Smoothed quantum-classical states in time-irreversible hybrid dynamics

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We consider a quantum system continuously monitored in time which in turn is coupled to an arbitrary dissipative classical system (diagonal reduced density matrix). The quantum and classical dynamics can modify each other, being described by an arbitrary time-irreversible hybrid Lindblad equation. Given a measurement trajectory, a conditional bipartite stochastic state can be inferred by taking into account all previous recording information (filtering). Here, we demonstrate that the joint quantum-classical state can also be inferred by taking into account both past and future measurement results (smoothing). The smoothed hybrid state is estimated without involving information from unobserved measurement channels. Its average over recording realizations recovers the joint time-irreversible behavior. As an application we consider a fluorescent system monitored by an inefficient photon detector. This feature is taken into account through a fictitious classical two-level system. The average purity of the smoothed quantum state increases over that of the (mixed) state obtained from the standard quantum jump approach.

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I. INTRODUCTION

In quantum mechanics the state of a system is described by a state vector, or more generally by a density-matrix operator in the case of open systems [1]. The environmental influence renders the system dynamics irreversible in time. In addition, the environment may be continuously monitored in time by some measuring device. A fundamental problem solved by the quantum jump approach [1–4] is the estimation of the system state conditioned on a given (single) measurement signal (trajectory). Given the stochastic nature of the measurement process, the system state inherits this property, while its average over measurement trajectories recovers the irreversible system dynamics.

The quantum jump approach delivers a stochastic system state that depends on all previous measurement results. The estimation is possible after knowing the system initial condition and its dynamics. Different refinements of this technique were known in a classical context (classical estimation theory) [5,6]. *Filtering* is a Bayesian estimation technique where the system state is conditioned on earlier measurements while *smoothing* means that both earlier and later measurements are considered. Hence, the standard quantum jump approach can be considered as a quantum extension of classical filtering. Different formulations of a "quantum version" of smoothing have been achieved recently [7–17].

The estimation of a *classical parameter* that affects the evolution of a quantum system using the results of (both earlier and later) measurements on that system was performed by Tsang in Ref. [7]. Specific physical applications of this approach have been analyzed [8]. Estimation of the result of a quantum measurement using past and future information was characterized by Gammelmark, Julsgaard, and Mølmer in Ref. [9]. A *past quantum state*, consisting of a pair of matrices, a density matrix, and an "effect operator," allows us to achieve a better estimation of a nonselective measurement performed over the system in the past. Extra analysis and specific implementations were posteriorly characterized [10–16]. In contrast with the previous results, a *smoothed quantum state* (positive density operator) consistent with past and future measurement information was introduced by Guevara and Wiseman in Ref. [17]. They considered a partially monitored optical quantum system. The smoothed quantum state can be estimated after knowing a pure state conditioned on both the observed (homodyne photocurrent) and unobserved (photon count) records. A significant recovering of the purity lost due to the unobserved signal is achieved.

Following the previous research lines, in this paper we demonstrate that a smoothed quantum-classical state can be consistently defined. It describes the estimated joint state of a dissipative (time-irreversible) hybrid quantum-classical arrangement when past and future measurement signals performed on the quantum subsystem are taken into account.

In contrast with previous analysis [7,9], a joint smoothed state is explicitly defined. In addition, here the quantum and classical dynamics are intrinsically correlated. Each one may modify the other. Their evolution is described by an arbitrary time-irreversible (hybrid) Lindblad rate equation [18,19], which corresponds to the more general bipartite evolution restricted by the requirements of Markovianicity and classicality (one of the reduced density matrices is diagonal in a fixed basis at all times). The smoothed joint state can be estimated after knowing a measurement trajectory, the initials conditions, and the characteristic parameters of the dynamics. By partial trace the bipartite smoothed state provides the partial (smoothed) states of the quantum system and the classical counterpart. As in the standard quantum jump approach, by averaging over realizations (past and futures ones) the joint irreversible dynamics is recovered.

As an application we consider a single fluorescent system monitored by an inefficient photon detector. By introducing a fictitious classical degree of freedom associated to the detector, the formalism applies to this situation. The purity of the smoothed state increases with respect to that obtained from the standard quantum jump approach. In contrast with previous approaches [17], for estimating the smoothed state it is not necessary to determine the (past) unobserved signal trajectory. This is a general feature of the present formalism.

The paper is outlined as follows. In Sec. II we introduce the underlying formalism that describes the hybrid quantum-classical evolution [18]. The corresponding filtered state, equivalent to the quantum jump approach [20], is also reviewed. Over this basis, the smoothed quantum-classical state is developed in Sec. III. As an example, in Sec. IV the formalism is applied to a fluorescent system monitored by an inefficient detector. The conclusions are provided in Sec. IV. Calculus details that support the main results are provided in the Appendices.

II. QUANTUM-CLASSICAL DYNAMICS

We consider a quantum system S interacting with classical (time-irreversible) degrees of freedom, denoted by C. The bipartite arrangement can be described by a hybrid quantum-classical density operator $|\rho_t)$. It is written as

$$|\rho_t) \equiv \sum_R (R|\rho_t)|R). \tag{1}$$

Here, the index R labels each state of C, which in turn has assigned a (column) vector $|R) = (0,\dots 1,\dots 0)^{\mathrm{T}}$. The (real) vectorial base $\{|R)\}$ satisfies $(R|R') = \delta_{RR'}$. Each (conditional) density operator $(R|\rho_t)$ is defined in the Hilbert space of S. Introducing the vector $(1| \equiv \sum_R (R| = (1,\dots,1)$, the unconditional density operator ρ_t of S follows from the hybrid (vectorial) operator as [18]

$$\rho_t = (1|\rho_t) = \sum_R (R|\rho_t). \tag{2}$$

The vector $|P_t)$ of classical probabilities $\{P_t[R]\}$ for the set of states $\{|R)\}$ can be written as

$$|P_t) = \sum_R P_t[R]|R) = \sum_R \mathrm{Tr}[(R|\rho_t)]|R). \tag{3}$$

$\mathrm{Tr}[\bullet]$ denotes trace operation in the Hilbert space of S. With the previous definitions, the vectorial hybrid operator can be rewritten as

$$|\rho_t) = \sum_R P_t[R]\frac{(R|\rho_t)}{\mathrm{Tr}[(R|\rho_t)]}|R). \tag{4}$$

Hence, $(R|\rho_t)/\mathrm{Tr}[(R|\rho_t)]$ is the quantum state of S *given that* C is in the state $|R)$.

The more general time-irreversible (Markovian) evolution equation describing the interaction between S and C is given by a (hybrid) Lindblad rate equation [18],

$$\frac{d|\rho_t)}{dt} = \hat{\mathcal{L}}|\rho_t), \tag{5}$$

where (separable) initial conditions are assumed, $|\rho_0) = \rho_0|P_0)$.

The evolution generator $\hat{\mathcal{L}}$ is a matrix of superoperators. This property is denoted by the upper hat symbol. From now on, superoperators of this kind are named as *vectorial superoperators*. $\hat{\mathcal{L}}$ may adopt very different structures [18,19], which may be used, for example, for describing radiation patterns in fluorescent systems coupled to classically fluctuating reservoirs [20,21] [see also Eq. (31) below]. The following analysis and results are valid for arbitrary $\hat{\mathcal{L}}$ structures, even beyond those associated to Lindblad rate equations. The specific hybrid nature of the quantum-classical arrangement becomes a fundamental ingredient in the results developed in Sec. III.

Filtered state

The quantum system S (or equivalently its environment) is continuously monitored in time. We assume that, up to time t, each recorded measurement realization consists of a set of random times $\overleftarrow{t} \equiv \{t_1,t_2,\dots t_n\}$, with $0 \leqslant t_i \leqslant t$. Each time t_i can be associated to the time at which S suffers a given transition. A filtered state is an estimation $|\rho_t^{\mathrm{st}})$ of the bipartite state conditioned on a given (past) measurement trajectory. In addition to the times $\overleftarrow{t}$, the initial bipartite state $|\rho_0)$ and its dynamics [Eq. (5)] are known. The quantum jump approach allows us to define a filtered state $|\rho_t^{\mathrm{st}})$, which relies on the closure condition

$$\overleftarrow{|\rho_t^{\mathrm{st}})} = |\rho_t), \tag{6}$$

that is, the average of $|\rho_t^{\mathrm{st}})$ over measurement trajectories (denoted with the over arrow $\leftarrow$) recovers the dynamics dictated by Eq. (5).

The filtered state can be written as (see Appendix A)

$$|\rho_t^{\mathrm{st}}) = \frac{\hat{\mathcal{U}}[t,0,\overleftarrow{t}]|\rho_0)}{\mathrm{Tr}[(1|\hat{\mathcal{U}}[t,0,\overleftarrow{t}]|\rho_0)]}. \tag{7}$$

Here, the (unconditional) vectorial propagator $\hat{\mathcal{U}}$ is defined as

$$\hat{\mathcal{U}}[t,t',\{\tau_i\}_1^n] \equiv e^{\hat{\mathcal{D}}(t-\tau_n)}\left\{\prod_{i=2}^{n}\hat{\mathcal{J}}e^{\hat{\mathcal{D}}(\tau_i-\tau_{i-1})}\right\}\hat{\mathcal{J}}e^{\hat{\mathcal{D}}(\tau_1-t')}. \tag{8}$$

The vectorial superoperators $\hat{\mathcal{D}}$ and $\hat{\mathcal{J}}$ recover the bipartite dynamics generator

$$\hat{\mathcal{L}} = \hat{\mathcal{D}} + \hat{\mathcal{J}}. \tag{9}$$

The vectorial superoperator $\hat{\mathcal{J}}$ [see, for example, explicit expressions (29) (unidimensional case) and (33) (vectorial case)] is chosen such that the transformation $|\rho) \to \hat{\mathcal{M}}|\rho)$, where

$$\hat{\mathcal{M}}|\rho) = \frac{\hat{\mathcal{J}}|\rho)}{\mathrm{Tr}[(1|\hat{\mathcal{J}}|\rho)]} \tag{10}$$

corresponds to the measurement transformation of the bipartite state given that a measurement record occurred. Similarly, a propagator $\hat{\mathcal{T}}(t,\tau)$ associated to the superoperator $\hat{\mathcal{D}} = \hat{\mathcal{L}} - \hat{\mathcal{J}}$,

$$\hat{\mathcal{T}}(t,\tau)|\rho) \equiv \frac{e^{\hat{\mathcal{D}}(t-\tau)}|\rho)}{\mathrm{Tr}[(1|e^{\hat{\mathcal{D}}(t-\tau)}|\rho)]}, \tag{11}$$

can be read as the (normalized) transformation of the conditional state between recording events happening successively at times τ and t.

Given the property defined by Eq. (6) one can associate a probability density $P_t[\overleftarrow{t}]$ for the occurrence of a given trajectory (defined by the set of times $\overleftarrow{t}$). It reads

$$P_t[\overleftarrow{t}] = \mathrm{Tr}[(1|\hat{\mathcal{U}}[t,0,\overleftarrow{t}]|\rho_0)]. \tag{12}$$

This weight, jointly with the definition (7), allows us to demonstrate that the requirement (6) is in fact fulfilled (Appendix A).

The joint filtered state $|\rho_t^{\rm st})$, Eq. (7), through the relations [see Eqs. (2) and (3)]

$$\rho_t^{\rm st} = \big(1|\rho_t^{\rm st}\big), \quad \big|P_t^{\rm st}\big) = \sum_R \mathrm{Tr}\big[\big(R\big|\rho_t^{\rm st}\big)\big]|R), \tag{13}$$

also allows us to estimate the partial states of S and C.

III. QUANTUM-CLASSICAL SMOOTHED STATE

The state estimation (7) is conditioned on measurement results previous to the time t, that is, the set $\overleftarrow{t}$. It is also possible to take into account posterior (future of t) recording events up to a given time $T > t$. The events between t and T are denoted by $\overrightarrow{t} \equiv \{t_{n+1},t_{n+2},\ldots t_N\}$, which satisfy $t \leqslant t_i \leqslant T$. The task now is to find the new estimation for the joint state (smoothed state) taking into account this extra information.

The joint probability density $P_T[\overleftrightarrow{t}]$ for a trajectory in $(0,T)$ with detection times $\overleftrightarrow{t} \equiv \overleftarrow{t} \cup \overrightarrow{t} = \{t_i\}_1^N$, from Eq. (12), can be written as

$$P_T[\overleftrightarrow{t}] = \mathrm{Tr}[(1|\hat{\mathcal{U}}[T,t,\overrightarrow{t}]\hat{\mathcal{U}}[t,0,\overleftarrow{t}]|\rho_0)]. \tag{14}$$

This object can also be expressed as

$$P_T[\overleftrightarrow{t}] = \sum_{R_t} P_T[\overleftrightarrow{t},R_t]. \tag{15}$$

Here, $P_T[\overleftrightarrow{t},R_t]$ is the joint probability density of the random variables $\overleftrightarrow{t}$ and R_t. The last one labels the state of the classical degrees of freedom at time t. From Eq. (14) we write

$$P_T[\overleftrightarrow{t},R_t] = \mathrm{Tr}[(1|\hat{\mathcal{U}}[T,t,\overrightarrow{t}]|R_t)(R_t|\hat{\mathcal{U}}[t,0,\overleftarrow{t}]|\rho_0)]. \tag{16}$$

This expression can be interpreted in terms of a (classical) measurement performed over C at time t. Using that $\sum_{R_t}|R_t)(R_t|$ is the identity matrix in the vectorial space of C, it follows that the normalization (15) is satisfied trivially.

By introducing the conditional probability $P_T[R_t|\overleftrightarrow{t}]$ of R_t *given* the set $\overleftrightarrow{t}$, Bayes's rule gives the relation

$$P_T[\overleftrightarrow{t},R_t] = P_T[R_t|\overleftrightarrow{t}]P_T[\overleftrightarrow{t}]. \tag{17}$$

Hence, from Eqs. (15) and (16) we get

$$P_T[R_t|\overleftrightarrow{t}] = \frac{\mathrm{Tr}[(1|\hat{\mathcal{U}}[T,t,\overrightarrow{t}]|R_t)(R_t|\hat{\mathcal{U}}[t,0,\overleftarrow{t}]|\rho_0)]}{\sum_R \mathrm{Tr}[(1|\hat{\mathcal{U}}[T,t,\overrightarrow{t}]|R)(R|\hat{\mathcal{U}}[t,0,\overleftarrow{t}]|\rho_0)]}. \tag{18}$$

This expression allows us to estimate the state of the classical degrees of freedom C at time t given that we know both past and future measurement results ($\overleftrightarrow{t} = \overleftarrow{t} \cup \overrightarrow{t}$) performed on the quantum system S in the time interval $(0,T)$.

Now, we introduce the un-normalized joint filtered state

$$\big|\tilde{\rho}_t^{\rm st}\big) \equiv \hat{\mathcal{U}}[t,0,\overleftarrow{t}]|\rho_0), \tag{19}$$

and the "effect vectorial-operator"

$$\big|E_t^{\rm st}\big) \equiv \hat{\mathcal{U}}^{\#}[T,t,\overrightarrow{t}]|\mathrm{I}). \tag{20}$$

Here $|\mathrm{I}) \equiv \mathrm{I}|1)$, where I is the identity matrix in the Hilbert space of S. Furthermore, $\hat{\mathcal{U}}^{\#}$ is the dual propagator of $\hat{\mathcal{U}}$. It is defined by the relation [19]

$$\mathrm{Tr}[(A|\hat{\mathcal{U}}|\rho)] = \mathrm{Tr}[(\rho|\hat{\mathcal{U}}^{\#}|A)], \tag{21}$$

where $|\rho)$ and $|A)$ are arbitrary vectorial states and operators, respectively. Using that $\mathrm{Tr}[(A|\hat{\mathcal{U}}\hat{\mathcal{V}}|\rho)] = \mathrm{Tr}[(\rho|\hat{\mathcal{V}}^{\#}\hat{\mathcal{U}}^{\#}|A)]$, from Eq. (8) it follows that

$$\hat{\mathcal{U}}^{\#}\big[t,t',\{\tau_i\}_1^n\big] = e^{\hat{\mathcal{D}}^{\#}(\tau_1-t')}\hat{\mathcal{J}}^{\#}\left\{\prod_{i=2}^n e^{\hat{\mathcal{D}}^{\#}(\tau_i-\tau_{i-1})}\hat{\mathcal{J}}^{\#}\right\}e^{\hat{\mathcal{D}}^{\#}(t-\tau_n)}, \tag{22}$$

where $\hat{\mathcal{D}}^{\#}$ and $\hat{\mathcal{J}}^{\#}$ are the dual operators to $\hat{\mathcal{D}}$ and $\hat{\mathcal{J}}$, respectively (see Ref. [19]).

Given that

$$\mathrm{Tr}[(1|\hat{\mathcal{U}}[T,t,\overrightarrow{t}]|\rho)] = \mathrm{Tr}[(\rho|\hat{\mathcal{U}}^{\#}[T,t,\overrightarrow{t}]|\mathrm{I})], \tag{23}$$

the probability (18) can be written in terms of $|\tilde{\rho}_t^{\rm st})$ and $|E_t^{\rm st})$ as

$$P_T[R_t|\overleftrightarrow{t}] = \frac{\mathrm{Tr}\big[\big(\tilde{\rho}_t^{\rm st}\big|R_t\big)\big(R_t\big|E_t^{\rm st}\big)\big]}{\sum_R \mathrm{Tr}\big[\big(\tilde{\rho}_t^{\rm st}\big|R\big)\big(R\big|E_t^{\rm st}\big)\big]}. \tag{24}$$

The structure of this equation is similar to that obtained in Refs. [7,9], where the pair $\{|\tilde{\rho}_t^{\rm st}),|E_t^{\rm st})\}$ can be named as a "vectorial past quantum state". Trivially, under the replacement $|\tilde{\rho}_t^{\rm st}) \to |\rho_t^{\rm st})$, the smoothed probability $P_T[R_t|\overleftrightarrow{t}]$ can also be written in terms of the normalized filtered state (7). Furthermore, for $T \to t$ (filtering), Eq. (13) is recovered, $\lim_{T\to t} P_T[R_t|\overleftrightarrow{t}] = P_t[R_t|\overleftarrow{t}] = \mathrm{Tr}[(\rho_t^{\rm st}|R_t)]$.

From $P_T[R_t|\overleftrightarrow{t}]$ it is possible to define a smoothed quantum-classical state $|\rho_{t,T}^{\rm st})$, that is, an estimation of the quantum-classical joint state taking into account both past and future measurement results. From Eq. (4), we write

$$\big|\rho_{t,T}^{\rm st}\big) = \sum_R P_T[R|\overleftrightarrow{t}]\frac{\big(R\big|\rho_t^{\rm st}\big)}{\mathrm{Tr}\big[\big(R\big|\rho_t^{\rm st}\big)\big]}\big|R\big), \tag{25}$$

where $|\rho_t^{\rm st})$ is the filtered state defined in Eq. (7) while $P_T[R|\overleftrightarrow{t}]$ follows from Eq. (24). This is the main result of this section. Notice that $|\rho_{t,T}^{\rm st})$ can be obtained after knowing the measurement results, the joint initial state, and the quantum-classical dynamics [see Eqs. (8) and (22)].

The previous result relies on the fact that the state of S *given* that C is in the state $|R)$ at time t is given by $(R|\rho_t^{\rm st})/\mathrm{Tr}[(R|\rho_t^{\rm st})]$. Hence, the smoothed probability $P_T[R|\overleftrightarrow{t}]$ is the correct weight of each contribution given that we know both past and future measurement results.

Similarly to the filtering case [Eq. (13)], the relations

$$\rho_{t,T}^{\rm st} = \big(1\big|\rho_{t,T}^{\rm st}\big), \qquad \big|P_{t,T}^{\rm st}\big) = \sum_R \mathrm{Tr}\big[\big(R\big|\rho_{t,T}^{\rm st}\big)\big]\,|R) \tag{26}$$

correspond to the smoothed estimations of the partial states of S and C, respectively.

In Appendix B we demonstrate that by averaging the smoothed joint state $|\rho_{t,T}^{\rm st})$ over future measurement results, $|\rho_{t,T}^{\rm st}) \to \overrightarrow{|\rho_{t,T}^{\rm st})}$, the filtered state $|\rho_t^{\rm st})$ is recovered:

$$\overrightarrow{\big|\rho_{t,T}^{\rm st}\big)} = \big|\rho_t^{\rm st}\big) \Rightarrow \overleftrightarrow{\big|\rho_{t,T}^{\rm st}\big)} = |\rho_t). \tag{27}$$

In addition, the second equality follows straightforwardly from the former one and Eq. (6). The overarrow $\leftrightarrow$ means average over both past and future measurement results. Thus, the average of both the joint smoothed and filtered states recovers the irreversible dynamics of the quantum-classical arrangement, Eq. (5).

IV. INEFFICIENT PHOTON DETECTION

The formalism developed in the previous section may have applications in different contexts. For example, the dynamics of fluorescent systems coupled to classically self-fluctuating reservoirs [20,21] can be described through different Lindblad rate equations. The classical degrees of freedom correspond to different configurational states of the environment. In addition, as the formalism can be applied independently of the physical origin of the classical counterpart, here we apply the previous results to a different physical situation.

We consider a single fluorescent two-level system (with states $|\pm\rangle$) coupled to a resonant laser field. The system-laser (time-reversible) coupling is proportional to Rabi frequency Ω, while its natural (time-irreversible) decay rate is γ. The evolution of its density matrix ρ_t is [1,4]

$$\frac{d\rho_t}{dt} = -\frac{i\Omega}{2}[\sigma_x,\rho_t]_- + \gamma(\sigma\rho_t\sigma^\dagger - \{\sigma^\dagger\sigma,\rho_t\}_+), \qquad (28)$$

where σ_x is the x-Pauli matrix, while $\sigma = |-\rangle\langle+|$, and $\sigma^\dagger = |-\rangle\langle+|$. Furthermore, $[p,q]_- \equiv pq - qp$, while $\{p,q\}_+ \equiv (pq+qp)/2$ denotes an anticommutator.

The scattered radiation field is measured by an inefficient photon detector the efficiency of which is η. The standard quantum jump approach covers this situation [4]. Its description can be recovered from Sec. II in the limit in which the classical system C has only one state (becoming irrelevant), while the quantum one is the two-level system described previously. The splitting defined by Eq. (9) is performed by introducing the (unidimensional) superoperators [4]

$$\mathcal{J}[\rho] = \gamma\eta\sigma\rho\sigma^\dagger, \qquad \mathcal{D} = \mathcal{L} - \mathcal{J}, \qquad (29)$$

where $\mathcal{L}$ follows from Eq. (28), $d\rho_t/dt = \mathcal{L}[\rho]$. Given that

$$\mathcal{M}[\rho] = \frac{\mathcal{J}[\rho]}{\mathrm{Tr}[\mathcal{J}\rho]} = |-\rangle\langle-|, \qquad (30)$$

the system resets to its ground state in each detection event. In consequence the emission process is a renewal one. A waiting-time distribution [2] gives the probability density of the time interval between consecutive events (see Appendix C). The filtered state ρ_t^{st} [Eq. (7)] is not pure, $1/2 \leqslant \mathrm{Tr}[(\rho_t^{\mathrm{st}})^2] \leqslant 1$. Nevertheless, for $\eta = 1$ (perfect detector), a pure state is obtained, $\mathrm{Tr}[(\rho_t^{\mathrm{st}})^2] = 1$ (strictly, this equality is valid in general after the first detection event). Our goal here is to get a new estimation of the system state using the smoothing technique described previously.

A. Quantum-classical representation

The measurement trajectory is given by the detection times obtained from the inefficient detector. Clearly, the system [Eq. (28)] does not include any classical degree of freedom. Nevertheless, one can introduce a fictitious classical system

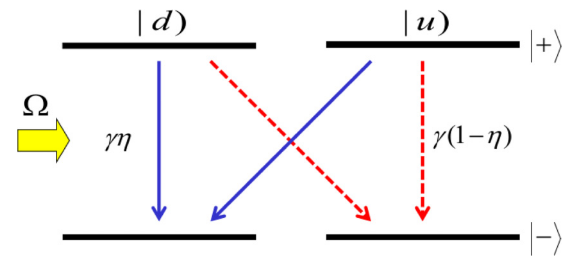

FIG. 1. Scheme levels corresponding to the evolution (31). The quantum system is characterized by the states $|\pm\rangle$, while the classical one is characterized by the states $|d)$ and $|u)$. The transition rates are $\gamma\eta$ (blue full lines) and $\gamma(1-\eta)$ (red dashed lines). The quantum system is coupled to an external laser field with Rabi frequency Ω.

that takes into account the imperfection of the detector while the (quantum) system dynamics remains the same. It is described by two (classical) states denoted by $|a)$, with $a = d$ (detected) and $a = u$ (undetected) (Fig. 1). The joint vectorial state $|\rho_t)$ is defined by the matrices $(a|\rho_t) = \rho_t^a$. Their evolution is given by the Lindblad rate equation

$$\frac{d\rho_t^a}{dt} = -\frac{i\Omega}{2}[\sigma_x,\rho_t^a]_- + \gamma_a(\sigma\rho_t^a\sigma^\dagger - \{\sigma^\dagger\sigma,\rho_t^a\}_+) - \gamma_b\{\sigma^\dagger\sigma,\rho_t^a\}_+ + \gamma_a\sigma\rho_t^b\sigma^\dagger, \qquad (31)$$

where the indices are $a = d,u$ while $b = u,d$. The decay and coupling rates are

$$\gamma_d \equiv \gamma\eta, \qquad \gamma_u \equiv \gamma(1-\eta). \qquad (32)$$

The initial conditions are taken as $\rho_0^d = |-\rangle\langle-|$, and $\rho_0^u = 0$.

The evolution (31) can be read as follows (see Fig. 1). The quantum system can suffers the transition $|+\rangle \to |-\rangle$ with rates $\gamma\eta$ and $\gamma(1-\eta)$ when the classical system is in the states $|d)$ and $|u)$, respectively. In addition, the transitions $|+\rangle|d) \to |-\rangle|u)$ and $|+\rangle|u) \to |-\rangle|d)$ happen with rates $\gamma(1-\eta)$ and $\gamma\eta$, respectively. Therefore, transitions with rate $\gamma\eta$ (detected events) collapse C to the state $|d)$, while transitions with rate $\gamma(1-\eta)$ (undetected events) collapse C to the state $|u)$. Independently of the state of the classical system, the fluorescent one is coupled to the external laser with Rabi frequency Ω.

Given the hybrid evolution (31), the system dynamics follows from Eq. (2), $\rho_t = \rho_t^d + \rho_t^u$. It is simple to check that $(d/dt)\rho_t$ obtained in this way obeys the Lindblad evolution (28), while the previous initial conditions imply that $\rho_0 = \rho_0^d + \rho_0^u = |-\rangle\langle-|$. Therefore, the fictitious classical degrees of freedom C associated to the Lindblad rate equation (31) do not affect the dynamics of the quantum system S defined by Eq. (28). We remark that this property is valid for any value of the characteristic parameters Ω, γ, and η.

The dynamics of C is strongly correlated with the behavior of S. It starts in the state $|d)$. Transitions between its states $|d) \leftrightarrow |u)$ only may happen when S is in the upper state $|+\rangle$. For $\eta = 1$ it remains in the initial condition, that is, the state $|d)$.

B. Quantum-classical filtered state

The filtered joint state $|\rho_t^{\mathrm{st}})$ [Eq. (7)] can be obtained after defining the splitting (9). We choose the vectorial superoperator $\hat{\mathcal{J}}$ such that $\hat{\mathcal{M}}$ represents the measurement transformation corresponding to the *detected photons*, that is, the transitions with rate $\gamma\eta$ in Fig. 1. Therefore, $[|\rho) = \rho^d|d) + \rho^u|u)]$

$$\hat{\mathcal{J}}|\rho) = \gamma_d[\sigma\rho^d\sigma^\dagger + \sigma\rho^u\sigma^\dagger]|d), \tag{33}$$

which in explicit form reads $\hat{\mathcal{J}}|\rho) = \gamma_d(\langle+|\rho^d|+\rangle + \langle+|\rho^u|+\rangle)|-\rangle\langle-||d)$. The measurement transformation [Eq. (10)] becomes

$$\hat{\mathcal{M}}|\rho) = |-\rangle\langle-||d). \tag{34}$$

Hence, S and C are reset to the states $|-\rangle\langle-|$ and $|d)$, respectively. On the other hand, the conditional evolution (11) is defined with $\hat{\mathcal{D}} = \hat{\mathcal{L}} - \hat{\mathcal{J}}$, where $\hat{\mathcal{L}}$ follows from the Lindblad rate equation (31). With these definitions ($\hat{\mathcal{J}}$ and $\hat{\mathcal{D}}$), the joint filtered state $|\rho_t^{\mathrm{st}})$ [Eq. (7)] can be determined after knowing the times of the detection events.

In a realistic experimental situation, the measurement trajectory is provided by the (inefficient) photon detector. Here, they are numerically implemented from a waiting-time distribution [2] that gives the probability density for the time intervals between consecutive events (Appendix C).

In Fig. 2 we show a realization of the S and C filtered states (dotted blue lines) [Eq. (13)] through the upper system population $\langle+|\rho_t^{\mathrm{st}}|+\rangle$ and the population $(d|P_t^{\mathrm{st}})$. The disruptive changes in the states are associated to the detection times. Furthermore, the corresponding purities are also shown, $\mathrm{Tr}[(\rho_t^{\mathrm{st}})^2]$ and purity$(d,u) = (d|P_t^{\mathrm{st}})^2 + (u|P_t^{\mathrm{st}})^2$. For a perfect detector $\eta = 1$, at any time these last two objects are equal to one.

A fundamental property of the quantum system realizations shown in Fig. 2 is that they are exactly the same as those obtained from the standard quantum jump approach applied to the Lindblad evolution (28). In fact, it is straightforward to demonstrate that

$$(1|\hat{\mathcal{D}}|\rho) = \mathcal{D}(1|\rho) = \mathcal{D}(\rho^d + \rho^u), \tag{35}$$

where $|\rho) = \rho^d|d) + \rho^u|u)$ and $\mathcal{D}$ is defined by Eq. (29). Given that $\rho^d + \rho^u$ gives the state of S, and given that $(1|\hat{\mathcal{M}}|\rho) = |-\rangle\langle-|$ [Eq. (34)] it follows that the realizations of $(1|\rho_t^{\mathrm{st}}) = \rho_t^{\mathrm{st}}$ coincide with the realizations of the standard quantum jump approach defined from the measurement superoperator (29) and (30). The measurement statistics is also the same (Appendix C). Thus, not only the irreversible evolution of the S but also the filtered state obtained from the Lindblad rate equation (31) are the same as those obtained from Eq. (28). This is the main property that sustains the ansatz given by Eq. (31).

C. Quantum-classical smoothed state

The representation of the fluorescent system monitored by an inefficient detector in terms of a Lindblad rate equation allows us to define a joint smoothed state. In fact, given the detection times, it follows from Eq. (25), while the partial states follow from Eq. (26).

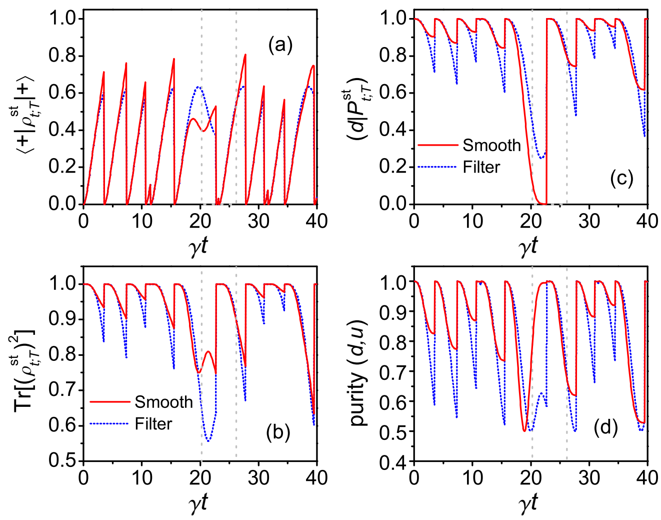

FIG. 2. Realizations for filtered (dotted blue lines) and smooth (full red lines) states associated to Eq. (31). (a) Upper system population $\langle+|\rho_{t,T}^{\mathrm{st}}|+\rangle$. (b) System purity $\mathrm{Tr}[(\rho_{t,T}^{\mathrm{st}})^2]$. (c) Classical population $(d|P_{t,T}^{\mathrm{st}})$. (d) purity$(d,u) \equiv (d|P_{t,T}^{\mathrm{st}})^2 + (u|P_{t,t}^{\mathrm{st}})^2$. The filtered states correspond to $T = t$. In all cases, the parameters are $\Omega/\gamma = 1$ and $\eta = 0.8$. For the smoothed realizations, T is chosen such that $\gamma(T-t) = 30$. The vertical dashed lines are the times of the undetected events.

In Fig. 2, for the same realization of measurement events, we also plot the smoothed states (full red lines) through the upper system population $\langle+|\rho_{t,T}^{\mathrm{st}}|+\rangle$ and the population $(d|P_{t,T}^{\mathrm{st}})$. The plotted purities are $\mathrm{Tr}[(\rho_{t,T}^{\mathrm{st}})^2]$ and purity$(d,u) = (d|P_{t,T}^{\mathrm{st}})^2 + (u|P_{t,T}^{\mathrm{st}})^2$.

The filtered and smoothed realizations develop disruptive events at the same (detection) times. Nevertheless, for both S and C, the smoothed purities are higher than the filtered purities. The increment of the smoothed purities is a consequence of the general result (27), that is, averaging the smoothed states over future measurements events one recovers the filtered states.

In an experimental situation it is impossible to determine when the detector fails. Nevertheless, given that here we are determining the measurement events in a numerical way (see Appendix C), it is possible to know when the undetected events happen. In Fig. 2 they are indicated by the vertical dashed lines. These times are not necessary for defining the joint smoothed state (25). Nevertheless, they allow us to understand some features of the smoothed states. While $\langle+|\rho_{t,T}^{\mathrm{st}}|+\rangle$ does not develop any special characteristic, around $\gamma t \simeq 20$ the smoothed realization of $(d|P_{t,T}^{\mathrm{st}})$ anticipates the behavior of the filtered population $(d|P_t^{\mathrm{st}})$. This signature is also observed in other quantum optical arrangements [17]. In addition, here the smoothed realization almost vanishes, property consistent with the fact that undetected measurement events lead to the (unobserved) transition $|P) \to |u)$ (see Fig. 1).

The recovering of the purity lost due to the inefficient detector can be quantified by averaging over an ensemble of measurement events. In Fig. 3 we plot the smoothed and filtered averaged purities of S, $\overleftrightarrow{\mathrm{Tr}[(\rho_{t,T}^{\mathrm{st}})^2]}$ and $\overleftarrow{\mathrm{Tr}[(\rho_t^{\mathrm{st}})^2]}$, respectively. The plots were obtained by averaging 5×10^3

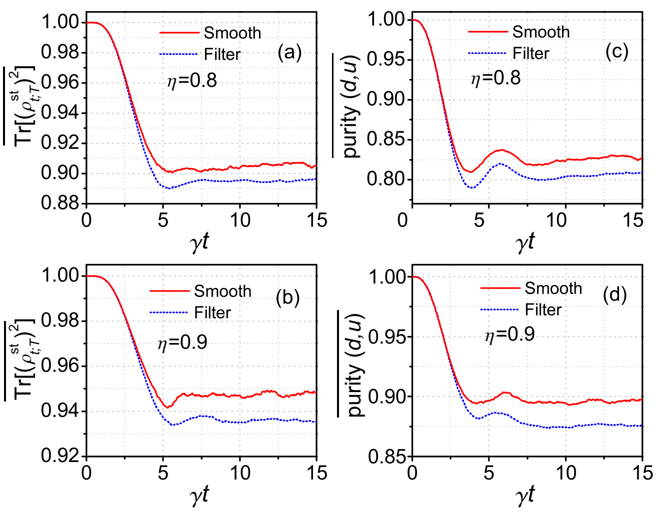

FIG. 3. Purities of the smoothed and filtered partial states when averaged over 5×10^3 realizations (see Fig. 2). (a) and (b) correspond to $\overleftrightarrow{\mathrm{Tr}[(\rho^{\mathrm{st}}_{t,T})^2]}$ (smoothed) and $\overleftarrow{\mathrm{Tr}[(\rho^{\mathrm{st}}_{t})^2]}$ (filtered). (c) and (d) correspond to $\overleftrightarrow{(d|\rho^{\mathrm{st}}_{t,T})^2+(u|\rho^{\mathrm{st}}_{t,T})^2}$ (smoothed) and $\overleftarrow{(d|\rho^{\mathrm{st}}_{t})^2+(u|\rho^{\mathrm{st}}_{t})^2}$ (filtered), both objects being denoted as $\overline{\mathrm{purity}(d,u)}$. The parameters are $\Omega/\gamma=1$, while η is indicated in each plot.

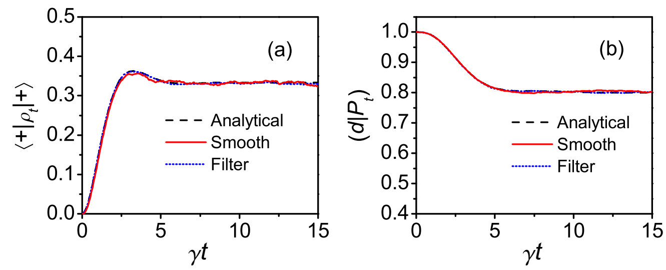

FIG. 4. Analytical solutions and average over realizations of the filtered and smoothed states. (a) Upper system population, $\langle +|\rho_t|+\rangle$ analytical solution, jointly with the smoothed $\overleftrightarrow{\langle +|\rho^{\mathrm{st}}_{t,T}|+\rangle}$ and filtered $\overleftarrow{\langle +|\rho^{\mathrm{st}}_{t}|+\rangle}$ averaged populations. (b) Classical population, $(d|P_t)$ analytical solution, jointly with the smoothed $\overleftrightarrow{(d|P^{\mathrm{st}}_{t,T})}$ and filtered $\overleftarrow{(d|P^{\mathrm{st}}_{t})}$ averaged populations. In all cases the averages were performed with 5×10^5 realizations. The parameters are $\Omega/\gamma=1$ and $\eta=0.8$.

realizations. Under smoothing, with $\eta=0.8$, about 10% of the purity lost is recovered when compared with the filtered purity. For $\eta=0.9$ the recovering is around 15%. These results are similar to that obtained in Ref. [17] with a different measurement arrangement.

In Fig. 3 the filtered and smoothed purities of C are also shown. While these objects refer to the fictitious classical system associated to the imperfection of the detector, the graphics consistently show similar properties to that of the quantum counterpart S.

D. Ensemble behavior

When averaged over an ensemble of realizations [Eqs. (6) and (27)] both the joint filtered and smoothed states must recover the dynamics given by the Lindblad rate equation (31). For the quantum subsystem, the evolution is exactly the same as that obtained from the standard Lindblad equation (28).

In order to check these properties, in Fig. 4 we plot the averaged smoothed $\overleftrightarrow{\langle +|\rho^{\mathrm{st}}_{t,T}|+\rangle}$ and filtered $\overleftarrow{\langle +|\rho^{\mathrm{st}}_{t}|+\rangle}$ system populations. Consistently, both averages recover the analytical solution $\langle +|\rho_t|+\rangle$ that follows from Eq. (28) [or Eq. (31)], which is independent of η. The curves are indistinguishable in the scale of the plots. The same property is valid for the smoothed $\overleftrightarrow{(d|P^{\mathrm{st}}_{t,T})}$ and filtered $\overleftarrow{(d|P^{\mathrm{st}}_{t})}$ averaged classical populations. The analytical solution of these objects follows from Eq. (31).

From Eq. (31), it is simple to check that $\lim_{t\to\infty}(d|\rho_t)=\eta\rho_\infty$ and $\lim_{t\to\infty}(u|\rho_t)=(1-\eta)\rho_\infty$ where $\rho_\infty=\lim_{t\to\infty}\rho_t$ is the stationary solution of Eq. (28), $\rho_\infty=\{\{\Omega^2,-i\gamma\Omega\},\{i\gamma\Omega,\gamma^2+\Omega^2\}\}/(\gamma^2+2\Omega^2)$. In Fig. 4, the (analytical) stationary values $\lim_{t\to\infty}\langle +|\rho_t|+\rangle=\Omega^2/(\gamma^2+2\Omega^2)=1/3$ and $\lim_{t\to\infty}(d|P_t)=\eta=0.8$ are also correctly achieved.

V. SUMMARY AND CONCLUSIONS

An extra class of smoothed quantum state was introduced. It describes a hybrid quantum-classical arrangement conditioned not only on earlier (filtering) but also on later measurements results (smoothing). The joint evolution is given by an arbitrary hybrid Lindblad rate equation. Hence, mutual influence between the quantum and classical subsystems is allowed. The measurement process is performed on the quantum system.

The results relies on a Bayesian analysis, which provides a better estimation of the classical system state [Eq. (24)], which in turn leads to an improved estimation of the joint state [Eq. (25)]. Partial smoothed states follow by tracing the partner system information [Eq. (26)]. The hybrid smoothed state can be determined after knowing the initial joint state, the hybrid evolution, and the measurement results. The estimation does not rely on unobserved information such as, for example, that provided by measurement processes performed on the classical subsystem.

By averaging the smoothed state over future measurement results the filtered state is recovered [Eq. (27)]. This property guarantees that a purer estimation of the joint state is always obtained. Furthermore, and similarly to the standard quantum jump approach, here the time-irreversible joint dynamics is recovered after averaging over both past and future measurement results.

The formalism was applied to a standard fluorescent system monitored by an inefficient photon detector. This situation was covered by introducing fictitious classical degrees of freedom (Fig. 1) associated to the imperfect photon detector. The joint quantum-classical dynamics [Eq. (31)] leads to the same quantum system dynamics. A significant recovering of the purity lost due to the inefficient recording process is achieved by taking into account future measurement results (Figs. 2 and 3). Consistently, the ensemble averages of both the filtered and smoothed realizations recover the quantum irreversible system dynamics (Fig. 4).

The present results can be extended and applied in different physical situations. For example, many hybrid measurement channels as well as application in single-molecule spectroscopy [20,21] can be straightforwardly handled by using the developed theoretical formalism. How different classical dynamics influences the smoothed state can also be studied in this context.

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APPENDIX A: VECTORIAL QUANTUM JUMP APPROACH

Here, we review the quantum jump approach formulated for quantum-classical hybrid dynamics [20] described through a Lindblad rate equation, Eq. (5).

Given the relation (9), the evolution of the joint state can be rewritten as

$$\frac{d|\rho_t)}{dt} = (\hat{\mathcal{D}} + \hat{\mathcal{J}})|\rho_t). \tag{A1}$$

This evolution can be "unraveled" in terms of measurements trajectories. By solving the previous Markovian evolution as $|\rho_t) = e^{\hat{\mathcal{D}}t}|\rho_0) + \int_0^t e^{\hat{\mathcal{D}}(t-\tau)}\hat{\mathcal{J}}|\rho_\tau)d\tau$, after successive iterations, it follows that

$$|\rho_t) = \hat{\mathcal{G}}(t)|\rho_0) = \sum_{n=0}^{\infty} \hat{\mathcal{G}}_n(t)|\rho_0). \tag{A2}$$

Here, $\hat{\mathcal{G}}_0(t) = e^{\hat{\mathcal{D}}t}$, while

$$\hat{\mathcal{G}}_n(t) = \int_0^t \overleftarrow{dt_n}\hat{\mathcal{U}}[t,0,\overleftarrow{t_n}], \tag{A3}$$

where the propagator $\hat{\mathcal{U}}$ is defined by Eq. (8). Furthermore, $\overleftarrow{t_n} = \{t_i\}_{i=1}^{i=n}$ are the integration variables corresponding to the nested integrals $\int_0^t \overleftarrow{dt_n} \equiv \int_0^t dt_n \dots \int_0^{t_3} dt_2 \int_0^{t_2} dt_1$.

Each contribution in Eq. (A2) can be rewritten as

$$\hat{\mathcal{G}}_n(t)|\rho_0) = \int_0^t \overleftarrow{dt_n} P_t[\overleftarrow{t_n}] \frac{\hat{\mathcal{U}}[t,0,\overleftarrow{t_n}]|\rho_0)}{\mathrm{Tr}[(1|\hat{\mathcal{U}}[t,0,\overleftarrow{t_n}]|\rho_0)]}, \tag{A4}$$

where $P_t[\overleftarrow{t_n}]$, similarly to Eq. (12), is defined as

$$P_t[\overleftarrow{t_n}] = \mathrm{Tr}[(1|\hat{\mathcal{U}}[t,0,\overleftarrow{t_n}]|\rho_0)]. \tag{A5}$$

This object can be read as the n-joint probability density of a measurement trajectory with n events, each one happening at times $\overleftarrow{t_n}$. Hence, from Eq. (A4), the corresponding conditional stochastic joint state $|\rho_t^{\mathrm{st}})$ (associated to the set $\overleftarrow{t_n}$) is

$$|\rho_t^{\mathrm{st}}) = \frac{\hat{\mathcal{U}}[t,0,\overleftarrow{t_n}]|\rho_0)}{\mathrm{Tr}[(1|\hat{\mathcal{U}}[t,0,\overleftarrow{t_n}]|\rho_0)]}, \tag{A6}$$

which recovers Eq. (7). In this way, Eq. (A2) can be read as an *addition over all possible trajectories* with n events happening at the arbitrary times $\overleftarrow{t_n}$. By construction, the fulfillment of condition (6) is guaranteed. Notice that $P_t[\overleftarrow{t_n}]$ satisfies the normalization

$$\sum_{n=0}^{\infty} \int_0^t \overleftarrow{dt_n} P_t[\overleftarrow{t_n}] = 1. \tag{A7}$$

The previous associations are consistent with the definitions of the measurement transformation Eq. (10) and conditional evolution Eq. (11). In fact, by using the mathematical principle of induction, it is possible to rewrite Eq. (A6) as

$$|\rho_t^{\mathrm{st}}) = \hat{\mathcal{T}}(t,t_n)\hat{\mathcal{M}} \cdots \hat{\mathcal{T}}(t_2,t_1)\hat{\mathcal{M}}\hat{\mathcal{T}}(t_1,0)|\rho_0). \tag{A8}$$

Therefore, the conditional state can in fact be written as successive applications of the measurement transformation $\hat{\mathcal{M}}$, while $\hat{\mathcal{T}}$ gives the normalized conditional propagation between measurement events.

Interestingly, by using the mathematical principle of induction it is also possible to prove that the n-joint probability, Eq. (A5), can be rewritten as

$$P_t[\overleftarrow{t_n}] = P_0[t,t_n;\hat{\mathcal{M}}|\rho_{t_n}^{\mathrm{st}})]w[t_n,t_{n-1};\hat{\mathcal{M}}|\rho_{t_{n-1}}^{\mathrm{st}})] \times \cdots w[t_2,t_1;\hat{\mathcal{M}}|\rho_{t_1}^{\mathrm{st}})]w[t_1,0;|\rho_0)]. \tag{A9}$$

In this expression, for $i \geqslant 1$

$$|\rho_{t_{i+1}}^{\mathrm{st}}) = \hat{\mathcal{T}}(t_{i+1},t_i)\hat{\mathcal{M}}|\rho_{t_i}^{\mathrm{st}}), \tag{A10}$$

while $|\rho_{t_1}^{\mathrm{st}}) = \hat{\mathcal{T}}(t_1,0)|\rho_0)$. The function $w[t,\tau;|\rho)]$ can be read as a waiting-time distribution [2], that is, given that at time τ the joint state is $|\rho)$, it gives the probability density for an interval $t-\tau$ between consecutive measurement events. It reads

$$w[t,\tau;|\rho)] \equiv \mathrm{Tr}[(1|\hat{\mathcal{J}}e^{\hat{\mathcal{D}}(t-\tau)}|\rho)]. \tag{A11}$$

On the other hand, $P_0[t,\tau;|\rho)]$ is the associated survival probability

$$P_0[t,\tau;|\rho)] = 1 - \int_\tau^t w[t',\tau;|\rho)]dt', \tag{A12}$$

being defined as

$$P_0[t,\tau;|\rho)] \equiv \mathrm{Tr}[(1|e^{\hat{\mathcal{D}}(t-\tau)}|\rho)]. \tag{A13}$$

Given an initial condition $|\rho_0)$, the dynamics defined by Eq. (A8) can be numerically implemented by getting the random measurement times from the survival probability [20]. As in the standard quantum jump approach it is also possible to write an explicit stochastic differential equation for the state $|\rho_t^{\mathrm{st}})$, the average of which over realizations recovers the deterministic evolution (A1).

APPENDIX B: AVERAGING OVER FUTURE MEASUREMENTS RESULTS

Here, we demonstrate that the average of the joint smoothed state over future realizations recovers the filtered state, $\overline{|\rho_{t,T}^{\mathrm{st}})} = |\rho_t^{\mathrm{st}})$, Eq. (27). Given the expression (25), this is equivalent to demonstrate that the average of the smoothed conditional probability $P_T[R_t|\overleftrightarrow{t}]$ over measurements performed in the future recovers the filtered conditional probability $P_t[R_t|\overleftarrow{t}] = \mathrm{Tr}[(R_t|\rho_t^{\mathrm{st}})]$. In an explicit way, the previous condition can be written as

$$P_t[R_t|\overleftarrow{t}] = \int \overrightarrow{dt}\, P_T[R_t|\overleftarrow{t}\,\overrightarrow{t}]P_T[\overrightarrow{t}|\overleftarrow{t}]. \tag{B1}$$

For clarity $P_T[R_t|\overleftrightarrow{t}]$ was denoted as $P_T[R_t|\overleftarrow{t}\,\overrightarrow{t}]$. Furthermore, the integral $\int \overrightarrow{dt}$ [see Eq. (B5) below] is an addition over all possible measurement trajectories in (t,T) *given* that we know one trajectory in $(0,t)$, which is defined by the set of times $\overleftarrow{t}$. In what follows we demonstrate the validity of Eq. (B1).

The conditional probability density $P_T[\overrightarrow{t}|\overleftarrow{t}]$ for the times $\overrightarrow{t}$ of future measurements given the past measurement times $\overleftarrow{t}$ fulfills the Bayes relation

$$P_T[\overleftrightarrow{t}] = P_T[\overrightarrow{t}|\overleftarrow{t}]P_t[\overleftarrow{t}]. \tag{B2}$$

Here, $P_t[\overleftarrow{t}]$ is defined by Eq. (12) while $P_T[\overleftrightarrow{t}]$ is defined by Eq. (14), which leads to

$$P_T[\overrightarrow{t}|\overleftarrow{t}] = \frac{\mathrm{Tr}[(1|\hat{\mathcal{U}}[T,t,\overrightarrow{t}]\hat{\mathcal{U}}[t,0,\overleftarrow{t}]|\rho_0)]}{\mathrm{Tr}[(1|\hat{\mathcal{U}}[t,0,\overleftarrow{t}]|\rho_0)]}. \tag{B3}$$

By using this result and Eq. (18) for $P_T[R_t|\overleftarrow{t}\,\overrightarrow{t}]$, Eq. (B1) becomes

$$P_t[R_t|\overleftarrow{t}] = \int d\overrightarrow{t}\,\frac{\mathrm{Tr}[(1|\hat{\mathcal{U}}[T,t,\overrightarrow{t}]|R_t)(R_t|\hat{\mathcal{U}}[t,0,\overleftarrow{t}]|\rho_0)]}{\mathrm{Tr}[(1|\hat{\mathcal{U}}[t,0,\overleftarrow{t}]|\rho_0)]}. \tag{B4}$$

The integral $\int d\overrightarrow{t}$ is an addition over all possible measurement trajectories in (t,T). Hence, it is given by

$$\int d\overrightarrow{t} = \sum_{N=0}^{\infty}\int_t^T d\overrightarrow{t_N} = \sum_{N=0}^{\infty}\int_t^T dt_N \cdots \int_t^{t_3} dt_2 \int_t^{t_2} dt_1, \tag{B5}$$

where the addition takes into account an arbitrary number of detection events in the time interval (t,T). By working in a Laplace domain, it is possible to demonstrate that

$$\int d\overrightarrow{t}\,\hat{\mathcal{U}}[T,t,\overrightarrow{t}] = \exp[(T-t)\hat{\mathcal{L}}], \tag{B6}$$

where $\hat{\mathcal{U}}$ is the propagator (8) and $\hat{\mathcal{L}}$ defines the Lindblad rate equation (5). Using this result and the trace conservation property

$$\mathrm{Tr}[(1|\exp[t\hat{\mathcal{L}}]|\rho)] = \mathrm{Tr}[(1|\rho)], \tag{B7}$$

Eq. (B4) becomes

$$P_t[R_t|\overleftarrow{t}] = \frac{\mathrm{Tr}[(R_t|\hat{\mathcal{U}}[t,0,\overleftarrow{t}]|\rho_0)]}{\mathrm{Tr}[(1|\hat{\mathcal{U}}[t,0,\overleftarrow{t}]|\rho_0)]}, \tag{B8}$$

$$= \mathrm{Tr}[(R_t|\rho_t^{\mathrm{st}})], \tag{B9}$$

where the expression (7) was used. The last equality demonstrates the validity of Eq. (B1) and in consequence also the validity of Eq. (27).

APPENDIX C: DETECTION TIMES

In an experimental situation, the detection times are determined from the photon detector. Instead, here they are obtained from the quantum jump approach. In fact, this formalism not only allows us to defining the filtered state, but also allows us to determine the measurement statistics. A (state-dependent) waiting-time distribution [2] gives the probability density for the time interval between consecutive events, Eq. (A11).

For a fluorescent system monitored with an inefficient detector $\eta < 1$ [Eqs. (28)–(30)], by working Eq. (A11) in a Laplace domain $[f(u) = \int_0^\infty dt f(t)e^{-ut}]$ we get $[(t-\tau) \to u,\ w[t,\tau;|\rho)] \to w_\eta(u)]$

$$w_\eta(u) = \frac{\gamma\eta\Omega^2}{u(u+\gamma)(2u+\gamma)+(2u+\gamma\eta)\Omega^2}. \tag{C1}$$

Given that the reset state (30) does not depend on the (previous) state of the system, $w_\eta(u)$ inherits this property, leading to a renewal point process. Exactly the same expression and property follow from Eq. (A11) calculated over the basis of the Lindblad rate equation (31) and the vectorial superoperator (33). This feature also demonstrates that the quantum-classical representation leads to the same quantum system dynamics.

Interestingly, the previous expression can be written as

$$w_\eta(u) = \frac{\eta w_1(u)}{1-(1-\eta)w_1(u)}, \tag{C2}$$

where $w_1(u) = w_\eta(u)|_{\eta=1}$, that is, the waiting-time distribution for perfect detection, $\eta = 1$. By using the geometric series it follows that

$$w_\eta(u) = \eta w_1(u)\sum_{n=0}^{\infty}[(1-\eta)w_1(u)]^n. \tag{C3}$$

In this way, $w_\eta(u)$ is determined from successive convolution terms, each one representing a time interval where n-fails detection events happen with probability $(1-\eta)^n$ and a detection event happen with probability η. Consequently, one can determine the random events from $w_\eta(u)$. Equivalently, each event is chosen in agreement with $w_1(u)$ (perfect detection) and each event is accepted or rejected with probabilities η and $(1-\eta)$, respectively. This last algorithm recovers the expected definition of an inefficient photon detector.

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