real symmetric matrix G , whose k, l element is given by G_{kl} . For instance, for the linear four-mode cluster in Figs. 2 and 3, we find

$$G_{L4} = \begin{pmatrix} \frac{3}{5} & 0 & -\frac{1}{5} & 0 \\ 0 & \frac{2}{5} & 0 & -\frac{1}{5} \\ -\frac{1}{5} & 0 & \frac{2}{5} & 0 \\ 0 & -\frac{1}{5} & 0 & \frac{3}{5} \end{pmatrix}, \quad (\text{B1})$$

where the subscript $L4$ stands for ‘‘linear four-mode.’’ With the geometric constraints, it is then straightforward to construct the vectors $\vec{\alpha}_k$ accordingly. For this task, it is advisable to start from the ‘‘symmetric center’’ of the cluster. In the present case

of the linear four-mode cluster, we start from nodes 2 and 3, in keeping with the geometric constraints $(G_{L4})_{22} = (G_{L4})_{33} = 2/5$ and that $(G_{L4})_{23} = 0$ from (B1), and take

$$\begin{aligned}\vec{\alpha}_2 &= (0 \quad \sqrt{\frac{2}{5}} \quad 0 \quad 0), \\ \vec{\alpha}_3 &= (0 \quad 0 \quad \sqrt{\frac{2}{5}} \quad 0).\end{aligned}\quad (\text{B2})$$

One can next choose to construct $\vec{\alpha}_1$ taking into account the geometric constraints $(G_{L4})_{1k}$ with $k = 1 \sim 3$, leaving out $(G_{L4})_{14}$ for later when constructing $\vec{\alpha}_4$. We get accordingly

$$\vec{\alpha}_1 = \left(\frac{1}{\sqrt{2}} \quad 0 \quad \frac{-1}{\sqrt{10}} \quad 0\right).\quad (\text{B3})$$

Taking into account the remaining constraints for $\vec{\alpha}_4$, one can find immediately

$$\vec{\alpha}_4 = \left(0 \quad \frac{-1}{\sqrt{10}} \quad 0 \quad \frac{1}{\sqrt{2}}\right).\quad (\text{B4})$$

With the $\vec{\alpha}_k$'s available, it is then straightforward to construct the k th row of the matrix U following (46), and thus the U matrix. Using (B2)–(B4), we obtain for the linear four-mode

cluster

$$U_{L4} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{2i}{\sqrt{10}} & \frac{-1}{\sqrt{10}} & 0 \\ \frac{i}{\sqrt{2}} & \frac{2}{\sqrt{10}} & \frac{i}{\sqrt{10}} & 0 \\ 0 & \frac{i}{\sqrt{10}} & \frac{2}{\sqrt{10}} & \frac{i}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{10}} & \frac{2i}{\sqrt{10}} & \frac{1}{\sqrt{2}} \end{pmatrix}.\quad (\text{B5})$$

For larger clusters, the procedure proceed similarly to that presented above, although with greater complexity due to the increased number of nodes and geometric constraints involved. We summarize here our results for the linear-optical cluster states that are considered in Sec. III B. For the linear six-mode cluster of Fig. 4(a), we find the matrix for geometric constraints

$$G_{L6} = \begin{pmatrix} \frac{8}{13} & 0 & \frac{-3}{13} & 0 & \frac{1}{13} & 0 \\ 0 & \frac{5}{13} & 0 & \frac{-2}{13} & 0 & \frac{1}{13} \\ \frac{-3}{13} & 0 & \frac{6}{13} & 0 & \frac{-2}{13} & 0 \\ 0 & \frac{-2}{13} & 0 & \frac{6}{13} & 0 & \frac{-3}{13} \\ \frac{1}{13} & 0 & \frac{-2}{13} & 0 & \frac{5}{13} & 0 \\ 0 & \frac{1}{13} & 0 & \frac{-3}{13} & 0 & \frac{8}{13} \end{pmatrix}.\quad (\text{B6})$$

Constructing the vectors $\vec{\alpha}_k$ accordingly, we obtain the U matrix

$$U_{L6} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{3}} & -\sqrt{\frac{3}{26}} & -\sqrt{\frac{2}{39}}i & 0 & 0 \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \sqrt{\frac{3}{26}}i & -\sqrt{\frac{2}{39}} & 0 & 0 \\ 0 & \frac{i}{\sqrt{3}} & \sqrt{\frac{6}{13}} & 2\sqrt{\frac{2}{39}}i & 0 & 0 \\ 0 & 0 & 2\sqrt{\frac{2}{39}}i & \sqrt{\frac{6}{13}} & \frac{i}{\sqrt{3}} & 0 \\ 0 & 0 & -\sqrt{\frac{2}{39}} & \sqrt{\frac{3}{26}}i & \frac{1}{\sqrt{3}} & \frac{i}{\sqrt{2}} \\ 0 & 0 & -\sqrt{\frac{2}{39}}i & -\sqrt{\frac{3}{26}} & \frac{i}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}.\quad (\text{B7})$$

In the case of the two-rail cluster of Fig. 4(b), we find the geometric constraints

$$G_{2R} = \begin{pmatrix} \frac{9}{17} & 0 & 0 & -\frac{5}{17} & 0 & \frac{1}{17} & \frac{1}{17} & 0 \\ 0 & \frac{21}{34} & -\frac{13}{34} & 0 & -\frac{3}{34} & 0 & 0 & \frac{1}{17} \\ 0 & -\frac{13}{34} & \frac{21}{34} & 0 & -\frac{3}{34} & 0 & 0 & \frac{1}{17} \\ -\frac{5}{17} & 0 & 0 & \frac{15}{34} & 0 & -\frac{3}{34} & -\frac{3}{34} & 0 \\ 0 & -\frac{3}{34} & -\frac{3}{34} & 0 & \frac{15}{34} & 0 & 0 & -\frac{5}{17} \\ \frac{1}{17} & 0 & 0 & -\frac{3}{34} & 0 & \frac{21}{34} & -\frac{13}{34} & 0 \\ \frac{1}{17} & 0 & 0 & -\frac{3}{34} & 0 & -\frac{13}{34} & \frac{21}{34} & 0 \\ 0 & \frac{1}{17} & \frac{1}{17} & 0 & -\frac{5}{17} & 0 & 0 & \frac{9}{17} \end{pmatrix},\quad (\text{B8})$$

which allows us to construct the corresponding U matrix

$$U_{2R} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{i}{\sqrt{3}} & \frac{i}{\sqrt{15}} & -\sqrt{\frac{10}{51}} & -\sqrt{\frac{6}{85}}i & 0 & 0 & 0 \\ \frac{i}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{15}} & \sqrt{\frac{5}{102}}i & -\sqrt{\frac{3}{170}} & 0 & 0 & 0 \\ \frac{i}{\sqrt{3}} & 0 & \sqrt{\frac{3}{5}} & \sqrt{\frac{5}{102}}i & -\sqrt{\frac{3}{170}} & 0 & 0 & 0 \\ 0 & \frac{i}{\sqrt{3}} & \frac{i}{\sqrt{15}} & \sqrt{\frac{15}{34}} & \sqrt{\frac{27}{170}}i & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\frac{27}{170}}i & \sqrt{\frac{15}{34}} & \frac{i}{\sqrt{15}} & \frac{i}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & -\sqrt{\frac{3}{170}} & \sqrt{\frac{5}{102}}i & \sqrt{\frac{3}{5}} & 0 & \frac{i}{\sqrt{3}} \\ 0 & 0 & 0 & -\sqrt{\frac{3}{170}} & \sqrt{\frac{5}{102}}i & \frac{-2}{\sqrt{15}} & \frac{1}{\sqrt{3}} & \frac{i}{\sqrt{3}} \\ 0 & 0 & 0 & -\sqrt{\frac{6}{85}}i & -\sqrt{\frac{10}{51}} & \frac{i}{\sqrt{15}} & \frac{i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}. \quad (\text{B9})$$

For the three-rail cluster in Fig. 4(c), we find

$$G_{3R} = \begin{pmatrix} \frac{32}{65} & 0 & 0 & 0 & -\frac{21}{65} & 0 & \frac{3}{65} & \frac{3}{65} & \frac{3}{65} & 0 \\ 0 & \frac{47}{65} & -\frac{18}{65} & -\frac{18}{65} & 0 & -\frac{4}{65} & 0 & 0 & 0 & \frac{3}{65} \\ 0 & -\frac{18}{65} & \frac{47}{65} & -\frac{18}{65} & 0 & -\frac{4}{65} & 0 & 0 & 0 & \frac{3}{65} \\ 0 & -\frac{18}{65} & -\frac{18}{65} & \frac{47}{65} & 0 & -\frac{4}{65} & 0 & 0 & 0 & \frac{3}{65} \\ -\frac{21}{65} & 0 & 0 & 0 & \frac{28}{65} & 0 & -\frac{4}{65} & -\frac{4}{65} & -\frac{4}{65} & 0 \\ 0 & -\frac{4}{65} & -\frac{4}{65} & -\frac{4}{65} & 0 & \frac{28}{65} & 0 & 0 & 0 & -\frac{21}{65} \\ \frac{3}{65} & 0 & 0 & 0 & -\frac{4}{65} & 0 & \frac{47}{65} & -\frac{18}{65} & -\frac{18}{65} & 0 \\ \frac{3}{65} & 0 & 0 & 0 & -\frac{4}{65} & 0 & -\frac{18}{65} & \frac{47}{65} & -\frac{18}{65} & 0 \\ \frac{3}{65} & 0 & 0 & 0 & -\frac{4}{65} & 0 & -\frac{18}{65} & -\frac{18}{65} & \frac{47}{65} & 0 \\ 0 & \frac{3}{65} & \frac{3}{65} & \frac{3}{65} & 0 & -\frac{21}{65} & 0 & 0 & 0 & \frac{32}{65} \end{pmatrix}, \quad (\text{B10})$$

and the U matrix can be constructed in the way described above. We get

$$U_{3R} = \begin{pmatrix} \frac{1}{2} & \frac{i}{\sqrt{3}} & \frac{i}{\sqrt{15}} & \frac{i}{\sqrt{35}} & -\frac{3}{2}\sqrt{\frac{7}{65}} & \frac{-6i}{\sqrt{455}} & 0 & 0 & 0 & 0 \\ \frac{i}{2} & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{15}} & \frac{-2}{\sqrt{35}} & \frac{i}{2}\sqrt{\frac{7}{65}} & \frac{-2}{\sqrt{455}} & 0 & 0 & 0 & 0 \\ \frac{i}{2} & 0 & \sqrt{\frac{3}{5}} & \frac{-2}{\sqrt{35}} & \frac{i}{2}\sqrt{\frac{7}{65}} & \frac{-2}{\sqrt{455}} & 0 & 0 & 0 & 0 \\ \frac{i}{2} & 0 & 0 & \sqrt{\frac{5}{7}} & \frac{i}{2}\sqrt{\frac{7}{65}} & \frac{-2}{\sqrt{455}} & 0 & 0 & 0 & 0 \\ 0 & \frac{i}{\sqrt{3}} & \frac{i}{\sqrt{15}} & \frac{i}{\sqrt{35}} & 2\sqrt{\frac{7}{65}} & \frac{8i}{\sqrt{455}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{8i}{\sqrt{455}} & 2\sqrt{\frac{7}{65}} & \frac{i}{\sqrt{35}} & \frac{i}{\sqrt{15}} & \frac{i}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & \frac{-2}{\sqrt{455}} & \frac{i}{2}\sqrt{\frac{7}{65}} & \sqrt{\frac{5}{7}} & 0 & 0 & \frac{i}{2} \\ 0 & 0 & 0 & 0 & \frac{-2}{\sqrt{455}} & \frac{i}{2}\sqrt{\frac{7}{65}} & \frac{-2}{\sqrt{35}} & \sqrt{\frac{3}{5}} & 0 & \frac{i}{2} \\ 0 & 0 & 0 & 0 & \frac{-2}{\sqrt{455}} & \frac{i}{2}\sqrt{\frac{7}{65}} & \frac{-2}{\sqrt{35}} & \frac{-2}{\sqrt{15}} & \frac{1}{\sqrt{3}} & \frac{i}{2} \\ 0 & 0 & 0 & 0 & \frac{-6i}{\sqrt{455}} & -\frac{3}{2}\sqrt{\frac{7}{65}} & \frac{i}{\sqrt{35}} & \frac{i}{\sqrt{15}} & \frac{i}{\sqrt{3}} & \frac{1}{2} \end{pmatrix}. \quad (\text{B11})$$

It is the results (B5), (B7), (B9), and (B11) for the U matrices that we adopt in our calculations for the linear-optical clusters in Sec. III B, such as in finding the excess-noise operators in (59), (64), (67), and (70).

APPENDIX C: DERIVATION FOR EQ. (73) AND THE NOISE-REDUCTION MECHANISM FOR MULTI-RAIL LINEAR-OPTICAL CLUSTERS

Here we derive the identity (73) for the noise correlators that is indispensable in calculating the quadrature correlators (27) for linear-optical clusters. Based on (73), we will also provide analysis for the noise reduction through multirail designs in measurement-based teleportation with linear-optical clusters.

To begin with, let us introduce a simplified notation by writing $\vec{\beta}_k \equiv \sum_{l \in N_k} \vec{\alpha}_l$ in (46), so that we have

$$\vec{u}_k = \vec{\alpha}_k + i\vec{\beta}_k. \quad (\text{C1})$$

The orthonormality condition of \vec{u}_k (or unitarity of U) (47) can then be put in a more succinct form,

$$\begin{aligned} \sum_m (\alpha_{k,m} \alpha_{l,m} + \beta_{k,m} \beta_{l,m}) &= \delta_{kl}, \\ \sum_m (\alpha_{k,m} \beta_{l,m} - \alpha_{l,m} \beta_{k,m}) &= 0, \end{aligned} \quad (\text{C2})$$

where $\alpha_{k,m}$ denotes the m th component of $\vec{\alpha}_k$ and similarly for $\beta_{k,m}$. For the unitary transformation (45) induced by a linear-optical network, if we use the notation of (C1) and write

in favor of the quadrature operators, we would get

$$\begin{aligned} \hat{q}_k &= \sum_l (\alpha_{k,l} \hat{q}_l - \beta_{k,l} \hat{p}_l), \\ \hat{p}_k &= \sum_l (\alpha_{k,l} \hat{p}_l + \beta_{k,l} \hat{q}_l). \end{aligned} \quad (\text{C3})$$

It then follows that the noise operators are here

$$\hat{\delta}_k = \sum_m \left(\alpha_{k,m} + \sum_{n \in N_k} \beta_{n,m} \right) \hat{p}_m, \quad (\text{C4})$$

which is (49) expressed in the notation of (C1). Since the initial cluster modes are uncorrelated, it follows that $\langle \hat{p}_m \hat{p}_{m'} \rangle = \langle \hat{p}_m^2 \rangle \delta_{mm'}$. The noise correlators can thus be written

$$\langle \hat{\delta}_k \hat{\delta}_l \rangle = \sum_m \left[\left(\alpha_{k,m} + \sum_{n \in N_k} \beta_{n,m} \right) \left(\alpha_{l,m} + \sum_{n' \in N_l} \beta_{n',m} \right) \langle \hat{p}_m^2 \rangle \right]. \quad (\text{C5})$$

In the case when all initial cluster modes are momentum-squeezed vacuum states with the same squeezing parameter r , we have $\langle \hat{p}_m^2 \rangle = e^{-2r}/4$ for every mode m . One can then deal with the summations in (C5) making use of the unitarity of U through (C2). Factoring out the constant $\langle \hat{p}_m^2 \rangle$ in (C5), we spell out the summations there and arrive at

$$\sum_m \left(\alpha_{k,m} \alpha_{l,m} + \alpha_{k,m} \sum_{n' \in N_l} \beta_{n',m} + \alpha_{l,m} \sum_{n \in N_k} \beta_{n,m} + \sum_{n \in N_k} \sum_{n' \in N_l} \beta_{n,m} \beta_{n',m} \right). \quad (\text{C6})$$

For the second term in the equation above, by exchanging the order of summations, we can write, with the aid of (C2),

$$\sum_{n' \in N_l} \left(\sum_m \alpha_{k,m} \beta_{n',m} \right) = \sum_{n' \in N_l} \left(\sum_m \alpha_{n',m} \beta_{k,m} \right). \quad (\text{C7})$$

Exchanging again the order of summations in the last expression, we can arrive at

$$\sum_m \beta_{k,m} \left(\sum_{n' \in N_l} \alpha_{n',m} \right) = \sum_m \beta_{k,m} \beta_{l,m}. \quad (\text{C8})$$

For the third term in (C6), it is just the second term there with k and l being exchanged. The result is thus the same as that of (C8). For the fourth term in (C6), again exchanging the order of summations there and using (C2), we can write

$$\sum_{n \in N_k} \sum_{n' \in N_l} \left(\sum_m \beta_{n,m} \beta_{n',m} \right) = \sum_{n \in N_k} \sum_{n' \in N_l} \left(\delta_{nn'} - \sum_m \alpha_{n,m} \alpha_{n',m} \right). \quad (\text{C9})$$

Exchanging again the order of summations in the second term above, we get

$$\sum_{n \in N_k} \sum_{n' \in N_l} \delta_{nn'} - \sum_m \left(\sum_{n \in N_k} \alpha_{n,m} \right) \left(\sum_{n' \in N_l} \alpha_{n',m} \right) = \sum_{n \in N_k} \sum_{n' \in N_l} \delta_{nn'} - \sum_m \beta_{k,m} \beta_{l,m}. \quad (\text{C10})$$

Using the results (C8) and (C10) for the respective terms in (C6), we thus find that the summations there become

$$\sum_{n \in N_k} \left(\sum_{n' \in N_l} \delta_{nn'} \right) + \sum_m (\alpha_{k,m} \alpha_{l,m} + \beta_{k,m} \beta_{l,m}) = M_{kl} + \delta_{kl}, \quad (\text{C11})$$

where we have used the first equation of (C2) in dealing with the second summation on the left-hand side. For the first summation in (C11), it yields exactly the M_{kl} defined in (74). Utilizing the result (C11) in (C5), we arrive finally at the identity (73).

In the light of the result (73), we will now demonstrate how noise reduction in the multirail teleportation can be optimized through a symmetric arrangement among the rails of a linear-optical cluster. In other words, we will show that the coefficient $\frac{-1}{N}$ in the expressions for \hat{q}_μ and \hat{q}_ν in (72) serves to minimize the excess noise in these output quadratures. Similar to (44) for canonical clusters, for CZ-gate teleportation through an N -rail linear-optical cluster, one can have \hat{q}_μ and \hat{q}_ν in the generic expressions

$$\hat{q}_\sigma = \hat{q}_\rho - \sum_k \eta_k \hat{\delta}_k, \quad (\text{C12})$$

where $(\sigma, \rho) = (\mu, \alpha)$ and (ν, β) , and the summation over k covers the same range as in (72). Note that here we have kept the minus sign in front of the summation for consistency with its two-rail counterpart (71). As before, the coefficients η_k in (C12) must satisfy the condition $\sum_k \eta_k = 1$. Now that the noise operators $\hat{\delta}_k$ for linear-optical clusters are no longer independent from each other, the excess noise in the teleported \hat{q}_σ becomes here $\langle (\hat{q}_\sigma - \hat{q}_\rho)^2 \rangle = \sum_{k,l} \eta_k \eta_l \langle \hat{\delta}_k \hat{\delta}_l \rangle$. For a uniformly squeezed linear-optical cluster state, we can write this expression as

$$\begin{aligned} & \sum_k \left(\eta_k^2 \langle \hat{\delta}_k^2 \rangle + \sum_{l \neq k} \eta_k \eta_l \langle \hat{\delta}_k \hat{\delta}_l \rangle \right) \\ &= \sum_k \left(3\eta_k^2 + \sum_{l \neq k} 2\eta_k \eta_l \right) \frac{e^{-2r}}{4}, \end{aligned} \quad (\text{C13})$$

where we have applied (73) in reaching the right-hand side. Since $\sum_k \eta_k = 1$, it follows that

$$\sum_{k,l} \eta_k \eta_l = \sum_k \left(\eta_k^2 + \sum_{l \neq k} \eta_k \eta_l \right) = 1. \quad (\text{C14})$$

We can therefore reduce the expression (C13) for the excess noise to the form

$$\left(\sum_k \eta_k^2 + 2 \right) \frac{e^{-2r}}{4}. \quad (\text{C15})$$

It is now clear that minimization for the excess noise here is completely identical to that for the case of canonical clusters [see below (44)]. Namely, it corresponds to locating the point over the N -dimensional hyperplane $\sum_k \eta_k = 1$ that is closest to the origin. Obviously, the result is the symmetric point $\eta_k = 1/N$ for all k , which means that the excess noise for \hat{q}_σ

in (C12) can be minimized if it is distributed equally over each of the multirails.

APPENDIX D: SINGLE-QND CONSTRUCTION FOR MULTIRAIL CV CANONICAL CLUSTERS

We discuss in this Appendix the fabrication of multirail CV canonical cluster states in the time-encoding single-QND approach [25]. For simplicity, we will demonstrate with the single-mode teleportation shown in Fig. 1(c), which is redrawn in Fig. 7 with the nodes numbered in accordance with their time sequence in the single-QND scheme (see below). Let us start by considering the teleportation through an extended network of QND gates illustrated in Fig. 7, where the teleportation is divided into three stages: (i) input coupling, (ii) multirail construction, and (iii) output generation. At stage (i), the input mode is coupled to qumode 1 through a QND gate. The input mode is then subject to a homodyne detection, while qumode 1 is directed toward the next stage. At the same time, in preparation for stage (ii), the momentum-squeezed mode 3 is also generated. At stage (ii), qumodes 1 and 3 entangle with additional momentum-squeezed qumodes $\{2, 4, \dots, (N+2)\}$ through a sequence of QND gates to form the multirails. The midrail modes $\{2, 4, \dots, (N+2)\}$ in the multirails are measured subsequently in preparation for the corrective operations over qumode 3 in stage (iii), while modes 1 and 3 are led to the next stage. At stage (iii), qumode 1 is measured, while qumode 3 is corrected in accordance with measurement outcomes of all other modes to generate the desired output mode μ .

As is clear from Fig. 7, stage (ii) of the teleportation consists of N repeated units that entangle modes 1 and 3 to the midrail modes $\{2, 4, \dots, (N+2)\}$ via pairs of QND gates, and subsequently detect the midrail modes. One can thus readily condense stage (ii) into the single-QND design shown in Fig. 8. Namely, by cycling qumodes 1 and 3 repeatedly through the QND gate while directing the midrail modes along a U -shaped optical path that would meet qumodes 1 and 3 at appropriate times, one would be able to implement the multirail structure in the time domain utilizing a single QND gate. Since there are N multirails in total, qumodes 1 and 3 must cycle N times before entering stage (iii). Therefore, the optical path for qumodes 1 and 3 can be fashioned into a helical one, so that the loop in Fig. 8 is just a single run of it. On the other hand, due to the lack of symmetry in the setup at stage (i), its transition into the highly symmetric stage (ii) poses the major challenge to the single-QND implementation here. We propose to overcome this difficulty through the design illustrated in Fig. 9, as we shall now explain.

First, we send in the input mode α toward the QND gate for coupling with qumode 1 at time $t = 0$ and then direct it to a homodyne detector [Fig. 9(a)]. Second, the input-mode-encoded qumode 1 then proceeds along a helical path and

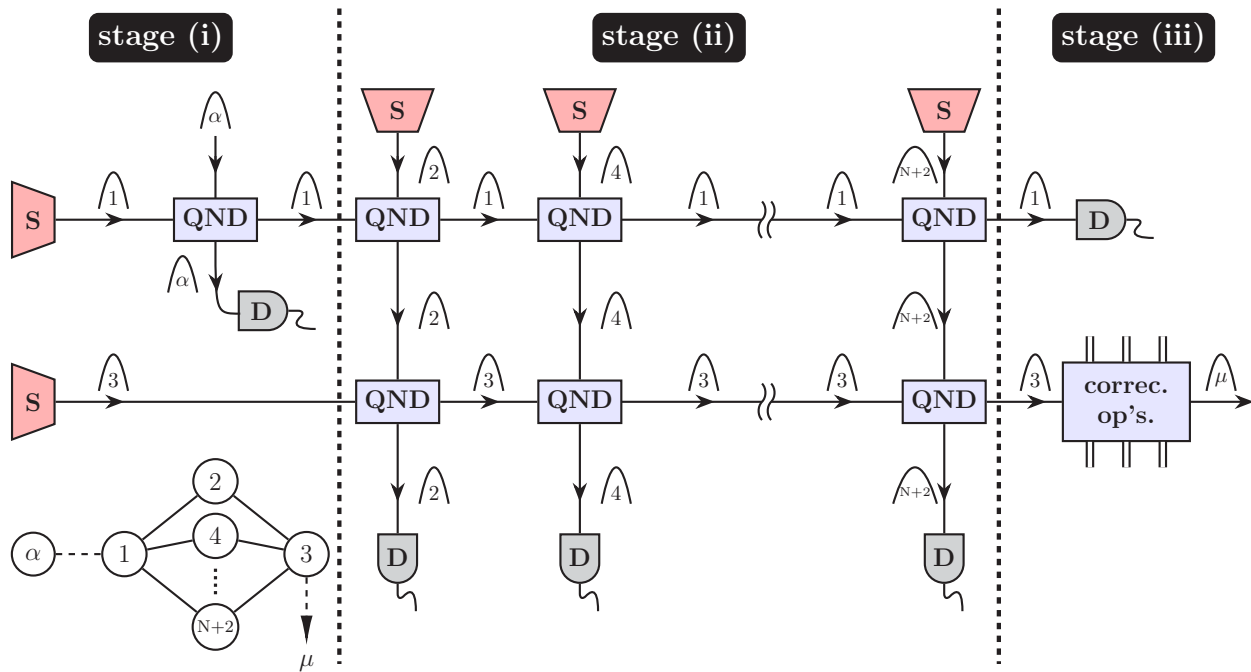


FIG. 7. A network of QND gates for the single-mode teleportation through an N -rail canonical cluster shown in the lower left corner of this figure. Here “S” are single-mode squeezers that generate momentum-squeezed vacuum modes, “QND” are QND gates that entangle the pairs of intersecting modes, and “D” are the homodyne detectors for the qumodes. Each qumode is denoted by a numbered pulse for the corresponding cluster node, and similarly for the input mode α and the output mode μ . The teleportation is divided into three stages (see text), which are indicated by vertical dotted lines. The gate “correc. op’s.” in stage (iii) represents corrective operations for the output mode, where the double wires on the sides stand for signals fed forward from measurement outcomes of other qumodes.

returns to the QND gate for entanglement with qumode 2 at time $t = T$. Qumode 2 then follows a U -shaped optical path that will lead it back to the QND gate [Fig. 9(b)]. Third, qumode 3 is then generated and propagates along the same optical path as qumode 1, so that it would meet with qumode 2 at the QND gate at time $t = 1.5 T$. After this, qumode 2 is then detected, while qumode 3 proceeds along the helical path with a time lag $1.5 T$ behind qumode 1 [Fig. 9(c)]. We are now connected to the fully implemented stage (ii) illustrated in Fig. 8, which corresponds roughly to a snapshot at $t \simeq 1.75 T$.

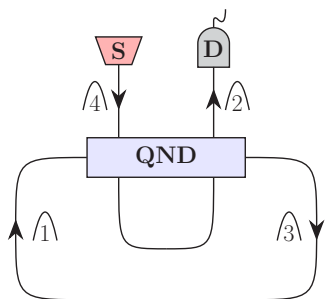


FIG. 8. A snapshot for the single-QND counterpart for stage (ii) of the multirail teleportation depicted in Fig. 7. Here the midrail modes $\{2, 4, \dots, (N + 2)\}$ are produced sequentially from the squeezer at a fixed interval, so that they can meet qumodes 1 and 3 at appropriate times at the QND gate. Note that for the optical paths the path lengths illustrated here are not in actual proportions and the loop for qumodes 1 and 3 is in fact part of a helical path; see text and also Fig. 9.

Finally, in order to separate qumodes 1 and 3 over the same helical path at stage (iii), one can prepare them with orthogonal polarizations at the outset. The single-QND counterpart for stage (iii) can then be achieved through a polarization beam splitter which would divert, for instance, qumode 1 to a homodyne detector, while passing qumode 3 to the corrective gates. In this case, of course, one must ensure that the QND gate should function impartially for both polarizations.

It is interesting to note that the scheme above can be generalized easily for single-qumode teleportation involving a longer linear cluster implemented with multirail structures. This can be done by repeating the single-QND procedures above while skipping at stage (iii) the corrective operations for qumode 3. That is, by taking the *uncorrected* qumode 3 as the new input state α in Fig. 9(a), one can furnish an additional multirail teleportation down the linear cluster. Depending on the length of the linear cluster, this iteration can be terminated via the stage (iii) implementation above once the desired number of runs is reached.

For two-mode operations, such as the CZ-gate teleportation of Fig. 4(d), one could extend the preceding single-QND design by either time multiplexing or polarization multiplexing similar to what was proposed for two-dimensional square lattices in the single-QND approach [25]. For instance, if we modify the node configurations in Fig. 4(d) slightly by connecting the input modes to nodes 1 and $(2N + 4)$, and output modes to nodes $(N + 2)$ and $(N + 3)$, we would then have two identical, independent single-qumode teleporting arms before reaching the output stage. Therefore, the single-QND implementation here can be achieved through two copies

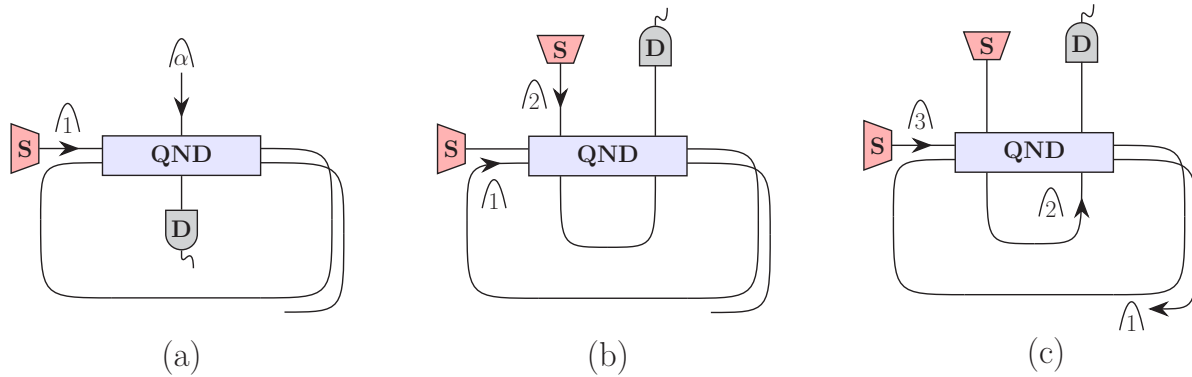


FIG. 9. Single-QND realization for the transition from stage (i) [panel (a)] to the initial steps of stage (ii) [panels (b), (c)] for the teleportation shown in Fig. 7. For clarity, we show only part of the helical path for modes 1 and 3.

of the pulse sequence above intercalating each other in time. At the output stage, however, prior to subjecting qumodes $(N + 2)$ and $(N + 3)$ to the respective corrective operations, one must

direct them back to the QND gate for entanglement. Since these two modes are separated in time, this can be achieved with the help of an active beam diverter.

- [1] This is aptly reflected in the quote from Galileo Galilei, “Measure what can be measured, and make measurable what cannot be measured,” although the source of which is unclear; see, e.g., T. Strohmer, *IEEE Signal Proc. Lett.* **19**, 887 (2012).
- [2] *Quantum Theory and Measurement*, edited by J. A. Wheeler and W. H. Zurek (Princeton University Press, Princeton, NJ, 1983).
- [3] V. B. Braginsky and F. Ya. Khalili, *Quantum Measurement* (Cambridge University Press, Cambridge, UK, 1992).
- [4] H. M. Wiseman and G. J. Milburn, *Quantum Measurement and Control* (Cambridge University Press, Cambridge, UK, 2010).
- [5] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, UK, 2000).
- [6] D. DiVincenzo, *Fortschr. Phys.* **48**, 771 (2000); reprinted also as Ch. 1 of Ref. [7].
- [7] *Scalable Quantum Computers: Paving the Way to Realization*, edited by S. L. Braunstein, H.-K. Lo, and P. Kok (Wiley, New York, 2001).
- [8] R. Raussendorf and H. J. Briegel, *Phys. Rev. Lett.* **86**, 5188 (2001).
- [9] M. A. Nielsen, *Rep. Math. Phys.* **57**, 147 (2006).
- [10] H. J. Briegel, D. E. Browne, W. Dür, R. Raussendorf, and M. Van den Nest, *Nat. Phys.* **5**, 19 (2009).
- [11] The term “gate teleportation” sometimes refers specifically to processes in which the gate operation is effected through sequence of Bell measurements, which is not necessarily the case for all measurement-based gate operations; see, for example, R. Jozsa, [arXiv:quant-ph/0508124](https://arxiv.org/abs/quant-ph/0508124). However, here we use the term “gate teleportation” in a broader sense, referring to processes where the gate operation is achieved through appropriate measurement sequence and corrective operations over a resource state.
- [12] J. Zhang and S. L. Braunstein, *Phys. Rev. A* **73**, 032318 (2006).
- [13] N. C. Menicucci, P. van Loock, M. Gu, C. Weedbrook, T. C. Ralph, and M. A. Nielsen, *Phys. Rev. Lett.* **97**, 110501 (2006).
- [14] *Quantum Information with Continuous Variables*, edited by S. L. Braunstein and A. K. Pati (Kluwer Academic, Dordrecht, Netherlands, 2003).
- [15] S. L. Braunstein and P. van Loock, *Rev. Mod. Phys.* **77**, 513 (2005).
- [16] *Quantum Information with Continuous Variables of Atoms and Light*, edited by N. J. Cerf, G. Leuchs, and E. S. Polzik (Imperial College Press, London, 2007).
- [17] C. Weedbrook, S. Pirandola, R. García-Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, *Rev. Mod. Phys.* **84**, 621 (2012).
- [18] S. L. Braunstein, *Phys. Rev. A* **71**, 055801 (2005).
- [19] By “inline squeezing,” we mean squeezing operations over nonvacuum input states, while “offline squeezing” refers to those with vacuum inputs.
- [20] P. van Loock, C. Weedbrook, and M. Gu, *Phys. Rev. A* **76**, 032321 (2007).
- [21] R. Ukai, N. Iwata, Y. Shimokawa, S. C. Armstrong, A. Politi, J.-I. Yoshikawa, P. van Loock, and A. Furusawa, *Phys. Rev. Lett.* **106**, 240504 (2011).
- [22] R. Ukai, S. Yokoyama, J.-I. Yoshikawa, P. van Loock, and A. Furusawa, *Phys. Rev. Lett.* **107**, 250501 (2011).
- [23] X. Su, S. Hao, X. Deng, L. Ma, M. Wang, X. Jia, C. Xie, and K. Peng, *Nat. Commun.* **4**, 2828 (2013).
- [24] N. C. Menicucci, S. T. Flammia, and O. Pfister, *Phys. Rev. Lett.* **101**, 130501 (2008); S. T. Flammia, N. C. Menicucci, and O. Pfister, *J. Phys. B* **42**, 114009 (2009); M. Chen, N. C. Menicucci, and O. Pfister, *Phys. Rev. Lett.* **112**, 120505 (2014).
- [25] N. C. Menicucci, X. Ma, and T. C. Ralph, *Phys. Rev. Lett.* **104**, 250503 (2010).
- [26] N. C. Menicucci, *Phys. Rev. A* **83**, 062314 (2011); S. Yokoyama, R. Ukai, S. C. Armstrong, C. Sornphiphatphong, T. Kaji, S. Suzuki, J. Yoshikawa, H. Yonezawa, N. C. Menicucci, and A. Furusawa, *Nat. Photon.* **7**, 982 (2013).
- [27] N. C. Menicucci, *Phys. Rev. Lett.* **112**, 120504 (2014).
- [28] Y.-C. Su, MS thesis, National Chung Cheng University, Taiwan, 2015.
- [29] D. F. Walls and G. J. Milburn, *Quantum Optics*, 2nd ed. (Springer-Verlag, Berlin, 2008).
- [30] R. Ukai, J.-I. Yoshikawa, N. Iwata, P. van Loock, and A. Furusawa, *Phys. Rev. A* **81**, 032315 (2010).

- [31] P. Kok and B. W. Lovett, *Introduction to Optical Quantum Information Processing* (Cambridge University Press, Cambridge, UK, 2010).
- [32] A. Furusawa and P. van Loock, *Quantum Teleportation and Entanglement* (Wiley-VCH, Weinheim, 2011).
- [33] Strictly speaking, the quadrature manipulation for the case of Fig. 2 belongs also not to the trivial case (a), since the corrective operations (14) involve Fourier transformations over output cluster nodes. Thus, the quadrature manipulation in this case belongs to category (c). Nevertheless, the Fourier transformations \hat{F}_k^2 in (14) merely induce sign changes in the quadrature operators for nodes $k = 2, 3$; the remaining quadrature manipulations there are relatively trivial in comparison with other cases considered in this paper.
- [34] J. Eisert and M. B. Plenio, *Int. J. Quantum Inform.* **01**, 479 (2003); G. Adesso and F. Illuminati, *J. Phys. A* **40**, 7821 (2007).
- [35] R. Simon, *Phys. Rev. Lett.* **84**, 2726 (2000).
- [36] R. F. Werner and M. M. Wolf, *Phys. Rev. Lett.* **86**, 3658 (2001).
- [37] Note that the number $1/4$ is dependent on our choice of setting $\hbar = 1/2$ in writing (1). Another choice with $\hbar = 2$ that is often adopted in the related literature would have this number being 1, instead; see Ref. [17].
- [38] A. Serafini, F. Illuminati, and S. De Siena, *J. Phys. B* **37**, L21 (2004); G. Adesso, A. Serafini, and F. Illuminati, *Phys. Rev. A* **70**, 022318 (2004).
- [39] G. Vidal and R. F. Werner, *Phys. Rev. A* **65**, 032314 (2002).
- [40] M. Reck, A. Zeilinger, H. J. Bernstein, and P. Bertani, *Phys. Rev. Lett.* **73**, 58 (1994).
- [41] For the case of Fourier-transformed input modes, the input-coupling equation (17) for canonical cluster states remain to have \hat{q}'' entries that will not be helpful for quadrature manipulations, despite the Fourier transformations. This is because these entries involve only either the input mode ρ or the cluster mode k , while for quadrature manipulations we need entries of the input-coupling equation that would mix the input and the cluster modes.
- [42] Y. Miwa, R. Ukai, J. I. Yoshikawa, R. Filip, P. van Loock, and A. Furusawa, *Phys. Rev. A* **82**, 032305 (2010); N. C. Menicucci, S. T. Flammia, and P. van Loock, *ibid.* **83**, 042335 (2011).
- [43] G. Ferrini, J. Roslund, F. Arzani, Y. Cai, C. Fabre, and N. Treps, *Phys. Rev. A* **91**, 032314 (2015).