Bound state in the continuum by spatially separated ensembles of atoms in a coupled-cavity array

P. T. Fong and C. K. Law

Department of Physics and Institute of Theoretical Physics, The Chinese University of Hong Kong, Shatin,
Hong Kong Special Administrative Region, China

We present an analytic solution of bound states in the continuum (BICs) for photons and atoms in a one-dimensional coupled-cavity array. These bound states are formed by two ensembles of two-level atoms confined in separated cavities of the array. We show that in the regime where the photon hopping rate between neighboring cavities is high compared with the collective Rabi frequency the BIC corresponds to a nonradiating collective atomic state in which the two ensembles of atoms are strongly entangled. On the other hand, in the low photon hopping rate regime, the BIC behaves as a quantum cavity in which photons can be trapped between the two ensembles of atoms.

I. INTRODUCTION

Bound states in the continuum (BICs) are spatially localized states with energies embedded in the continuous spectrum of a system. The first example of a BIC for a single quantum particle was proposed by von Neumann and Wigner [1]. Later, Stillinger and Herrick explored the possibilities of BICs in two-electron systems [2,3]. A formal theory of BICs based on interfering resonances was developed by Friedrich and Wintgen [4]. It is now known that a BIC is a wave phenomenon that can be realized in a variety of physical wave systems [5]. In particular, BICs have been observed in various (classical) photonic systems such as waveguides [6–8], photonic crystals [9,10], and photonic Lieb lattices [11,12]. There also exist surface bound states in the continuum in photonic structures [13,14]. Since photonic BICs correspond to a high-quality confinement of light at specific frequencies, they have useful applications in optical devices such as filters [15] and lasers [16].

Although the concept of BICs was originally proposed as a solution of the Schrödinger equation, there have been relatively few studies of BICs in quantum systems. A difficulty is a lack of quantum systems processing such a kind of bound states in general. However, recent investigations in waveguide QED have found interesting examples of BICs for dressed photon-atom systems [17–24]. Waveguide QED generally refers to photon-atom interactions in one-dimensional photonic structures, such as superconducting qubits in transmission lines [25,26] and trapped atoms in a nanophotonic waveguide [27,28]. Since atoms can be strongly coupled to tightly confined waveguide modes, waveguide QED has been a useful platform for studying photon-photon interaction [29] and quantum information processing [30–32]. In particular, a coupled-cavity array can have energy bands equivalent to that of the tight-binding model, and interesting effects such as photon scattering [33,34], single-photon transport [35–37], and photon-atom bound states outside the continuum [38,39] have been discussed in literature.

We note that BICs comprising a single excitation (or photon) have been studied theoretically in a coupled-cavity array [22–24]. Such BICs are typically formed by two separated two-level atoms, such that a single photon at a resonance frequency can be perfectly trapped in the space

between the two atoms. In this regard, the two atoms behave as a quantum cavity for a single photon.

In this paper we show that BICs comprising multiple excitations or photons can be formed by two spatially separated ensembles of two-level atoms in a coupled-cavity array. This extends the previous studies [22–24] to situations of multiple photons. We note that the problem of two-particle BICs has been studied in various Bose-Hubbard models [40–42] and interesting interference effects due to quantum statistics of particles have been reported [43]. Here we present a type of multiparticle BICs formed by two distinct constituents, namely, photons and atoms in waveguide QED, and the atomic degree of freedom is described by collective spin variables. As we shall see below, our BICs exhibit various features depending on the photon hopping rate between neighboring cavities and the photon-atom interaction strength. Apart from the quantum-cavity effect for multiple photons, there exists a subradiance regime where excitations are mostly stored in a collective atomic state that does not radiate. We note that subradiance is a collective quantum phenomenon that has attracted research interest recently in experimental [44–46] and theoretical studies [47–51]. Our paper provides an example of subradiance by two ensembles of atoms.

Our paper is organized as follows. We first describe the system and the corresponding Hamiltonian in Sec. II. Then, by making use of a decoupling condition, we obtain an exact analytic solution of BICs with a general excitation number in Sec. III. In Sec. IV, we focus on the properties of BICs in a triple-cavity system. Such a system allows us to analyze the structures of BICs as well as their formation in detail. In particular, we indicate how subradiant states can be formed dynamically by free evolution in the subradiance regime. We also provide a linear theory of the system in the quantum-cavity regime and determine how atomic decoherence affects the storage time of photons. The conclusions and remarks are given in Sec. V.

II. THE MODEL

We consider a one-dimensional coupled-cavity array formed by $N + 1$ (where $N > 1$) cavities, in which the leftmost and rightmost cavities each contain M identical two-level atoms (Fig. 1). We assume that each cavity mode
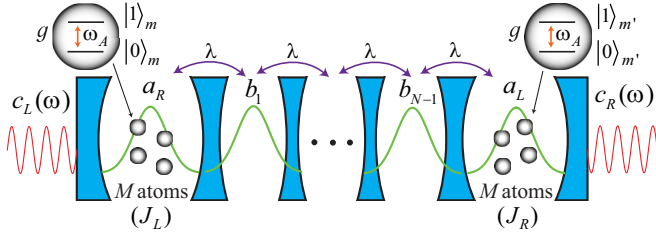
FIG. 1. A schematic drawing of the one-dimensional coupled-cavity array. The leftmost and rightmost cavities each contain M identical two-level atoms. The coupling parameters and operators defined in the text are indicated.

has the same resonance frequency ω_c which is close to the atomic transition frequency ω_A. The full Hamiltonian of the system is given by $\mathcal{H} = H_1 + H_2$ where ($\hbar = 1$)

$$
\begin{aligned}
H_1 = {} & \omega_c(a_L^\dagger a_L + a_R^\dagger a_R) + \omega_A(J_L^z + J_R^z) + \sum_{n=1}^{N-1} \omega_c b_n^\dagger b_n \\
& + g(a_L^\dagger J_L^- + a_R^\dagger J_R^- + a_L J_L^+ + a_R J_R^+) \\
& + \lambda(a_L^\dagger b_1 + a_R^\dagger b_{N-1} + a_L b_1^\dagger + a_R b_{N-1}^\dagger) \\
& + \lambda \sum_{n=1}^{N-2}(b_n^\dagger b_{n+1} + b_{n+1}^\dagger b_n),
\end{aligned}
\tag{1}
$$

$$
\begin{aligned}
H_2 = {} & \int_0^\infty d\omega\, \omega[c_L^\dagger(\omega)c_L(\omega) + c_R^\dagger(\omega)c_R(\omega)] \\
& + \int_0^\infty d\omega[\eta_L(\omega)c_L^\dagger(\omega)a_L + \eta_R(\omega)c_R^\dagger(\omega)a_R + \text{H.c.}].
\end{aligned}
\tag{2}
$$

Here H_1 describes the photon-atom interaction (second line) and tight-binding-type coupling between neighboring cavities (third and fourth line), and H_2 describes outside field modes and their interaction with the fields in the end cavities. Specifically, a_L and a_R are annihilation operators for the leftmost and rightmost cavity modes, respectively, b_n ($n = 1, 2, \ldots, N-1$) are annihilation operators associated with the cavities in the middle, and λ (assumed real) is the coupling constant between adjacent cavities [52]. In our model, we assume that all atoms in the respective cavities experience the same photon-atom coupling strength g, and so atoms are treated collectively. For the M atoms in the leftmost cavity, we define

$$
J_L^z = \frac{1}{2}\sum_{m=1}^{M}(|1\rangle_m\langle 1| - |0\rangle_m\langle 0|),
\tag{3}
$$

$$
J_L^+ = \sum_{m=1}^{M}|1\rangle_m\langle 0| = (J_L^-)^\dagger.
\tag{4}
$$

Here $|1\rangle_m$ and $|0\rangle_m$ denote the excited and ground state of the mth atom. For the atoms in the rightmost cavity, J_R^z and J_R^+ are defined similarly but with the summation index m taken from $M+1$ to $2M$. Finally for the H_2, we have used $c_L(\omega)$ and $c_R(\omega)$, respectively, for the annihilation operators of continuous field modes at the frequency ω outside the leftmost

and rightmost cavities, and $\eta_L(\omega)$ and $\eta_R(\omega)$ are coupling strengths [53].

The Hamiltonian H_1 can be simplified by using the normal modes of the cavity chain (excluding the two end cavities). The annihilation operators associated with such normal modes are given by

$$
B_k = \sqrt{\frac{2}{N}}\sum_{n=1}^{N-1}b_n\sin\frac{kn\pi}{N},
\tag{5}
$$

with $k = 1, 2, \ldots, N-1$. Such operators satisfy $[B_k, B_{k'}^\dagger] = \delta_{kk'}$. In this way, H_1 reads

$$
\begin{aligned}
H_1 = {} & \omega_c(a_L^\dagger a_L + a_R^\dagger a_R) + \omega_A(J_L^z + J_R^z) \\
& + \sum_{k=1}^{N-1}\Omega_k B_k^\dagger B_k + \left[a_L^\dagger\left(gJ_L^- + \sum_{k=1}^{N-1}\lambda_k^L B_k\right) + \text{H.c.}\right] \\
& + \left[a_R^\dagger\left(gJ_R^- + \sum_{k=1}^{N-1}\lambda_k^R B_k\right) + \text{H.c.}\right]
\end{aligned}
\tag{6}
$$

where

$$
\lambda_k^L = \lambda\sqrt{\frac{2}{N}}\sin\frac{k\pi}{N},
\tag{7}
$$

$$
\lambda_k^R = \lambda\sqrt{\frac{2}{N}}\sin\frac{k(N-1)\pi}{N} = (-1)^{(k+1)}\lambda_k^L,
\tag{8}
$$

$$
\Omega_k = \omega_c + 2\lambda\cos\frac{k\pi}{N}
\tag{9}
$$

are defined. For later purposes, we introduce the operator

$$
\mathcal{N} = a_L^\dagger a_L + a_R^\dagger a_R + J_L^z + J_R^z + M + \sum_{k=1}^{N-1}B_k^\dagger B_k
\tag{10}
$$

which corresponds to the total number of excitations stored in the cavity array and atoms.

III. SOLUTIONS OF THE BIC

Owing to the coupling to the continuum, the full Hamiltonian $\mathcal{H}$ has a continuous energy spectrum with energy $E \geqslant -M\omega_A$. A BIC represents a localized eigenstate with its energy embedded in the continuous energy spectrum. Here we adopt a strict localization condition that all excitations are trapped inside the atoms and the cavity array, while all the field modes outside [described by $c_L(\omega)$ and $c_R(\omega)$] are in the vacuum state $|\text{vac}\rangle_{\text{out}}$. For a given excitation number K, the BIC takes the form

$$
|\Psi_{\text{BIC}}^{(K)}\rangle = |\beta_K\rangle|\text{vac}\rangle_{\text{out}}
\tag{11}
$$

where $|\beta_K\rangle$ is a trapped state describing the state of atoms and photons inside the cavity array. Specifically, $|\beta_K\rangle$ is a common eigenvector of H_1 and $\mathcal{N}$:

$$
H_1|\beta_K\rangle = E_b|\beta_K\rangle,
\tag{12}
$$

$$
\mathcal{N}|\beta_K\rangle = K|\beta_K\rangle,
\tag{13}
$$

where K is an integer, and E_b is the energy eigenvalue.

In this paper we present a solution of $|\beta_K\rangle$ with a given excitation number K in the form

$$|\beta_K\rangle = \sum_{m=0}^{K} \sum_{n=0}^{K-m} \alpha_{mn} B_q^{\dagger m} (J_L^+)^n (J_R^+)^{K-m-n} |\phi\rangle. \qquad (14)$$

Here α_{mn} are some coefficients, and $|\phi\rangle$ is the state in which all the field modes in the cavity array are in the vacuum state and all the atoms are in their $|0\rangle$ state. In writing Eq. (14), we have assumed that there exists a mode described by B_q which has the frequency $\Omega_q = \omega_A$. For example, such a mode occurs when $q = N/2$ for an even N and $\omega_c = \omega_A$. The effect of a nonzero detuning ($\Omega_q \neq \omega_A$) will be discussed in the next section.

Noting that $|\beta_K\rangle$ defined by Eq. (14) has zero photons in the end cavities, $|\beta_K\rangle$ is an eigenvector of H_1 with the energy eigenvalue $E_b = (K - M)\omega_A$ if the following conditions are satisfied:

$$(g J_L^- + \lambda_q^L B_q)|\beta_K\rangle = 0, \qquad (15)$$

$$(g J_R^- + \lambda_q^R B_q)|\beta_K\rangle = 0. \qquad (16)$$

Physically, these two conditions can be interpreted as a destructive interference between the photon emission by atoms [described by $g a_\mu^\dagger J_\mu^-$ in Eq. (6)] and the photon tunneling [described by $\lambda_q^\mu a_\mu^\dagger B_q$ in Eq. (6)]. Together with the vacuum field outside, the conditions (15) and (16) ensure that the fields in the end cavities remain in the vacuum state. Since the end cavity modes are never excited, no energy can escape to the continuum, and hence all the excitations are trapped in the atoms and the B_q mode. Unlike BICs based on destructive interference in classical electromagnetic wave systems [5,54], here the interference corresponds to the inhibition of quantum transitions in a finite dimensional Hilbert space.

The conditions Eqs. (15) and (16) require $M \geqslant K$. To determine the explicit form of $|\beta_K\rangle$, we rewrite Eq. (14) as

$$|\beta_K\rangle = \sum_{m=0}^{K} \sum_{n=0}^{K-m} c_{m,n} |m,n,K-m-n\rangle |0_L,0_R\rangle \qquad (17)$$

where $|0_L,0_R\rangle$ denotes the vacuum state of the two end cavities, $c_{m,n}$'s are coefficients, and $|m,n,K-m-n\rangle$ is a common eigenvector of $B_q^\dagger B_q$, J_μ^2, and J_μ^z ($\mu = L,R$) such that

$$B_q^\dagger B_q |m,n,r\rangle = m |m,n,r\rangle, \qquad (18)$$

$$J_L^z |m,n,r\rangle = \left(n - \frac{M}{2}\right) |m,n,r\rangle, \qquad (19)$$

$$J_R^z |m,n,r\rangle = \left(r - \frac{M}{2}\right) |m,n,r\rangle. \qquad (20)$$

In words, m is the photon number in the B_q mode, and n and r are the number of excited atoms in the left and right end cavities, respectively.

By using the conditions (15) and (16), we obtain the recursive relations:

$$c_{m+1,n-1} = -\frac{g}{\lambda_q^L} \frac{\sqrt{n(M-n+1)}}{\sqrt{m+1}} c_{m,n}, \qquad (21)$$

$$c_{m+1,n} = -\frac{g}{\lambda_q^R} \frac{\sqrt{(K-m-n)(M-K+m+n+1)}}{\sqrt{m+1}} c_{m,n}. \qquad (22)$$

This leads to the solution of $c_{m,n}$ given by

$$c_{m,n} = (\chi)^m \left(\frac{\lambda_q^R}{\lambda_q^L}\right)^n \sqrt{\frac{(M-K+n+m)!}{(K-n-m)!m!}}$$

$$\times \sqrt{\frac{(M-n)!}{M!}} \sqrt{\frac{K!}{(M-K)!n!}} c_{0,0}, \qquad (23)$$

where the parameter χ (assumed positive) is defined by

$$\chi = -g/\lambda_q^R = (-1)^q g/\lambda_q^L \qquad (24)$$

and the value of $c_{0,0}$ is determined by the normalization condition. Note that $(\lambda_q^R/\lambda_q^L) = (-1)^{q+1}$ depends on the integer mode index q.

Equation (23) is a main result of this paper. For a given M and K, the structure of the BIC is controlled by the parameter χ since the m-photon amplitude scales with χ^m. In the following we discuss the features in $\chi \gg 1$ and $\chi \ll 1$ regimes.

A. Subradiance regime

In the $\chi \ll 1$ limit, the zero photon amplitudes $c_{0,n}$ contribute most to the BIC. Therefore the BIC is mainly formed by excited atoms in a collective state that does not radiate, i.e., a subradiant state. By keeping $c_{0,n}$ terms only, we have

$$|\beta_K\rangle \approx \sum_{n=0}^{K-m} c_{0,n} |0,n,K-n\rangle |0_L,0_R\rangle. \qquad (25)$$

In particular, the state with the excitation number $K = M$ is a maximally entangled state in which the atomic excitations in the two ensembles are perfectly corrected:

$$|\beta_M\rangle \approx \frac{1}{\sqrt{M+1}} \sum_{n=0}^{M} (\pm 1)^n |0,n,M-n\rangle |0_L,0_R\rangle, \qquad (26)$$

where $+1$ and -1 are for odd and even q, respectively. As we shall see in the next section the right-hand side of Eq. (26) is a singlet state.

Note that the specific requirement of the smallness of χ for the subradiant regime depends on the excitation number K. To ensure that the zero photon amplitudes $c_{0,n}$ are dominant, we require $|c_{m+1,n}|/|c_{m,n}| \ll 1$. By Eq. (22), it can be shown that $|c_{m+1,n}|/|c_{m,n}| < \chi\sqrt{(1+M-r)r}$ with r defined in Eq. (20). Noting that r is bounded by $0 \leqslant r \leqslant K-n-m$, $|c_{m+1,n}|/|c_{m,n}| \ll 1$ would need $\chi(M+1)/2 \ll 1$ for $K = M$, and $\chi\sqrt{M} \ll 1$ for $K \ll M$.
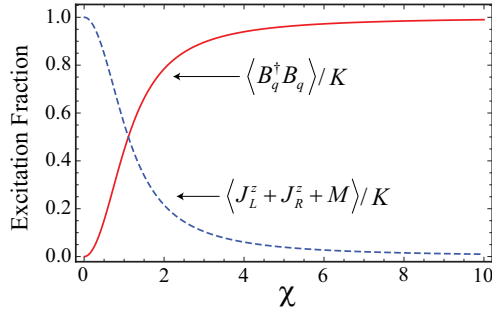
FIG. 2. The average number of photons (solid line) and excited atoms (dashed line) normalized by the total number of excitation as a function of χ for the BIC with $M = K = 2$.

B. Quantum-cavity regime

In the regime where $\chi \gg 1$, $c_{K,0}$ in Eq. (23) is much larger than all other amplitudes. Hence $|\beta_K\rangle$ is approximately a Fock state:

$$|\beta_K\rangle \approx |K,0,0\rangle|0_L,0_R\rangle, \tag{27}$$

which indicates that almost all excitations are stored as photons in the the field mode B_q. In other words, the two ensembles of atoms effectively form a quantum cavity for multiple photons.

In Fig. 2, we illustrate the transition between two regimes by showing the expectation values of photon number in the mode B_q and the atomic excitation number for the state $|\beta_M\rangle$ as a function of χ. For the case with the excitation number $K = M = 2$ used in the figure, $\chi > 5$ is sufficient to have more than 90% of photonic excitations. On the other hand $\chi < 0.5$ would give more than 90% of excitation stored in the atoms.

IV. TRIPLE-CAVITY CONFIGURATION

In this section we examine the BICs in a triple-cavity configuration ($N = 2$ in Fig. 1) in which there is only one normal mode $B_1 = b_1$ for the cavity in the middle. Such a configuration is conceptually simpler and it provides useful physical insights about the formation of BICs.

A. Effective Hamiltonian in the subradiant regime

First we introduce the following operators:

$$a_- = \frac{a_L - a_R}{\sqrt{2}}, \tag{28}$$

$$v_\pm = \frac{a_L + a_R \pm \sqrt{2}b_1}{2}, \tag{29}$$

which satisfy $[a_-,a_-^\dagger] = [v_+,v_+^\dagger] = [v_-,v_-^\dagger] = 1$ and $[a_-,v_\pm^\dagger] = [v_+,v_-^\dagger] = 0$. These operators correspond to creation and annihilation operators associated with the normal modes of the three coupled cavities. Accordingly, the

Hamiltonian H_1 in Eq. (1) can be rewritten as

$$H_1 = \omega_A S^z + \omega_c a_-^\dagger a_- + \frac{g}{\sqrt{2}}(S^- a_-^\dagger + \text{H.c.})$$
$$+ (\omega_c + \sqrt{2}\lambda)v_+^\dagger v_+ + (\omega_c - \sqrt{2}\lambda)v_-^\dagger v_-$$
$$+ \frac{g}{2}[(J_L^- - \tilde{J}_R^-)(v_+^\dagger + v_-^\dagger) + \text{H.c.}]. \tag{30}$$

Here we have defined $S^z = J_L^z + \tilde{J}_R^z$ and $S^\pm = J_L^\pm + \tilde{J}_R^\pm$, with $\tilde{J}_R^\pm = -J_R^\pm$ and $\tilde{J}_R^z = J_R^z$. Note that S^i ($i = z,\pm$) obey the commutation relations of angular momentum operators: $[S^+,S^-] = 2S^z$ and $[S^z,S^\pm] = \pm S^\pm$. Since there are M identical two-level atoms in each of the end cavity, S^i's are equivalent to the addition of angular momentum operators for two spin-$M/2$ systems.

Let $|s,m_s\rangle$ be a common eigenvector of S^2 and S^z, then

$$S^2|s,m_s\rangle = s(s+1)|s,m_s\rangle, \tag{31}$$

$$S^z|s,m_s\rangle = m_s|s,m_s\rangle, \tag{32}$$

where the allowed quantum numbers are $s = 0,1,2,\ldots,M$ and $m_s = -s, -s + 1, \ldots, s$.

Now returning to the Hamiltonian (30) and assuming $\omega_c \approx \omega_A$, we see that the atoms and $v_\pm$ modes are essentially uncoupled in the subradiance regime because the detunings $\pm\sqrt{2}\lambda$ are much larger than g in magnitude. Therefore we may neglect the far off resonant interaction terms in the last line of the Eq. (30). Such an approximation is a type of rotating wave approximation when viewing the system in the interaction picture. By keeping resonant terms of Eq. (30) (i.e., discarding $v_\pm$ modes), atoms and the a_- mode photons are coupled via the effective Hamiltonian

$$H_1' = \omega_A S^z + \omega_c a_-^\dagger a_- + \frac{g}{\sqrt{2}}(S^- a_-^\dagger + \text{H.c.}), \tag{33}$$

which is the Hamiltonian of the Tavis-Cummings model [55,56]. Since $[S^2,H_1'] = 0$, the quantum number s is a constant of motion.

According to the effective Hamiltonian H_1', atoms in the collective state $|s,m_s = -s\rangle$ are nonradiating, since they cannot emit a photon to the a_- mode by lowering the quantum number m_s. This is connected to the BIC in the previous section, where the $|\beta_K\rangle$ given in Eq. (25) can be rewritten as

$$|\beta_{M-s}\rangle \approx |s,m_s = -s\rangle|0_b\rangle|0_L,0_R\rangle, \tag{34}$$

with $|0_b\rangle$ being the vacuum field state in the b_1 mode. For example, Eq. (26) corresponds to the singlet state with $s = 0$. A general derivation of Eq. (34) can be obtained by using Eq. (25) with $c_{0,n}$ obtained in Eq. (23) and Clebsch-Gordan coefficients.

B. Evolution to subradiant states

Consider the system in the subradiance regime with atoms initially prepared in a superposition of $|s,m_s\rangle$ states and no photon in the cavities; the initial state is then given by

$$|\psi(0)\rangle = \sum_{s=0}^{M} \sum_{m_s=-s}^{s} C_{s,m_s}|s,m_s\rangle|0_b\rangle|0_L,0_R\rangle, \tag{35}$$

where C_{s,m_s} are some coefficients. As the system evolves, atoms would be de-excited by emitting a photon into the a_- mode according to H_1', and hence the quantum number m_s is lowered by 1. Furthermore, since the a_- mode is damped because of its coupling to the outside modes [governed by H_2 in Eq. (2)], de-excitation of atoms will continue until the quantum number m_s reaches the lowest possible value $m_s = -s$. Therefore the system would eventually be trapped as a mixture of $|s, -s\rangle$ for different s's.

In terms of the reduced density matrix ρ obtained by taking the trace over the outside field modes, the final ρ would be a mixed state:

$$\rho \approx \sum_{s=0}^{M} p_s |s, -s\rangle\langle s, -s| \otimes |0\rangle_{TT}\langle 0|, \qquad (36)$$

where $p_s = \sum_{m_s=-s}^{s} |C_{s,m_s}|^2$ and $|0\rangle_T \equiv |0_b\rangle|0_L, 0_R\rangle$ for brevity.

To verify the above result, we calculate the time evolution of ρ by the master equation method. Assuming that the coupling strengths $\eta_L(\omega)$ and $\eta_R(\omega)$ in Eq. (2) are frequency independent, the outside field modes are equivalent to Markovian oscillator baths at zero temperature. This leads to the usual Markovian master equation:

$$\dot{\rho} = -i[H_1, \rho] + \sum_{\mu=L,R} \frac{\gamma_c}{2} \mathcal{D}[a_\mu]\rho, \qquad (37)$$

where H_1 is the original Hamiltonian in Eq. (1) without using the approximation in the previous subsection, and the superoperator $\mathcal{D}$ is defined by

$$\mathcal{D}[X]\rho = (2X\rho X^\dagger - X^\dagger X\rho - \rho X^\dagger X). \qquad (38)$$

In writing Eq. (37), we have used $\eta_L(\omega) = \eta_R(\omega) = \sqrt{\gamma_c/2\pi}$, with γ_c being the leakage rate of the end cavities. Note that decoherence effects on atoms due to interactions with noncavity modes have been omitted in the master equation. This is justified as long as the relevant relaxation rates of atoms are sufficiently small so that the steady state of ρ defined by Eq. (37) can be established before decoherence effects become significant.

We have solved the master equation numerically for the $M = 2$ case, and the numerical steady state agrees well with the approximation given in Eq. (36). Specifically, we consider the two atoms in the left (right) cavity are initially prepared in the excited (ground) states, and all the field modes are in the vacuum state. In this case, $s = 0, 1, 2$ are allowed quantum numbers.

In Fig. 3 we illustrate the numerical results by showing the time dependence of the probability in the trapped state $|\beta_i\rangle$, which is $P_i = \langle\beta_i|\rho|\beta_i\rangle$, obtained from the master equation. We see that P_2 is a constant because the initial system with an excitation number $K = 2$ has a partial overlap with trapped state $|\beta_2\rangle$, and it will remain a constant. As the system evolves, P_0 and P_1 increase from zero and become steady, and so the system is trapped in the lower states $|\beta_1\rangle$ and $|\beta_0\rangle$. Note that $|\beta_0\rangle$ is just the trivial ground state of the system. For the parameters used in Fig. 3, we have verified that $|\beta_2\rangle$ and $|\beta_1\rangle$ are well approximated by Eq. (34) and the steady value of P_i agrees well with p_{M-i} defined after Eq. (36) within 1% discrepancy.
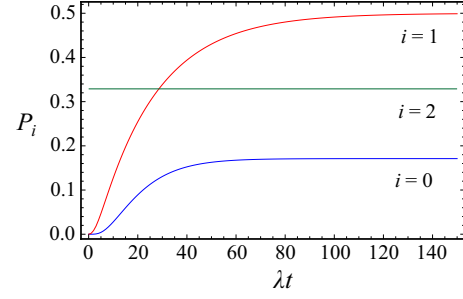

FIG. 3. Occupation probabilities of the states $|\beta_i\rangle$ for $i = 0, 1, 2$ as a function of dimensionless time λt. The evolution is obtained by solving the master equation (37) numerically with the parameters $M = 2$, $N = 2$, $g/\lambda = 0.1$, $\gamma_c/\lambda = 1$ and the initial state has an excitation number $K = 2$ (see the text).

Our discussion in this subsection so far has assumed $\omega_A = \omega_c$, which is a condition required for the BIC solution (Sec. III). A nonzero detuning $\delta \equiv \omega_c - \omega_A$ would lead to an incomplete destructive interference, and so photons would escape to the continuum through the end cavities. However, we find that the rate of such a loss can be significantly suppressed in the $\chi \ll 1$ limit. Specifically, consider a system in a certain trapped state $|\beta_i\rangle$ with small magnitude of δ compared with λ; we then find that the probability loss rate γ_d is proportional to $\gamma_c K (2M - K + 1)\delta^2\chi^2/\lambda^2$, which decreases with χ^2 [57]. Hence in the $\chi \ll 1$ limit the trapped states in the detuned system can be maintained in a relatively long (but finite) time scale.

C. Linear analysis of the quantum-cavity regime

In this subsection we turn to the quantum-cavity regime where $\chi \gg 1$. The fact that atoms are weakly excited in this regime allows us to make the following approximation:

$$J_\mu^- \approx \sqrt{M}d_\mu, \quad J_\mu^+ \approx \sqrt{M}d_\mu^\dagger, \quad J_\mu^z = d_\mu^\dagger d_\mu - M/2, \quad (39)$$

where d_μ and $d_\mu^\dagger$ ($\mu = L, R$) are annihilation and creation operators satisfying the bosonic commutation relation $[d_\mu, d_\nu^\dagger] = \delta_{\mu\nu}$. Such an approximation can be derived by keeping the leading term in the Holstein-Primakoff transformation [58], which is justified because $\langle J_\mu^z \rangle \ll M/2$ for BICs in the $\chi\sqrt{M} \gg 1$ regime.

By using approximation (39), H_1 in Eq. (1) can be expressed as

$$\begin{aligned} H_1 = {}& \omega_c(a_L^\dagger a_L + a_R^\dagger a_R + F_0^\dagger F_0 + F_+^\dagger F_+ + F_-^\dagger F_-) \\ & + a_L^\dagger(\xi_+ F_+ + \xi_- F_-) + a_L(\xi_+ F_+^\dagger + \xi_- F_-^\dagger) \\ & + a_R^\dagger(\xi_+ F_+ - \xi_- F_-) + a_R(\xi_+ F_+^\dagger - \xi_- F_-^\dagger), \quad (40) \end{aligned}$$

where $\omega_A = \omega_c$ is assumed, and

$$F_+ = \frac{g\sqrt{M}(d_L + d_R) + 2\lambda b_1}{\sqrt{2}\sqrt{g^2M + 2\lambda^2}}, \qquad (41)$$

$$F_- = \frac{d_L - d_R}{\sqrt{2}}, \qquad (42)$$

$$F_0 = \frac{g\sqrt{M}b_1 - \lambda d_L - \lambda d_R}{\sqrt{g^2M + 2\lambda^2}} \qquad (43)$$

are annihilation operators associated with three orthogonal polaritonic modes defined by the atomic oscillators and the middle cavity, and they satisfy the commutation relations $[F_\alpha, F_{\alpha'}^\dagger] = \delta_{\alpha\alpha'}$ for bosons. In addition, the coupling coefficients $\xi_\pm$ are given by $\xi_+ = \sqrt{g^2 M + 2\lambda^2}/\sqrt{2}$ and $\xi_- = \sqrt{g^2 M}/\sqrt{2}$.

By Eq. (43), we see that the polaritonic mode F_0 does not couple to a_R and a_L, and hence F_0 is isolated from the continuous modes outside. The quantum cavity therefore refers to the F_0 mode in which energy can be confined. In particular, the F_0 mode is mainly photonic because $F_0 \approx b_1$ in the $\chi \gg 1$ limit.

The performance of the quantum cavity is limited by various loss mechanisms. Here we examine the loss due to the spontaneous emission of photons from atoms into noncavity modes. Without loss of generality, a nonzero detuning $\delta \equiv \omega_c - \omega_A$ is included in the following discussion. Assuming atoms experience a collective decay, the damping can be introduced by adding an imaginary part to the atomic frequency. Specifically, we make use of the linearized Hamiltonian and replace ω_A by $\omega_A - iM\gamma_A/2$, where γ_A is the single-atom spontaneous decay rate (the factor M is due to the collective spontaneous decay). Then by the Heisenberg equations of motion one can obtain a close set of differential equations for the expectation values of a_μ, d_μ, and b_1:

$$i\langle \dot{a}_\mu \rangle = \lambda\langle b_1 \rangle + g\sqrt{M}\langle d_\mu \rangle + \left(\omega_c - i\frac{\gamma_c}{2}\right)\langle a_\mu \rangle, \quad (44)$$

$$i\langle \dot{b}_1 \rangle = \omega_c\langle b_1 \rangle + \lambda\langle a_L \rangle + \lambda\langle a_R \rangle, \quad (45)$$

$$i\langle \dot{d}_\mu \rangle = g\sqrt{M}\langle a_\mu \rangle + \left(\omega_c - \delta - i\frac{M\gamma_A}{2}\right)\langle d_\mu \rangle, \quad (46)$$

where $\mu = L, R$ and the outside modes are assumed to be in the vacuum state. Alternatively, one can obtain these equations from the master equation with collective atomic damping included.

We solve the eigenvalues of the linear system defined by the right-hand side of Eqs. (44)–(46). The eigenvalue with the smallest imaginary part (denoted as Γ) corresponds to the trapped mode, and Γ is its decay rate. After some calculations, we have

$$\Gamma \approx \frac{M^2\gamma_A g^2 + \delta^2\gamma_c}{M^2 g^4 + \delta^2\gamma_c^2/4}\lambda^2, \quad (47)$$

if $g \gg \max(\gamma_A, \gamma_c, \lambda)$. We characterize the performance of the quantum cavity by the scaled quality factor $\tilde{Q} \equiv \gamma_c/\Gamma$, which is the (dimensionless) photon storage time of the quantum cavity in units of the free cavity decay time γ_c^{-1}. A quantum cavity with $\tilde{Q} \gg 1$ means that its photon storage time is much longer than that in the free cavity.

In Fig. 4, we illustrate the dependence of $\tilde{Q}$ as a function of δ. At resonance $\delta = 0$, $\tilde{Q} = \chi^2\gamma_c/\gamma_A$ is a maximum and it can be much larger than 1 in the $\chi \gg 1$ regime. In addition, by the width of the peak at half maximum, the high-quality factor can be maintained in a range of detunings when $|\delta| < Mg\sqrt{\gamma_A/\gamma_c}$.
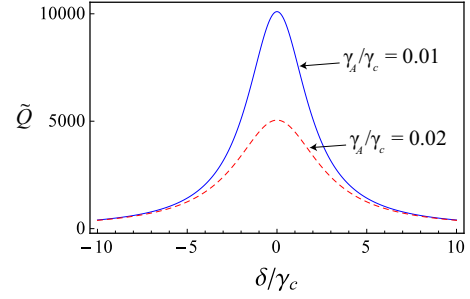

FIG. 4. Scaled quality factor $\tilde{Q} = \gamma_c/\Gamma$ as a function of dimensionless detuning δ/γ_c. The parameters are $M = 2, N = 2, g/\lambda = 10, \gamma_c/\lambda = 1$.

V. CONCLUDING REMARKS

To conclude, we have obtained an analytic solution of a type of BIC formed by photons and atoms in a coupled-cavity array. These BICs originate from a destructive quantum interference effect which prevents excitations from being coupled to outside field modes, even though the energy lies in the continuous spectrum of the whole system. In addition, we identify a subradiance regime and a quantum-cavity regime in which most excitations are, respectively, atomic and photonic.

We have also examined the triple-cavity configuration in detail. In the subradiance regime we found that the dynamics is governed by the Tavis-Cummings Hamiltonian, which suggests that a mixture of nonradiating atomic states can be generated as a steady state of the free-evolution problem. In the quantum-cavity regime, our analysis of the linearized system, which includes atomic damping, has indicated that photons can be stored in the quantum cavity with a lifetime significantly longer than γ_c^{-1}. Such an efficient trapping of multiple photons may be used to explore quantum effects of photon-atom interactions and applications in the cavity QED [59] and circuit QED [60].

Finally, we would like to add two remarks on experimental aspects of our model. First, like many other BICs based on parameter tuning [5], the existence of our BIC relies on the control of certain parameters, for example, $\omega_A = \Omega_q$. If these parameters are close but not exactly equal to the required values, a BIC would couple to the outside world and therefore has a finite lifetime (a leaky resonance). The dynamics can be analyzed by time-dependent perturbation theory, where the deviations of parameters are treated as a perturbation to the Hamiltonian. For a sufficiently weak perturbation, the system can still be trapped in the BIC with a time that can be long enough to produce observable effects. In other words, one may still study the BICs via leaky resonance in experiments without operating at perfect parameters. We have demonstrated this feature in Fig. 4, where the quantum cavity is able to store photons with a time much longer than the free cavity lifetime $1/\gamma_c$ even though the detuning is of the same size as the free cavity linewidth. Similarly, in the subradiance regime, the loss rate γ_d due to a finite detuning can be significantly suppressed in the $\chi \ll 1$ limit.

Second, we note that decoherence of atoms is one of the main obstacles for experiments. For example, a spontaneous decay of an atom due to interaction with noncavity modes

and various dephasing mechanisms would kick the system out of the BIC. In particular, the quantum correlation between the two atomic ensembles, which keeps atoms from radiating, would be degraded. Therefore λ and g should be significantly larger than decoherence rates of the system so that effects of BICs (such as Fig. 3) can be established and observed before appreciable decoherence takes place. The recent parameters achievable in circuit QED are promising, because λ and g in the 10–100-MHz range can be much higher than decoherence rates (about 10–50 KHz) [60]. In addition, the Tavis-Cummings model and its quantized energy spectrum have been demonstrated experimentally in circuit QED [56].

[1] J. von Neumann and E. Wigner, Phys. Z. **30**, 465 (1929).
[2] F. H. Stillinger, Jr., J. Chem. Phys. **45**, 3623 (1966).
[3] F. H. Stillinger and D. R. Herrick, Phys. Rev. A **11**, 446 (1975).
[4] H. Friedrich and D. Wintgen, Phys. Rev. A **32**, 3231 (1985).
[5] C. W. Hsu, B. Zhen, A. D. Stone, J. D. Joannopoulos, and M. Soljačić, Nat. Rev. Mats. **1**, 16048 (2016).
[6] F. Dreisow, A. Szameit, M. Heinrich, R. Keil, S. Nolte, A. Tünnermann, and S. Longhi, Opt. Lett. **34**, 2405 (2009).
[7] Y. Plotnik, O. Peleg, F. Dreisow, M. Heinrich, S. Nolte, A. Szameit, and M. Segev, Phys. Rev. Lett. **107**, 183901 (2011).
[8] S. Weimann, Y. Xu, R. Keil, A. E. Miroshnichenko, A. Tünnermann, S. Nolte, A. A. Sukhorukov, A. Szameit, and Y. S. Kivshar, Phys. Rev. Lett. **111**, 240403 (2013).
[9] C. W. Hsu, B. Zhen, J. Lee, S.-L. Chua, S. G. Johnson, J. D. Joannopoulos, and M. Soljačić, Nature (London) **499**, 188 (2013).
[10] R. Gansch, S. Kalchmair, P. Genevet, T. Zederbauer, H. Detz, A. M. Andrews, W. Schrenk, F. Capasso, M. Lončar, and G. Strasser, Light: Sci. and Appl. **5**, e16147 (2016).
[11] R. A. Vicencio, C. Cantillano, L. Morales-Inostroza, B. Real, C. Mejía-Cortés, S. Weimann, A. Szameit, and M. I. Molina, Phys. Rev. Lett. **114**, 245503 (2015).
[12] S. Mukherjee, A. Spracklen, D. Choudhury, N. Goldman, P. Öhberg, E. Andersson, and R. R. Thomson, Phys. Rev. Lett. **114**, 245504 (2015).
[13] M. I. Molina, A. E. Miroshnichenko, and Y. S. Kivshar, Phys. Rev. Lett. **108**, 070401 (2012).
[14] G. Corrielli, G. D. Valle, A. Crespi, R. Osellame, and S. Longhi, Phys. Rev. Lett. **111**, 220403 (2013).
[15] D. C. Marinica, A. G. Borisov, and S. V. Shabanov, Phys. Rev. Lett. **100**, 183902 (2008).
[16] A. Kodigala, T. Lepetit, Q. Gu, B. Bahari, Y. Fainman, and B. Kanté, Nature (London) **541**, 196 (2017).
[17] T. Tufarelli, F. Ciccarello, and M. S. Kim, Phys. Rev. A **87**, 013820 (2013).
[18] C. Gonzalez-Ballestero, F. J. García-Vidal, and E. Moreno, New J. Phys. **15**, 073015 (2013).
[19] P. Facchi, M. S. Kim, S. Pascazio, F. V. Pepe, D. Pomarico, and T. Tufarelli, Phys. Rev. A **94**, 043839 (2016).
[20] Ş. Ekin Kocabaş, Opt. Lett. **41**, 2533 (2016).
[21] P. Facchi, S. Pascazio, F. V. Pepe, and K. Yuasa, arXiv:1705.01967.
[22] S. Longhi, Eur. Phys. J. B **57**, 45 (2007).
[23] L. Zhou, H. Dong, Y.-x. Liu, C. P. Sun, and F. Nori, Phys. Rev. A **78**, 063827 (2008).
[24] C. Chen, C.-J. Yang, and J.-H. An, Phys. Rev. A **93**, 062122 (2016).
[25] A. Blais, R.-S. Huang, A. Wallraff, S. M. Girvin, and R. J. Schoelkopf, Phys. Rev. A **69**, 062320 (2004).

[26] A. F. van Loo, A. Fedorov, K. Lalumière, B. C. Sanders, A. Blais, and A. Wallraff, Science **342**, 1494 (2013).
[27] A. Goban, C.-L. Hung, S.-P. Yu, J. D. Hood, J. A. Muniz, J. H. Lee, M. J. Martin, A. C. McClung, K. S. Choi, D. E. Chang, O. Painter, and H. J. Kimble, Nat. Commun. **5**, 3808 (2014).
[28] P. Lodahl, S. Mahmoodian, and S. Stobbe, Rev. Mod. Phys. **87**, 347 (2015).
[29] D. Roy, C. M. Wilson, and O. Firstenberg, Rev. Mod. Phys. **89**, 021001 (2017).
[30] A. Blais, J. Gambetta, A. Wallraff, D. I. Schuster, S. M. Girvin, M. H. Devoret, and R. J. Schoelkopf, Phys. Rev. A **75**, 032329 (2007).
[31] V. Paulisch, H. J. Kimble, and A. González-Tudela, New J. Phys. **18**, 043041 (2016).
[32] H. Zheng, D. J. Gauthier, and H. U. Baranger, Phys. Rev. Lett. **111**, 090502 (2013).
[33] L. Zhou, Z. R. Gong, Y.-x. Liu, C. P. Sun, and F. Nori, Phys. Rev. Lett. **101**, 100501 (2008).
[34] Ş. E. Kocabaş, Phys. Rev. A **93**, 033829 (2016).
[35] W.-B. Yan and H. Fan, Phys. Rev. A **90**, 053807 (2014).
[36] J. Lu, L. Zhou, L.-M. Kuang, and F. Nori, Phys. Rev. A **89**, 013805 (2014).
[37] W. Qin and F. Nori, Phys. Rev. A **93**, 032337 (2016).
[38] G. Calajó, F. Ciccarello, D. Chang, and P. Rabl, Phys. Rev. A **93**, 033833 (2016).
[39] T. Shi, Y.-H. Wu, A. González-Tudela, and J. I. Cirac, Phys. Rev. X **6**, 021027 (2016).
[40] J. M. Zhang, D. Braak, and M. Kollar, Phys. Rev. Lett. **109**, 116405 (2012).
[41] G. D. Valle and S. Longhi, Phys. Rev. B **89**, 115118 (2014).
[42] S. Longhi and G. Della Valle, J. Phys.: Condens. Matter **25**, 235601 (2013).
[43] A. Crespi, L. Sansoni, G. D. Valle, A. Ciamei, R. Ramponi, F. Sciarrino, P. Mataloni, S. Longhi, and R. Osellame, Phys. Rev. Lett. **114**, 090201 (2015).
[44] S. Filipp, A. F. van Loo, M. Baur, L. Steffen, and A. Wallraff, Phys. Rev. A **84**, 061805(R) (2011).
[45] T. Bienaimé, N. Piovella, and R. Kaiser, Phys. Rev. Lett. **108**, 123602 (2012).
[46] W. Guerin, M. O. Araújo, and R. Kaiser, Phys. Rev. Lett. **116**, 083601 (2016).
[47] R. Wiegner, J. von Zanthier, and G. S. Agarwal, Phys. Rev. A **84**, 023805 (2011).
[48] M. O. Scully, Phys. Rev. Lett. **115**, 243602 (2015).
[49] H. Cai, D.-W. Wang, A. A. Svidzinsky, S.-Y. Zhu, and M. O. Scully, Phys. Rev. A **93**, 053804 (2016).
[50] M. Hebenstreit, B. Kraus, L. Ostermann, and H. Ritsch, Phys. Rev. Lett. **118**, 143602 (2017).
[51] B. Bellomo, G. L. Giorgi, G. M. Palma, and R. Zambrini, Phys. Rev. A **95**, 043807 (2017).

[52] A lattice with inhomogeneous complex hopping strengths can be a model of parity-time (PT) symmetric systems and some recent studies have explored BICs in such PT symmetric systems. See, for example, S. Longhi and G. D. Valle, Phys. Rev. A **89**, 052132 (2014); S. Longhi, Opt. Lett. **39**, 1697 (2014).

[53] It is worth noting that $\eta_L(\omega)$ and $\eta_R(\omega)$ can be different functions. Our BICs, which are determined by the parameters of H_1 only, are independent of these functions. If the two ensembles of atoms are placed at cavities other than the two end cavities, i.e., any two positions in the array, the corresponding Hamiltonian will take the same form, since two occupied cavities can be relabeled as end cavities. In this case the coupling to fields beyond the two occupied cavities will have a modified $\eta_L(\omega)$ and $\eta_R(\omega)$, but the form of BIC solution would be the same.

[54] Y. Yang, C. Peng, Y. Liang, Z. Li, and S. Noda, Phys. Rev. Lett. **113**, 037401 (2014).

[55] M. Tavis and F. W. Cummings, Phys. Rev. **170**, 379 (1968).

[56] J. M. Fink, R. Bianchetti, M. Baur, M. Göppl, L. Steffen, S. Filipp, P. J. Leek, A. Blais, and A. Wallraff, Phys. Rev. Lett. **103**, 083601 (2009).

[57] The calculation is done by treating the detuning as a perturbation term in the Hamiltonian. By the first-order time-dependent perturbation theory, the trapped state picks up a photon amplitude in end cavities proportional to $\delta g\sqrt{K(2M - K + 1)}/\lambda^2$, and hence the photon probability in the end cavity scales as $\delta^2 g^2 K(2M - K + 1)/\lambda^4$. The probability loss rate is estimated by the product of this probability and the cavity field decay rate.

[58] T. Holstein and H. Primakoff, Phys. Rev. **58**, 1098 (1940).

[59] For a recent review of cavity QED see A. Reiserer and G. Rempe, Rev. Mod. Phys. **87**, 1379 (2015).

[60] For a recent review of a circuit QED network see S. Schmidt and J. Koch, Ann. Phys. **525**, 395 (2013).