Bifurcation structure of cavity soliton dynamics in a vertical-cavity surface-emitting laser with a saturable absorber and time-delayed feedback

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We consider a wide-aperture surface-emitting laser with a saturable absorber section subjected to time-delayed feedback. We adopt the mean-field approach assuming a single longitudinal mode operation of the solitary vertical-cavity surface-emitting laser (VCSEL). We investigate cavity soliton dynamics under the effect of time-delayed feedback in a self-imaging configuration where diffraction in the external cavity is negligible. Using bifurcation analysis, direct numerical simulations, and numerical path-continuation methods, we identify the possible bifurcations and map them in a plane of feedback parameters. We show that for both the homogeneous and localized stationary lasing solutions in one spatial dimension, the time-delayed feedback induces complex spatiotemporal dynamics, in particular a period doubling route to chaos, quasiperiodic oscillations, and multistability of the stationary solutions.

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I. INTRODUCTION

Cavity solitons (CSs) are spatially localized light structures in the transverse plane of a nonlinear resonator that result from the balance of nonlinearity and diffraction (for reviews, see [1-11]). CSs belong to the class of dissipative structures found far from equilibrium; the losses in the system have to be balanced by external energy input. CSs normally require a region in the parameter space where a spatially periodic pattern and a stable homogeneous steady state coexist [12–14], so that in such a "pinning region," one or more peaks of the patterned state are surrounded by the homogeneous steady state. Recently, vertical-cavity surface-emitting lasers (VCSELs) have attracted considerable interest for CS studies and applications because they are inherently made with a short (single longitudinal mode) cavity, which can be transversely quite large [15]. In the first demonstrations of the existence of CSs in broad-area VCSELs, external coherent light with an appropriate frequency is injected to create the required pinning region and CSs have been found both below [16,17] and above [18] the lasing threshold. Utilizing the specific polarization properties of VCSELs [19], spatially localized structures have also been created in 40- μ m-diameter VCSELs [20], as well as in 80-µm-diameter VCSELs lasing on a high transverse-order flower mode [21,22]. For practical CS applications, the need for an external optical injection

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is a hindrance and a very attractive way to avoid it is the implementation of a saturable absorber in the VCSEL structure [23]. Hence, CS properties and dynamics in VCSELs with saturable absorbers have been extensively studied both theoretically [23,24] and experimentally [25,26].

The impact of time-delayed feedback on CS dynamics has been theoretically investigated for the cases of a driven nonlinear optical resonator [27,28] and broad-area VCSELs [29–31]. Delayed optical feedback is known to strongly modify the dynamical behavior of semiconductor lasers leading to external cavity mode hopping, periodic or aperiodic dynamics, and even coherence collapse [32,33]. Optical feedback impacts the VCSEL's modal properties and dynamics in quite the same way as those of traditional edge-emitting semiconductor lasers [34,35] with the additional peculiarity of introducing polarization switching and two-polarization mode dynamics [36–38]. Recently, first studies of CS behavior in optically injected broad-area VCSELs subjected to time-delayed optical feedback have appeared [29–31,39]. These studies elucidated the role of the strength and the phase of the time-delayed feedback for the creation of a drift bifurcation that causes the CSs to spontaneously move. For a saturable absorber VCSEL, a period-doubling route to temporal chaos of a single CS has been theoretically predicted for certain feedback parameters [40]. More recently, it has been shown that delayed feedback can induce pinning and depinning of cavity solitons when the resonator is illuminated by an inhomogeneous spatial Gaussian pumping beam [41]. It has to be noted that timedelayed feedback in spatially extended complex systems has a broader relevance than just laser physics and nonlinear optics.

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It concerns all fields of natural science [42], for instance, chemical reaction-diffusion systems [43–46].

Previously, oscillatory dynamics of CSs have been observed in systems without optical feedback: in the Lugiato-Lefever model [47] of a driven optical nonlinear cavity [48–50] and in a model of a VCSEL with a saturable absorber extended beyond the mean-field approximation [51]. Furthermore, a period-doubling route to chaos has been predicted for localized structures in the Lugiato-Lefever equation [52,53] in a forced and damped van der Pol model [54]. Spatiotemporal chaos has also been reported for the Lugiato-Lefever equation [55]. Experimentally, such oscillatory dynamics of localized structures have been observed in an optically pumped VCSEL with a saturable absorber [56]. More recently, two-dimensional dissipative optical rogue waves have been predicted to occur in VCSELs with [57] or without [58] delayed feedback.

In this paper, we carry out a detailed investigation of the bifurcation structure of time-delayed feedback-induced complex dynamics in a VCSEL with a saturable absorber. Using bifurcation analysis, direct numerical simulations, and numerical path-continuation methods, we show that the feedback impacts the homogeneous lasing solution and the localized CS solutions in a similar way, causing oscillatory dynamics with either a period-doubling or quasiperiodic route to chaos, as well as multistability of the stationary solutions. The paper is organized as follows: In Sec. II we introduce the mean-field model of a broad-area VCSEL with a saturable absorber and time-delayed feedback. In Sec. III we discuss its stationary spatially homogeneous and localized solutions with an emphasis on the impact of the feedback parameters on the branches of stationary solutions. In Sec. IV we reveal the underlying bifurcation structure and in Sec. VI we carry out direct numerical simulations to see the dynamical behavior under time-delayed feedback. In Sec. VI we perform numerical path-continuation calculations and map the bifurcation structure of the system in a plane of feedback parameters. Finally, we conclude in Sec. VII.

II. MODEL SYSTEM

We investigate a model system for a wide-aperture semiconductor laser, specifically a vertical-cavity surface-emitting laser (VCSEL) that consists of a gain section and a saturable absorber section sandwiched between two distributed Bragg reflectors (DBRs) that form the optical cavity. It is subjected to time-delayed optical feedback by forming an external cavity using a distant mirror to reflect the field. We adopt the Rosanov [59] and Lang-Kobayashi [60] approximation to model the delayed feedback. In this approximation, the feedback field is sufficiently attenuated and it can be modeled by a single delay term with a spatially homogeneous coefficient. In addition, we assume that the laser operates in a single-longitudinal mode. The system of dimensionless model equations reads [29–31,40]

$$\partial_t E = \left[(1 - i \alpha) N + (1 - i \beta) n - 1 + i \nabla_{\perp}^2 \right] E$$
$$+ \eta e^{i \varphi} E(t - \tau), \tag{1}$$

$$\partial_t N = b_1 [\mu - N (1 + |E|^2)],$$
(2)

$$\partial_t n = b_2 [-\gamma - n (1 + s |E|^2)],$$
(3)

where E = E(r,t), r = (x,y) is the slowly varying electromagnetic field envelope, and N = N(r,t) (n = n(r,t)) measures the state inversion of the carriers in the gain (absorber) section. Time is scaled to the photon lifetime and space is scaled to the diffraction length. Here, b_1 (b_2) is the ratio of the gain (absorber) carrier lifetime to the photon lifetime, μ is the gain current, and γ is the absorber voltage. Furthermore, α (β) is the linewidth enhancement factor of the gain (absorber) section and s is the ratio of the saturation intensities of the gain and absorber. Finally, η is the relative strength of the time delayed feedback, $\tau = 2L_{\text{ext}}/c$ is the delay time with c the speed of light and L_{ext} the external cavity length, and φ is a delay phase parameter that describes a phase shift on the time scale of the fundamental lasing frequency, due to, e.g., moving the mirror for a distance shorter than the wavelength.

Equations (1)–(3) have two types of homogeneous steadystate solutions. The trivial off solution reads [24]

$$E = 0, \quad N = \mu, \quad n = -\gamma. \tag{4}$$

It becomes unstable at the lasing threshold $\mu_{th} = 1 + \gamma$. From this point, a nontrivial branch of spatially homogeneous lasing solutions [so-called continuous waves (cw) solutions] emerges

$$E = |E|e^{i\omega t}, \quad N = \frac{\mu}{1+|E|^2}, \quad n = \frac{-\gamma}{1+s|E|^2},$$
 (5)

where ω is the frequency shift of the slowly varying field envelope in units of the photon lifetime.

Notice that for a stable lasing solution, the intensity must be large enough to overcome the saturable absorber which causes the cw branch to initially be unstable and lean towards lower gain. It then folds in a saddle-node bifurcation at

$$\mu_{\text{fold}} = \frac{\left(\sqrt{s-1} + \sqrt{\gamma}\right)^2}{s}.$$
 (6)

This gives rise to a regime below the lasing threshold where stable lasing solutions coexist with a stable off solution. Note that the time-delayed feedback can shift both bifurcation points.

The chosen parameter values for this article reflect an experimental setup of a VCSEL with a typical power output of the order of 1 mW at a pumping current of the order of 1 mA and a photon lifetime around 1 ps. Unless specified otherwise, they are

$$\alpha = 2, \ \beta = 0, \ b_1 = 0.04, \ b_2 = 0.02, \ s = 10,$$

 $\mu = 1.42, \ \gamma = 0.5, \ \tau = 100.$

III. STATIONARY SOLUTIONS

First we analyze the stationary solutions of the system (1)–(3) by purely analytical means and standard pathcontinuation techniques. Here, stationary means the field profile is constant in time while it rotates uniformly in the complex phase.

A. Continuous waves

The stationary cw solutions take the form

$$E = |E| e^{i(kx - \omega t)}, \quad \partial_t |E| = 0, \tag{7}$$

with the wave number k and the frequency shift ω and the corresponding carrier densities

$$N = \frac{\mu}{1+I}, \quad n = \frac{-\gamma}{1+sI},\tag{8}$$

with the field intensity $I = |E|^2$. For a stationary field amplitude, the delayed field is merely shifted in phase, $E(t - \tau) = e^{i\omega\tau}E(t)$. The model equations (1)–(3) then simplify to

$$0 = \frac{\mu (1 - i \alpha)}{1 + I} - \frac{\gamma (1 - i \beta)}{1 + s I} - 1 + i(\omega - k^2) + \eta e^{i(\omega \tau + \varphi)}.$$
 (9)

Separating the real and imaginary parts, we have

$$k^{2} = \omega - \frac{\alpha \mu}{1+I} + \frac{\beta \gamma}{1+sI} + \eta \sin(\omega \tau + \varphi), \quad (10a)$$

$$0 = \frac{\mu}{1+I} - \frac{\gamma}{1+sI} - 1 + \eta \cos(\omega \tau + \varphi).$$
 (10b)

With this, the solutions can be found graphically; see Fig. 1. The second equation yields all possible intensities I as a function of ω (cf. Fig. 1, upper line in blue). With I, we can calculate the right-hand side (rhs) of the first equation (inclined line in green). Exact solutions (black dots) are found wherever this line intersects with the lower flat line in red at the value of k^2 . For increasing values of η , additional solutions appear in a series of saddle-node bifurcations.

Note that linear stability analysis of the cw solutions show modulation instability for all wave numbers k for the investigated domain size.



FIG. 1. A graphical scheme to determine cw solutions of Eqs. (1)–(3). The intensity *I* as a function of the frequency shift ω is shown as the upper line in blue from solving Eq. (10b). The rhs of Eq. (10a) as a function of ω is shown as the inclined line in green and the lhs in red which is equal to zero in this example. Where these lines intersect, a solution exists, indicated with black dots. Increasing delay causes stronger oscillations in the rhs curve, inducing a series of saddle-node bifurcations as the lhs is crossed at additional points. The corresponding intensities can be read from the intensity curve. Here, $\varphi = 0$.

B. Cavity solitons

To find a branch of one-dimensional cavity solitons (CSs), we assume a stationary complex profile A(x) of the field envelope that rotates with a constant frequency shift ω like the cw solutions [23],

$$E(x,t) = A(x) e^{-i\omega t}.$$
(11)

This profile consists of an amplitude profile a(x) and a phase profile $\varphi(x)$,

$$A(x) = a(x)e^{i\varphi(x)},$$
(12a)

$$q = \partial_x \varphi, \tag{12b}$$

$$k = \frac{1}{a}\partial_x a, \qquad (12c)$$

$$f(|A|^2) = \frac{(1-i\alpha)\mu}{1+|A^2|} - \frac{(1-i\beta)\gamma}{1+s|A^2|} - 1.$$
 (12d)

Because the whole system is phase invariant, only the derivative of the phase profile is important, reducing the number of necessary variables to three. With this, we can write the system in the form

$$\partial_x a = ak,$$
 (13a)

$$\partial_x q = -2qk + \operatorname{Re}[f(a^2)] + \eta \cos(\omega \tau + \varphi),$$
 (13b)

$$\partial_x k = -\omega + q^2 - k^2 - \operatorname{Im}[f(a^2)] -\eta \sin(\omega\tau + \varphi), \qquad (13c)$$

following [61]. We can now treat it as a boundary value problem and apply standard path-continuation packages such as, e.g., AUTO-07P [62,63], to obtain a branch of stationary solutions.

C. Effective phase

To get an alternative view of the influence of the timedelayed feedback on the solution structure, we introduce an effective phase parameter ϑ [64],

$$\vartheta = (\omega \tau + \varphi) \operatorname{mod} 2\pi.$$
(14)

This yields the branches of localized solutions that have the same angle of interference with the delayed field. They form a tube-shaped manifold of all possible solutions of the system for a given delay strength. This view is not directly accessible experimentally. For any combination of τ and φ , one can also get the actual branches by solving Eq. (14) implicitly. In particular, it can be shown that the number of multistable solutions grows linearly with τ [64].

D. Solution structure

Figure 2 shows the obtained solution structures of cw solutions (cyan and blue with lower intensity) and CSs (magenta and red with higher intensity) for $\eta = 0.5\%$ and $\varphi = 0$ with μ as the control parameter. The left (right) panel shows the intensities (frequency shifts) of the corresponding solutions. For CSs, the intensity in the center is shown. Note



FIG. 2. Solution structure of the stationary solutions of Eqs. (1)–(3) for a feedback strength of $\eta = 0.5\%$. Left (right) panel shows the intensities $|E|^2$ (frequency shifts ω) as a function of the gain μ . Actual solutions for a fixed delay time $\tau = 100$ and delay phase $\varphi = 0$ are drawn in blue (red) for cw solutions (CSs) with the lower (higher) intensities. Both exhibit a similar snaking shape in both $|E|^2$ and ω due to a series of saddle-node bifurcations induced by time-delayed feedback. For reference, the central C-shaped curves in cyan (magenta) for cw solutions (CSs) show the solutions without delay. Changing φ moves the snaking curve periodically along the tube-shaped manifold of solutions, represented with dashed lines.

that both the cw solutions and CSs have a very similar solution structure.

The C-shaped curves represent the solution manifolds. Any point between the outer dashed C-shaped curves is a solution for appropriate delay parameters. The left (right) curves represent fully constructive (destructive) interference, i.e., $\vartheta = 0$ ($\vartheta = \pi$). The central solid curves represent the solution branch without delay. In the presence of delay, this curve effectively gets shifted in μ as the feedback either helps or hinders the field in the cavity. Aside from the shift, the changes to the curves are minimal.

The snaking curves show the actual solution branches for $\tau = 100$ and $\varphi = 0$. They form through a series of saddle-node bifurcations induced by the delayed feedback. Along the branches, the stability of the stationary solutions alternates. For the intensities, the positive (negative) slopes are stable (unstable), and vice versa for the corresponding frequency shifts. Note that a similar multistability effect was experimentally observed in a broad-area VCSEL with frequency-selective feedback [65].

An animation of Fig. 2 showing the effect of the feedback parameters on the solution structure is available in the Supplemental Material [66].

IV. PHASE BIFURCATION AND MULTISTABILITY

Due to translational and phase-shift symmetries of Eqs. (1)–(3), the point spectrum of the corresponding onedimensional linear eigenvalue problem has two zero eigenvalues corresponding to the even phase-shift neutral mode and the odd translational neutral mode. Note that in the limit of instantaneous medium response and for $\eta = 0$, the linear operator describing the stability of a CS solution of Eqs. (1)–(3) possesses an additional zero eigenvalue that corresponds to the



FIG. 3. A bifurcation diagram for the folds with the delay strength η and effective phase ϑ as control parameters. The solid blue line shows a branch of solutions that fulfill the condition (17) for a phase bifurcation. The dashed green (dotted red) line shows the fold continuation in AUTO-07P (DDE-BIFTOOL). The two continuations yield equivalent results. The folds can be attributed to a phase instability induced by the time-delayed feedback.

Galilean transformation symmetry,

$$E(x,t) \to E(x-vt,t)e^{ivx/2-iv^2t/4}$$
. (15)

However, the latter is typically broken for a nonvanishing delayed feedback term, leading to a shift of the corresponding real eigenvalue from the origin into the complex plane [61,64]. As a result, a stationary solution may lose its stability with respect to a drift bifurcation, giving rise to a CS moving with a constant velocity v. Drift or phase bifurcations can occur when the eigenvalue of the corresponding neutral mode ψ_0 becomes doubly degenerate with geometrical multiplicity one. There, the critical real eigenvalue passes through zero at the bifurcation point, so that the corresponding critical eigenfunction at this point is proportional to the neutral mode. This critical eigenvalue can either be a delay-induced branch of zero eigenvalue or correspond to a Galilean mode. In [64], a general expression for the onset of drift and phase bifurcations was derived,

$$\eta \tau = -\frac{\langle \boldsymbol{\psi}_0^{\top} | \boldsymbol{\psi}_0 \rangle}{\langle \boldsymbol{\psi}_0^{\top} | \boldsymbol{B} | \boldsymbol{\psi}_0 \rangle},\tag{16}$$

where ψ_0 is a neutral eigenfunction, ψ_0^{\dagger} is the corresponding adjoint eigenfunction, and *B* is the rotation matrix describing the phase shift due to the delay. Note that both drift- and phase-bifurcation thresholds tend to zero in the limit of large delays. While in the case of the drift bifurcation a pitchfork bifurcation takes place, the phase bifurcation corresponds to a saddle-node bifurcation where a pair of solutions merge and disappear. Note that this fold condition follows directly from Eq. (14) and can be written as

$$\frac{d\omega}{d\vartheta} = \frac{1}{\tau}.$$
(17)

Figure 3 shows the branch of stationary localized solutions satisfying the fold condition (17) (solid blue line) along with results from a fold continuation performed in AUTO-07P (dashed green line) and in DDE-BIFTOOL (dotted red line). One can see that all three calculations yield the same result. This demonstrates the equivalence of the continuations and identifies a phase bifurcation as the cause of the delay-induced multistability.



FIG. 4. Space-time plots of one-dimensional simulations of Eqs. (1)–(3) calculated for a delay time $\tau = 100$, delay phase $\varphi = 0$, and various delay strengths η . Without delay, the system equilibrates quickly to a stationary lasing localized structure. With increasing η , it becomes Hopf unstable and oscillates in intensity. Further increasing η causes the periodic orbit to undergo a series of period-doubling bifurcations that leads into chaos. The rightmost panel shows the chaotic behavior that is characterized by strong intensity spikes.

V. DIRECT NUMERICAL SIMULATIONS

In addition to the drift bifurcation leading to traveling CSs [23,64,67] and the phase bifurcation giving rise to the multistability of CSs' solutions, time-delayed feedback can also induce Andronov-Hopf bifurcations. Indeed, in [40] it has been shown that the inclusion of the feedback term leads to the formation of breathing CSs with a period-doubling route to chaos. In order to analyze transitions between different oscillating solutions of a single CS, one-dimensional direct numerical simulations of the system have been performed using the classical Runge-Kutta method on an equidistant mesh combined with a pseudospectral method for spatial derivatives. Note that interpolation of the delay term to reach the same order of convergence as the time stepping scheme is not needed in this case because the delay strength is small. Figure 4 shows exemplary space-time plots of the field intensity for different values of η at $\varphi = 0.5 \pi$. Without delayed feedback, the system forms localized lasing structures with a steady intensity profile. Increasing the feedback causes an Andronov-Hopf bifurcation and the intensity continuously oscillates. These oscillations undergo a period-doubling bifurcation leading to chaos, as demonstrated in [40]. In the right panel, one can see irregular oscillations with strong spikes of intensity that are otherwise never achieved in this system. This identifies the behavior as chaotic in distinction to quasiperiodic oscillations that would have a clearly limited interval of intensity. Indeed, for certain delay parameters, there is also a torus bifurcation which, however, is not explicitly represented in this figure.

To characterize the different kinds of temporal behavior, we trace the extrema of the intensity field in time. For CSs, the intensity value in the center was traced. Plotting these extrema after the system has reached a final state



FIG. 5. Bifurcation diagram with the delay strength η as the control parameter calculated for (a) $\varphi = 0$ and (b) $\varphi = 1.48\pi$. In direct numerical simulations, a time series is analyzed after the system has had sufficient time to settle. The intensity extrema of the time series are shown as red (blue) dots for the case of the spatially (non)extended system. For increasing η , we see either a period-doubling or quasiperiodic route to chaos. The bifurcation points move with changing delay phase φ . In a range of φ , there exist two separate windows with a route to chaos. These are actually connected through φ , i.e., the window appears below the first saddle-node bifurcation on the lower branch and later moves off to the right for increasing φ . Due to the periodicity of φ , a new window appears before the other one vanishes.

of operation yields a bifurcation diagram with η as the control parameter. Figure 5(a) shows an exemplary bifurcation diagram for $\varphi = 0$. Note that several windows of stationary behavior and chaotic dynamics can be observed. The location of these windows moves with the feedback phase φ and can be associated with the aforementioned delay-induced multistability of the stationary CS solution. In particular, at $\varphi \approx \pi$ the period-doubling route of the lower branch starts to appear in coexistence with the upper branch. For increasing φ , the bifurcation points move towards lower η . At the saddle-node bifurcation at $\varphi \approx 1.5 \pi$, they change direction and continue towards higher η —now as the upper branch. Here, the overlap of the two period-doubling routes is most pronounced [see Fig. 5(b), where an exemplary bifurcation diagram for $\varphi = 1.48 \pi$ is presented]. An animation showing the dynamics of the bifurcation diagram as one changes the feedback phase is available in the Supplemental Material [66].



FIG. 6. The left panel shows a bifurcation diagram obtained in DDE-BIFTOOL with the delay phase φ as the control parameter. Stationary solutions are drawn as thin lines in solid black for stable, dotted red for the unstable connection between the folds, and dashed blue for Andronov-Hopf instability. The periodic orbits connecting the Andronov-Hopf bifurcations are represented as thick lines in solid black for stable, dashed blue for period doubling, dotted red for cyclic fold and dash-dotted magenta for torus bifurcation. Corresponding data from direct numerical simulations is plotted as small cyan dots for comparison. The simulations are in agreement with the continuation. The right panels show the Floquet multipliers corresponding to the unstable periodic orbits. Values outside the unit circle indicate the respective instabilities, i.e., real and larger than one means cyclic fold, real and smaller than minus one means period doubling, and a complex pair with an absolute value larger than one means torus bifurcation.

Since the cw solutions are unstable to spatial modulation, they are not attainable in the same simulations as the CSs. We can, however, obtain their intensity in a spatially nonextended model, i.e., omitting the diffraction in the transverse plane. The resulting bifurcation diagram for the homogeneous lasing solution resembles the diagram for CSs in appearance. In particular, Andronov-Hopf bifurcations as well as saddle-node bifurcations of CSs and of the homogeneous lasing solutions occur at similar values of η . Therefore, the dynamical behavior of the homogeneous lasing solutions can be first studied in detail using, e.g., standard path-continuation tools for delay differential equations [68] as this analysis is much simpler than the complete analysis of the spatially distributed problem.

VI. DELAY CONTINUATION

We use DDE-BIFTOOL with the extensions for periodic orbits and rotational symmetry to analyze the dynamic solutions of the system. DDE-BIFTOOL [68] is a path-continuation toolbox for delay differential equations (DDEs) in MATLAB. Since DDE-BIFTOOL is designed to continuate delay differential equations, Eqs. (1)–(3) can be approximated by a set of coupled delay differential equations. However, the underlying algorithms' execution times scale badly with the system dimension. In particular, calculations for a single equation in space have shown an effective limit to spatial resolution of 64 mesh points on contemporary desktop hardware [41]. For the 4d system of interest, we estimate a limit of only 16 mesh points in space, which is hardly sufficient. We therefore look only at the cw case since CSs are expected to behave similarly as was demonstrated before.

Figure 6 shows the results of the analysis with DDE-BIFTOOL for $\eta = 1\%$ on the full interval of φ in the left panel. The stationary solution is stable (thin solid black line) only for a small interval of φ , while most of it is Hopf unstable (thin dashed blue line). Between the two folds, the branch is unstable (thin dotted red line). The periodic orbits are represented by thick lines (solid black when stable) through the extrema of their intensity profiles in time. They connect the Andronov-Hopf bifurcations. On two separate intervals, they are unstable due to period doubling (dashed blue line) and a torus bifurcation (dash-dotted magenta line), respectively. There is also a cyclic fold before the torus bifurcation, with the unstable part shown as a dotted red line. For comparison, the results from direct numerical simulations are shown as cyan dots. Both the continuation and the simulations are in good agreement.

The right panels of Fig. 6 show the respective Floquet multipliers of representative unstable periodic orbits. In the upper panel, the torus bifurcation is identified by a complex pair of Floquet multipliers outside the unit circle. In the middle panel, the cyclic fold is identified by a real Floquet multiplier larger than one. In the lower panel, the period doubling is identified by a real Floquet multiplier smaller than minus one.

Finally, Fig. 7 shows the full bifurcation diagram for cw stationary solutions and their periodic orbits in the (φ, η) plane. For the stationary solutions, the Andronov-Hopf bifurcation is shown as a thin dashed red line and the saddle-node bifurcation as a thin solid green line. For the periodic orbits, the period-doubling bifurcation is shown as a thick dashed blue line, the cyclic folds as a thick solid cyan line, and the torus bifurcation as a thick dotted black line. The torus bifurcation branch connects the cyclic fold with the crossing point of the



FIG. 7. Bifurcation diagram for cw solutions obtained with DDE-BIFTOOL with the feedback strength η and feedback phase φ as control parameters. The folds (Hopf thresholds) of the stationary solutions are drawn as a thin solid green line (dashed red line). The cyclic folds of the periodic orbits are shown as a thick solid cyan line and the period-doubling (torus) thresholds as a thick dashed blue (dotted black) line. Areas are colored corresponding to the various instabilities present with the torus region being hatched.

stationary fold with the Andronov-Hopf bifurcation. The colored areas represent the respective combination of delayinduced instabilities with the torus region being hatched. One can see that increasing η leads to complex spatiotemporal behavior of the homogeneous lasing solution, including multistability and coexistence of stationary states with periodic and aperiodic dynamics.

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VII. CONCLUSION

To conclude, we carried out a detailed investigation of the bifurcation structure of time-delayed feedback-induced complex dynamics in a broad-area VCSEL with a saturable absorber. Using bifurcation analysis and direct numerical simulations, we have shown that the feedback impacts the homogeneous lasing solution and the localized CS solutions in a similar way, causing multistability of the stationary solutions as well as oscillatory dynamics with either a period-doubling or quasiperiodic route to chaos. We have demonstrated that this multistability is caused by a feedback-induced phase bifurcation of the stationary solution. The threshold of the phase bifurcation was obtained by a combination of analytical and numerical path-continuation methods. The similarity between the bifurcation scenarios of the lasing homogeneous solutions and the CS solutions allows us to perform a complete mapping of the saddle-node, Andronov-Hopf, period-doubling, secondary Hopf (torus), and cyclic fold of periodic orbits' bifurcations in a plane of the feedback parameters, namely, the phase and strength of the feedback.

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