Dynamics of Rényi entropy and applications in detecting quantum non-Markovianity

Hongting Song*
Qian Xuesen Laboratory of Space Technology, China Academy of Space Technology, Beijing 100094, China

Yuanyuan Mao†
Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China and School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China

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Exploiting the master equations in the Lindblad form, we establish a sufficient and necessary condition for a Markovian dynamics to be unital. Based on this condition, we analyze the dynamical property of quantum Rényi entropy and propose a characterization of quantum non-Markovianity for unital dynamics in terms of Rényi entropy, which contains the previous criteria of non-Markovianity via von Neumann entropy and linear entropy as particular cases. The effectiveness of this measure in capturing the backflow of information from the environment is illustrated through several typical unital dynamics.

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I. INTRODUCTION

While Shannon entropy and related information measures [1], such as mutual information and relative entropy, capture many operational quantities in information and communication theory, they also have limitations in certain circumstances. In particular, in nonasymptotic or nonergodic settings where the law of large numbers does not readily apply, other entropic measures such as the minimum, the maximum, or the collision entropy have some advantage. Since Rényi entropy nicely unifies these different and isolated entropies [2], the investigation of Rényi entropy has significant value in both theory and practice.

Recall that Rényi investigated an axiomatic approach to derive Shannon entropy in his seminal paper [2] based on previous work by Feinstein and Fadeev and found five natural requirements, namely, (i) continuity, (ii) unitary invariance, (iii) normalization, (iv) additivity, and (v) arithmetic mean, for Shannon entropy. By relaxing the requirement (v) of an arithmetic mean to a general mean, he proposed a family of entropies now named after him. These requirements can be easily generalized to the quantum realm for functionals on state space. Consider a density matrix ρ that is positive semidefinite and satisfies the normalization condition $\mathrm{tr}\rho = 1$. The quantum Rényi entropy of order $\alpha \in (0,1) \cup (1,\infty)$ can be given as

$$S_\alpha(\rho) = \frac{1}{1-\alpha} \log_2 \mathrm{tr}\rho^\alpha,$$

which is non-negative and is additive on tensor-product states. It contains several important kinds of quantum information measure as limiting or special cases. Specifically, the limiting cases of quantum Rényi entropy when $\alpha \to 1$ and $\alpha \to \infty$ are known to be exactly the von Neumann entropy and minimum entropy, respectively, while the special cases of quantum Rényi entropy when $\alpha = 2$ and $\alpha = 1/2$ are alternative forms of linear entropy and quantum uncertainty defined via skew information [3,4].

Quantum Rényi entropies are not only appealing from a theoretical perspective, but also useful in the practice of quantum information processing. They have been widely used as a technical tool solving various problems in quantum information theory such as entanglement characterization [5–7], quantum communication protocols [8,9], and localization properties [10]. Recently, the time evolution of the Rényi entropy under the Lindblad equation has been investigated and a compact general formula for the lower bound of the entropy changing rate has been studied [11,12]. In this work we propose a sufficient and necessary condition for a Markovian dynamics to be unital and further derive the monotonicity of quantum Rényi entropy under this class of dynamics. Exploiting this property, a measure of quantum non-Markovianity for unital dynamics is put forward.

Unlike the well-defined concepts of Markovianity and non-Markovianity in the classical regime, their quantum versions are somewhat ambiguous, subtle, and often controversial in some sense. Although various criteria have been proposed in recent literature to qualitatively or quantitatively characterize quantum non-Markovianity based on different considerations, such as divisibility [13–16], information flow [17,18], correlations [14,19,20], and fidelity [21], a universal definition of quantum non-Markovianity is still absent and worth pursuing.

The alternative measure of quantum non-Markovianity via quantum Rényi entropy proposed here does not need auxiliary systems or extra states and thus makes it easy to compute and tractable in experiment. Its effectiveness in detecting quantum non-Markovianity is illustrated through several examples. It is compatible with the previous criteria for the phase damping channel but different from the measures based on divisibility, quantum mutual information, and the Fisher information matrix for random unitary dynamics to detect non-Markovianity. It is worth noting that for all the examples considered in this work, our criterion is in accord with the criterion via information flow, and this can be used to partially support the results in Ref. [22].

The paper is organized as follows. In Sec. II we briefly review the definition of quantum unital dynamics, derive a sufficient and necessary condition for a quantum Markovian dynamics to be unital, and further investigate the monotonic

*songhongting@qxslab.cn
†mao@amss.ac.cn

property of quantum Rényi entropy under Markovian unital dynamics. In Sec. III a candidate to measure quantum non-Markovianity via quantum Rényi entropy is proposed. Through several examples, the effectiveness of this measure to detect quantum non-Markovianity is illustrated in Sec. IV. Finally, in Sec. V we summarize with concluding remarks.

II. UNITAL DYNAMICS AND EVOLUTION OF RÉNYI ENTROPY

Physically, quantum dynamics describe the transmission in space or the evolution in time for a general open system. Mathematically, dynamics can be depicted by $\Lambda = \{\Lambda_t : t \geqslant 0\}$, which is a family of completely positive and trace-preserving linear maps on quantum state space [23]. As a special class, a unital dynamics maps the identity operator to itself, i.e., $\Lambda_t(\mathbf{1}) = \mathbf{1}$, which is also called as a doubly stochastic completely positive map. Apart from the practical relevance, quantum unital channels exhibit many special properties of theoretical interest, such as contractivity [24], and for two-dimensional systems, a unital channel can always be expressed as a convex combination of unitary channels, which is useful to simplify problems [25]. Typical examples are the depolarizing channel, the phase damping channel, and the two-Pauli channel of Bennett *et al.* [26]. Notably the problems on general channels for sufficiently large dimensions can often be tackled by considering their unital counterparts [27–29].

Recall that the conventional quantum Markovian processes can be described by master equations of the Lindblad form [11]

$$\frac{\partial \rho_t}{\partial t} = -i[H,\rho_t] + \tfrac{1}{2}\sum_i \gamma_i (2L_i\rho_t L_i^\dagger - L_i^\dagger L_i \rho_t - \rho_t L_i^\dagger L_i), \tag{1}$$

in which H is a Hermitian operator and $[\cdot,\cdot]$ is the commutator. Here we assume that the Lindblad operators L_i and nonnegative numbers γ_i are time independent. Now we derive a sufficient and necessary condition for this Markovian dynamics to be unital.

Proposition. A quantum dynamics $\Lambda_t(\rho) = \rho_t$ described by the Lindblad equation (1) is unital if and only if the Lindblad operators satisfy

$$\sum_i \gamma_i [L_i^\dagger, L_i] = 0. \tag{2}$$

Proof. To prove this proposition, first we prove the "if" part. Assume that the evolving state at some time $t \geqslant 0$ is $\rho_t = \mathbf{1}/n$, where n is the dimension of the Hilbert space that ρ_t lies in. Then we have

$$\frac{\partial \rho_t}{\partial t} = -i\left[H, \frac{\mathbf{1}}{n}\right] - \frac{1}{n}\sum_i \gamma_i [L_i^\dagger, L_i] = 0,$$

which indicates that the maximally mixed state is invariant under this dynamics.

For the "only if" part, assume a specific Markovian process that preserves identity, i.e., with initial state $\rho = \mathbf{1}/n$, the evolving state $\rho_t = \Lambda_t(\mathbf{1}/n) = \mathbf{1}/n$ for any $t \geqslant 0$. Substituting this into the Lindblad equation, we obtain

$$i\left[H, \frac{\mathbf{1}}{n}\right] + \frac{1}{n}\sum_i \gamma_i [L_i^\dagger, L_i] = \frac{1}{n}\sum_i \gamma_i [L_i^\dagger, L_i] = 0,$$

which completes the proof.

Now, as a generalization of the monotonicity of von Neumann entropy [30], we can directly derive the monotonic property of Rényi α entropy under Markovian unital dynamics from this proposition and the results in Ref. [12], in which Abe established a lower bound for the time derivative of general Rényi α entropy, namely,

$$\frac{dS_\alpha(\rho_t)}{dt} > \sum_i \gamma_i \langle [L_i^\dagger, L_i] \rangle_\alpha(t), \tag{3}$$

where $\langle A \rangle_\alpha(t)$ stands for the α average of A, i.e., $\langle A \rangle_\alpha(t) = \frac{\mathrm{tr}(A\rho_t^\alpha)}{\mathrm{tr}\rho_t^\alpha}$. Due to the linearity of the α average, substituting Eq. (2) into Eq. (3), we have

$$\frac{dS_\alpha(\rho_t)}{dt} > \sum_i \gamma_i \langle [L_i^\dagger, L_i] \rangle_\alpha(t) = \left\langle \sum_i \gamma_i [L_i^\dagger, L_i] \right\rangle_\alpha (t) = 0,$$

which leads to the monotonic property that Rényi α entropy always increases while undergoing Markovian unital dynamics.

It is worth mentioning in the Appendix of Ref. [31], the monotonic property was proved using another method.

III. QUANTUM NON-MARKOVIANITY VIA RÉNYI ENTROPY

With the help of the monotonicity of Rényi α entropy under unital Markovian operations, we propose a method to characterize the quantum non-Markovianity for unital dynamics. Any reduction of the Rényi α entropy under the unital dynamics acts as a witness of non-Markovianity.

Precisely, for a unital dynamics $\Lambda = \{\Lambda_t : t \geqslant 0\}$, it is defined to be Markovian via Rényi α entropy, referred to as Rényi Markovian, if for any initial state ρ, the inequality $\frac{d}{dt}S_\alpha(\Lambda_t(\rho)) \geqslant 0$ holds for all $t \geqslant 0$. Any violation of this monotonicity of the Rényi α entropy is regarded as an indication of quantum Rényi non-Markovianity, and the corresponding measure of the amount of non-Markovianity is given as

$$N_R(\Lambda) = \max_\rho \int_{dS_\alpha(\rho_t)/dt<0} -\frac{dS_\alpha(\rho_t)}{dt} dt,$$

where $\rho_t = \Lambda_t(\rho)$ and the maximization is taken over all the initial states ρ.

Due to the freedom of the values of α, this work contains some previous results on quantum Markovianity as particular cases [32,33]. When $\alpha \to 1$, it turns out to be the result in Ref. [32]; when $\alpha = 2$, it is just the one based on the Brukner-Zeilinger invariant information [33], which is directly related to the total ordinary variance of the state [34]. Once $\alpha = 1/2$, we can directly get the non-Markovianity witness via quantum uncertainty [3].

Unlike the criterion based on quantum mutual information, this measure does not need an auxiliary system, and unlike the one based on the information flow, only one state is needed in this measure. For the single-qubit case, this criterion can be further expressed as follows, which only depends on the determinant of the state. Since quantum Rényi α entropy can be written as

$$S_\alpha(\rho_t) = \frac{1}{1-\alpha}\log_2\left[\lambda_t^\alpha + (1-\lambda_t)^\alpha\right],$$

with $\lambda_t = \frac{1+\sqrt{1-4|\rho_t|}}{2}$ being an eigenvalue of the state and $|\rho_t|$ being the determinant of the state, the derivative can be directly calculated as

$$\frac{d}{dt}S_\alpha(\rho_t) = h(\alpha,t)\frac{d}{dt}|\rho_t|,$$

with

$$h(\alpha,t) = \frac{\alpha}{(\alpha-1)\ln 2}\frac{\lambda_t^{\alpha-1}-(1-\lambda_t)^{\alpha-1}}{\lambda_t^\alpha+(1-\lambda_t)^\alpha}\frac{1}{\sqrt{1-4|\rho_t|}},$$

which is always positive without reference to the values of α. Then the sign of the derivative of the Rényi entropy only depends on the sign of the derivative of determinant. Once there is some time interval (t_i,t_{i+1}) in which the determinant of the state, i.e., $|\rho_t|$, is a decreasing function, the dynamics is Rényi non-Markovian and the measure can be rewritten as

$$N_R(\Lambda) = \max_\rho \int_{d|\rho_t|/dt<0} -h(\alpha,t)\frac{d}{dt}|\rho_t|dt.$$

IV. EXAMPLES

Since phase damping and random unitary channels are two typical unital dynamics, we investigate the applications of this proposed characterization of quantum non-Markovianity by these two examples. A comparison with the existing measures of quantum non-Markovianity, such as those based on information flow, divisibility, quantum mutual information, and quantum Fisher information matrix, is also presented.

Example 1. Consider a two-level system linearly interacting with a thermal reservoir. The total Hamiltonian of this composite system is

$$H = \omega_0\sigma_z + \sum_i \omega_i a_i^\dagger a_i + \sum_i \sigma_z(g_i a_i + g_i^* a_i^\dagger),$$

where ω_0 is the energy gap in the qubit system, ω_i is the frequency of the ith reservoir mode, a_i and $a_i^\dagger$ are the annihilation and creation operators, respectively, and g_i is the coupling constant. In this case, the dynamics of the qubit system can be captured by the time-local master equation

$$\frac{d}{dt}\rho_t = \gamma(t)(\sigma_z\rho_t\sigma_z - \rho_t),$$

where $\gamma(t)$ is the time-dependent dephasing rate determined by the spectral density of the reservoir. It is obvious that this dynamics is unital, and for arbitrary initial state $\rho = \begin{pmatrix} a & b \\ b^* & d\end{pmatrix}$ with $a,d\in\mathbb{R}$, $b\in\mathbb{C}$, and $a+d=1$, the evolving state can be expressed as

$$\rho_t = \Lambda_t(\rho) = \begin{pmatrix} a & bf(t) \\ b^*f(t) & d\end{pmatrix},$$

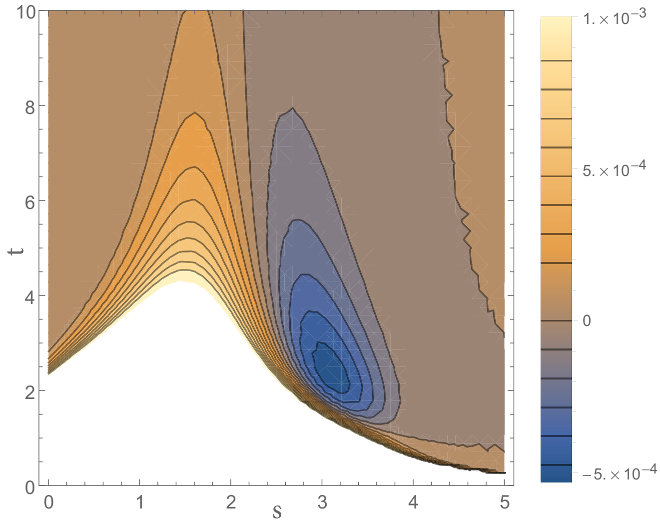

FIG. 1. Contour plot of the behavior of the derivative about the determinant with respect to time while varying the Ohmicity parameter s. When $s \in (2,4)$, the derivative is negative and the dynamics exhibits Rényi non-Markovianity. All quantities are dimensionless.

where $f(t) = \exp[-2\int_0^t \gamma(s)ds]$. The determinant of the state is

$$|\rho_t| = ad - |b|^2 f^2(t),$$

with the derivative

$$\frac{d}{dt}|\rho_t| \propto f^2(t)\gamma(t).$$

According to the approach introduced in the preceding section, this unital dynamics is Rényi Markovian if and only if $\frac{d}{dt}|\rho_t| \geqslant 0$, which is equivalent to $\gamma(t) \geqslant 0$ for all time t. Once there is some time $t_0 > 0$ such that $\gamma(t_0) < 0$, this unital dynamics is Rényi non-Markovian. This result is in accord with the characterizations of quantum non-Markovianity from the perspective of the information flow, divisibility, mutual information, and quantum Fisher information matrix. The quantitative measure for Rényi non-Markovianity is

$$N_R(\Lambda) = \max_\rho \int_{\gamma(t)<0} -4|b|^2 h(\alpha,t) f^2(t)\gamma(t)dt.$$

Consider a particular type of reservoir characterized by the Ohmic spectral density function as

$$J(\omega) = \frac{\omega^s}{\omega_c^{s-1}}e^{-\omega/\omega_c},$$

where s is the Ohmicity parameter and ω_c is the cutoff spectral frequency. Taking different values of s, one can obtain sub-Ohmic ($s<1$), Ohmic ($s=1$), and super-Ohmic ($s>1$) spectral densities, respectively. The dephasing rate in this case can be explicitly expressed as

$$\gamma(t) = \omega_c[1+(\omega_c t)^2]^{-s/2}\Gamma(s)\sin[s\arctan(\omega_c t)],$$

where $\Gamma(s)$ is the Euler Gamma function.

We plot part of the dynamical behavior of the time derivative of the determinant, i.e., $\frac{d}{dt}|\rho_t|$, as a function of time t and parameter s in Fig. 1. It can be seen that for some intervals such as $s \in (2,4)$, the dynamics is Rényi non-Markovian due to the reduction of $|\rho_t|$. That is to say, the non-Markovian behavior appears when the qubit interacts with a super-Ohmic reservoir. From this example, we see that this measure is compatible with previous criteria and effectively captures the backflow of information from the environment to the system.

Example 2. Consider a qubit system suffering from a random unitary channel governed by the master equation

$$\frac{d}{dt}\rho_t = \sum_{i=1}^{3} \gamma_i(t)(\sigma_i \rho_t \sigma_i - \rho_t), \quad t \geqslant 0,$$

where $\gamma_i(t), i = 1,2,3$, are suitable real functions. The dynamics can be rewritten in the equivalent form

$$\Lambda_t(\rho) = \sum_{i=0}^{3} p_i(t)\sigma_i \rho \sigma_i, \quad t \geqslant 0,$$

with $p_0 = [1 + \sum_{j=1}^3 \Gamma_j(t)]/4$ and $p_i = \Gamma_i(t)/2 + [1 - \sum_{j=1}^3 \Gamma_j(t)]/4$, $i = 1,2,3$, in which $\Gamma_i(t) = \exp\{2\int_0^t [\gamma_i(s) - \sum_{j=1}^3 \gamma_j(s)]ds\}$. Obviously, this dynamics is unital.

Previous work has shown the following [35,36]. (i) The dynamics is Markovian in the sense of divisibility if and only if $\gamma_i(t) \geqslant 0$, $i = 1,2,3$. (ii) The dynamics is Markovian in the sense of information flow if and only if $\gamma_i(t) + \gamma_j(t) \geqslant 0$, $i \neq j$ and $i,j = 1,2,3$. (iii) The dynamics is Markovian in the sense of mutual information if and only if $\sum_{i=0}^3 \dot{p}_i(t) \log_2 p_i(t) \leqslant 0$.

Now we derive the condition for this dynamics to be Rényi Markovian. For any initial state ρ, the evolving state can be expressed as

$$\rho_t = \begin{pmatrix} a_t & b_t \\ {b_t}^* & 1 - a_t \end{pmatrix},$$

where

$$a_t = [p_0(t) + p_3(t)]a + [p_1(t) + p_2(t)]d,$$
$$b_t = [p_0(t) - p_3(t)]b + [p_1(t) - p_2(t)]b^*.$$

The determinant of this state can be calculated as

$$|\rho_t| = \tfrac{1}{4}[1 - (1 - 4ad)\Gamma_3^2(t) - (b + b^*)^2\Gamma_1^2(t) + (b - b^*)^2\Gamma_2^2(t)]$$

Since

$$\frac{d}{dt}|\rho_t| = (1 - 4ad)[\gamma_1(t) + \gamma_2(t)]\Gamma_3^2(t) + (b + b^*)^2[\gamma_2(t) + \gamma_3(t)]\Gamma_1^2(t) - (b - b^*)^2[\gamma_1(t) + \gamma_3(t)]\Gamma_2^2(t)$$

and $ad \leqslant \frac{1}{4}$, we can get that if and only if $\gamma_i(t) + \gamma_j(t) \geqslant 0$, $i \neq j$ and $i,j = 1,2,3$, for all $t \geqslant 0$, this dynamics is Rényi Markovian.

In this case, the criterion proposed here is in accord with the one based on information flow but different from those based on divisibility and quantum mutual information. In particular, it is weaker than the criterion via divisibility. There exist examples as constructed in Refs. [37,38] that are Markovian under our criterion but exhibit non-Markovianity from the perspective of divisibility.

Example 3. Now we continue with the above example and restrict the discussion to the particular case of $p_i(t) = q_i[1 - p_0(t)]$, $i = 1,2,3$, and $q_1 = q_2 = q \in [0,1/2]$, and then $q_3 = 1 - 2q \in [0,1]$. For an arbitrary state ρ, the output state of this random unitary dynamics turns out to be

$$\rho_t = \begin{pmatrix} a_t & b_t \\ b_t^* & 1 - a_t \end{pmatrix}$$

with

$$a_t = \{p_0(t) + (1 - 2q)[1 - p_0(t)]\}a + 2q[1 - p_0(t)]d,$$
$$b_t = \{p_0(t) - (1 - 2q)[1 - p_0(t)]\}b,$$

whose determinant is

$$|\rho_t| = 2q[1 - p_0(t)]\{p_0(t) + (1 - 2q)[1 - p_0(t)]\} + \{1 - 4q[1 - p_0(t)]\}^2 ad - \{p_0(t) - (1 - 2q)[1 - p_0(t)]\}^2|b|^2.$$

Since

$$\frac{d}{dt}|\rho_t| = [2q(4q - 1)(1 - 4ad) + 4(1 - q)(1 - 2q)|b|^2]\dot{p}_0(t) - [8(1 - q)^2|b|^2 + 8q^2(1 - 4ad)]p_0(t)\dot{p}_0(t),$$

we can get that $\Lambda = \{\Lambda_t : t \geqslant 0\}$ is Rényi Markovian if and only if $\dot{p}_0(t) \leqslant 0$ and

$$p_0(t) \geqslant \max\left\{1 - \frac{1}{2(1 - q)}, 1 - \frac{1}{4q}\right\}$$

for all time $t \geqslant 0$, which is in accord with the criterion based on information flow [17] but different from the characterizations by mutual information and Fisher information matrix [39]. This result exemplifies the statement that the measure introduced in Ref. [17] captures the non-Markovian behavior of the unital part of quantum dynamics [22].

V. CONCLUSION

In this paper we mainly investigated the dynamical property of quantum Rényi entropy under Lindblad equations and established a relation between Markovian unital dynamics and positive changing rate of Rényi entropy. Based on this, an alternative measure of quantum non-Markovianity is proposed from the perspective of quantum Rényi entropy for unital dynamics. It contains the previous work on quantum non-Markovianity via von Neumann entropy and Brukner-Zeilinger invariant information as limiting or special cases. This measure can effectively capture the backflow of information from environment to the system and detect the non-Markovianity for some typical unital dynamics.

The method adopted here can be extended to any resource theory. Using the monotonicity of the resource under free operations, we can define the corresponding quantum Markovianity and non-Markovianity just like Ref. [40].

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