Theory of the Lamb shift in muonic tritium and the muonic ^{3}He ion

Savely G. Karshenboim*
Max-Planck-Institut für Quantenoptik, Garching 85748, Germany
and Pulkovo Observatory, St. Petersburg 196140, Russia

Evgeny Yu. Korzinin and Valery A. Shelyuto
D. I. Mendeleev Institute for Metrology, St. Petersburg 190005, Russia

Vladimir G. Ivanov
Pulkovo Observatory, St. Petersburg 196140, Russia
(Received 2 December 2016; published 3 August 2017)

Due to recent experimental efforts on light muonic atoms, we consider the quantum electrodynamics (QED) theory of two-body muonic atoms, namely, of those with $A = 3$. Here we present the results for the Lamb shift and fine structure up to the order $\alpha^5 m$ for the $n = 2$ levels, which in particular include (1) pure QED contributions, (2) the coefficient to the r_N^2 term with QED corrections (where r_N is the rms nuclear charge radius), and (3) the general expressions for the nuclear-structure contributions consistent with the presented QED theory. We revisit theory for the muonic helium-3 ion by rechecking all the relevant theoretical contributions and develop a theory of muonic tritium. We also reestimate the uncertainty of the nuclear-structure contribution.

DOI: 10.1103/PhysRevA.96.022505

I. INTRODUCTION

Since recently the CREMA Collaboration has started a program of the experiments on laser spectroscopy of the $n = 2$ energy levels in light muonic atoms. The first result has been achieved on muonic hydrogen [1] and the measurement on muonic deuterium followed [2], while the evaluation of the data on muonic-helium ions is expected [3]. While the accurate theory of ordinary hydrogenlike atoms is well established (see, e.g., [4,5]), the theory for the muonic atoms (bound systems of a nucleus and a muon) requires a special consideration. Since the muon is orbiting around the nucleus at an essentially lower orbit than that of the electron in an ordinary hydrogenlike atom, the theory of the energy levels in a two-body muonic atom differs from that for an ordinary atom.

The various intervals for $n = 2$ involve the hyperfine structure (HFS) (the nuclear-spin-dependent part of the spectrum) and the Lamb shift (the nuclear-spin-independent part of the spectrum). Their theory is quite different, and here we focus on the theory of the Lamb shift in muonic tritium and helium-3. The Lamb shift is basically the effect on the s states (we discuss that more accurately below). In the meanwhile, the experiments are targeting various $2s - 2p$ transitions. The measured transitions in the case of muonic hydrogen are shown in Fig. 1, and the same transitions are expected for muonic helium [3]. (Still, there may be some additional transition(s) measured as well.)

With the results of two transitions in hand one can extract the "experimental" values of the $2s$ HFS interval and of the $2s - 2p$ Lamb shift. To make such a separation, a complete theory of the $2p$ levels is also required.

We consider in this paper the Lamb shift in light muonic atoms with the nuclear spin $1/2$, while the case of the bosonic nucleus will be considered elsewhere. The muonic hydrogen theory has been revisited by us in [6] (where references to the earlier original works and reviews on muonic hydrogen can be found). Here we consider the muonic tritium atom and the muonic helium-3 ion. Both muonic atoms have the same nuclear spin as the muonic hydrogen and therefore their level structure (see Fig. 2) and certain QED and nuclear-structure effects are described by the similar equations. We follow here the logic of our review on muonic hydrogen [6]. Along with the results on muonic tritium and helium-3 ion we present also the results on the muonic hydrogen for comparison. The Lamb shift is considered in the main body of the paper, while the required theory of the $2p$ intervals is given in the Appendix.

The theoretical calculations require the input data, which are the parameters that describe the nuclei and the muon. The numerical values of the most important constants we have used are $R_\infty = 13\,605.693\,009(84)$ meV and $m_\mu/m_e = 206.768\,282\,6(46)$ [4]. The most important atomic parameters are summarized in Table I. Here, R_N stands for the rms charge radius of a nucleus, while r_N is its numerical value in fermi; m_r is the reduced mass of the muon, M is the nuclear mass. The relativistic units in which $\hbar = c = 1$ are applied throughout the paper.

In our calculations we have to deal with the nuclear magnetic moment and the related g factor. We need the value of the g factor of the nucleus in the terms of particle physics. (We denote it η to avoid any confusion with the g factor in nuclear physics, which differs.) In particle physics the g factor is defined as

$$\mu_x = \eta_x \frac{Z_x e}{2m_x} \mathbf{s}_x, \tag{1}$$

i.e., it expresses the magnetic moment in terms of the Bohr magneton for the x particle with mass m_x, charge Z_x, etc. It is convenient to split it into the Dirac part and the anomalous magnetic moment κ_N:

$$\eta = 2(1 + \kappa_N). \tag{2}$$

*savely.karshenboim@mpq.mpg.de

 ©2017 American Physical Society

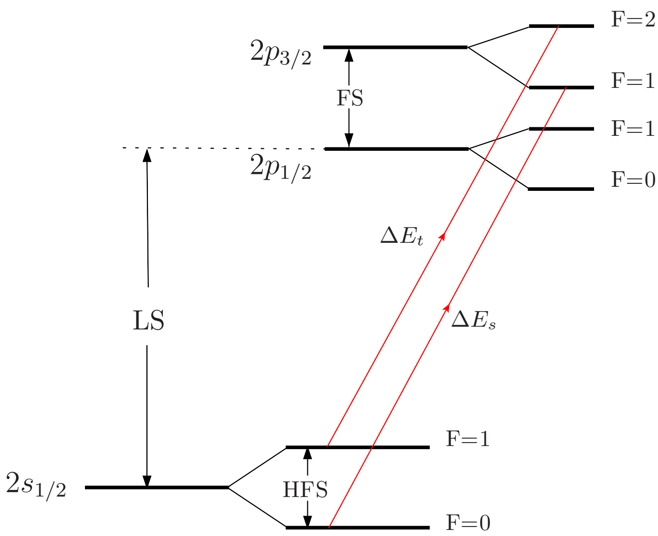

FIG. 1. Transitions measured in muonic hydrogen within the CREMA experiment at PSI (following [1]), not to scale.

We may also need some rough values of the nuclear radii to estimate the importance of various corrections, proportional to r_N^2. We put into Table I the results from scattering, evaluated in [7]. In our estimation we use round values: 0.84 fm for the proton (which is consistent with muonic hydrogen value [1]), 1.7 fm for the triton, and 2.0 fm for the helion.

The relevant contributions for the muonic helium-3 ion have been in principle already known (see the related references while we discuss particular terms), while they are unknown for muonic tritium. We systematically recheck all the QED contributions for muonic helium-3 and calculate the missing terms. Most of the muonic tritium results are calculated here.

II. OVERVIEW OF QED THEORY AND THE LEVEL STRUCTURE

The level structure of the muonic tritium and muonic helium-3 ion (see Fig. 2) is very similar to that in muonic hydrogen (cf. [6,10,11]). The only difference is that while the magnetic moment of the proton or triton is directed along its spin, the magnetic moment of the helion, the nucleus of the helium-3 atom, has the opposite direction. That changes the sign of the hyperfine contribution and reverses the position of the hyperfine components. In particular, while in hydrogen and tritium the $2s$ singlet lies below the triplet, in the helium-3 ion the singlet is above the triplet.

The labels LS*, FS*, and HFS* roughly describe the Lamb shift, and the fine and hyperfine-structure intervals. There is a small additional term Δ which may be included in those intervals differently, and we present our definitions below along with a brief overview of QED theory.

In particular, we define the Lamb-shift, and the fine- and hyperfine-structure intervals as

$$\begin{aligned}
\Delta E_{\rm HFS}(2p_{3/2}) &\equiv \Delta E(2p_{3/2}(F=2) - 2p_{3/2}(F=1)),\\
\Delta E_{\rm HFS}(2p_{1/2}) &\equiv \Delta E(2p_{1/2}(F=1) - 2p_{1/2}(F=0)),\\
\Delta E(2p_{3/2}) &\equiv \tfrac{5}{8}\Delta E(2p_{3/2}(F=2))\\
&\quad + \tfrac{3}{8}\Delta E(2p_{3/2}(F=1)),\\
\Delta E(2p_{1/2}) &\equiv \tfrac{3}{4}\Delta E(2p_{1/2}(F=1))\\
&\quad + \tfrac{1}{4}\Delta E(2p_{1/2}(F=0)),\\
\Delta E_{\rm FS}(2p) &\equiv \Delta E(2p_{3/2}) - \Delta E(2p_{1/2}),\\
\Delta E_{\rm HFS}(2s) &\equiv \Delta E(2s_{1/2}(F=1) - 2s_{1/2}(F=0)),\\
\Delta E(2s) &\equiv \tfrac{3}{4}\Delta E(2s_{1/2}(F=1))\\
&\quad + \tfrac{1}{4}\Delta E(2s_{1/2}(F=0)),\\
\Delta E_{\rm L}(2p_{1/2} - 2s) &\equiv \Delta E(2p_{1/2}) - \Delta E(2s).
\end{aligned} \tag{3}$$

The values of all the actual $n = 2$ energy levels can be therefore presented in the terms of the Lamb-shift interval $\Delta E_{\rm L}(2p_{1/2} - 2s)$, fine-structure interval $\Delta E_{\rm FS}(2p)$, and the HFS intervals $\Delta E_{\rm HFS}(2s)$, $\Delta E_{\rm HFS}(2p_{1/2})$, and $\Delta E_{\rm HFS}(2p_{3/2})$, and the value of the $2s$ binding energy $\Delta E(2s)$.

In some papers a separate term, named Δ, is introduced and the actual energy levels are presented in the terms of

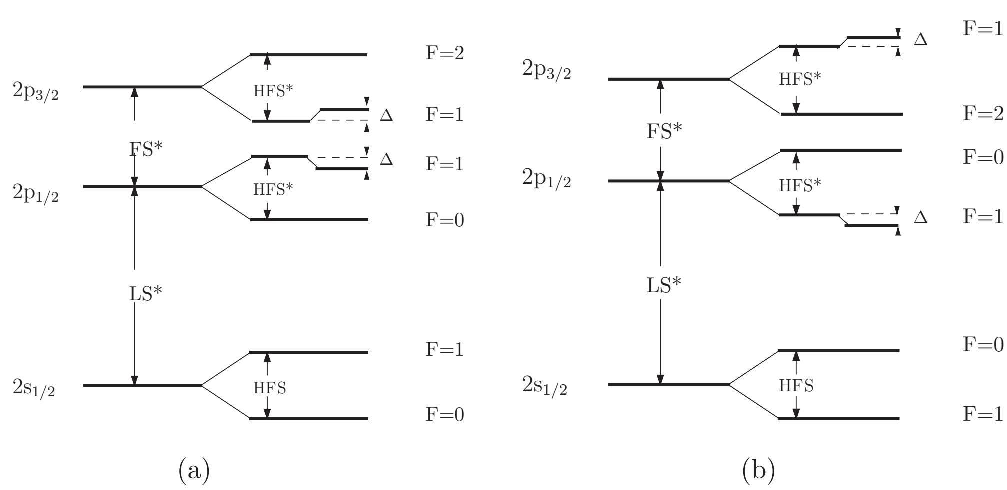

FIG. 2. Level structure at $n = 2$ in muonic tritium (a) and the helium-3 ion (b) (cf. [6,10,11]), not to scale. Note, the magnetic moment of the helion, the nucleus of the He-3 atom, is negative, and the HFS structure is "reversed." The states with a higher value of total angular momentum F are below the states with the smaller F. Here, LS*, FS*, and HFS* are for the solution of the Schrödinger-Coulomb equation and for the calculation of the diagonal terms of the relativistic perturbation. See this section, Sec. III A, and the Appendix for more explanations.

TABLE I. Parameters of light muonic atoms with the nuclear spin 1/2 [4]: charge Z and mass M of the nucleus, the reduced muon mass m_r, the characteristic atomic momentum $\overline{p}$ for the $n = 2$ energy levels, the rms nuclear charge radii from scattering [7], a value of which can be useful for rough preliminary estimations, the nuclear g factor in the terms of particle physics [see Eq. (1)] and the related anomalous magnetic moment κ_N, and the minimal nuclear excitation energy ΔE_N [8,9].

	μH	μT	μ^3He
Z	1	1	2
M [u]	1.007 276	3.015 501	3.014 932
m_r [u]	0.101 948 55	0.109 316 94	0.109 316 19
$\overline{p} = Z\alpha m_r/2$ [MeV/c]	0.346 495	0.371 538	0.378 395
m_μ/M	0.112 61	0.037 615	0.037 6224
r_N	0.895(18)	1.744(87)	1.959(34)
η_N	5.585 69	17.8363	−6.368 31
κ_N	1.792 85	7.918 17	−4.184 15
ΔE_N [MeV]	134.98	6.257	5.493

the six intervals mentioned above (but defined somewhat differently—cf. LS*, FS*, and HFS* in Fig. 2) and the "mixing term" Δ. We briefly discuss this term below in this section.

In classification of the QED contributions we completely follow our review on muonic hydrogen [6] and we give the muonic hydrogen results for comparison. First of all, we note that the muon-to-nucleus mass ratio is not so small as in ordinary atoms and we have to really deal with a two-body problem. For most of the contributions we follow the effective Dirac equation (EDE) approach, while for some we use the Breit-type equation approach (see [6] for detail). Both approaches present an efficient way to take into account two-body effects for the relativistic contributions.

The result for the binding energy of the solution of the EDE for the nl_j states is of the form [12,13]

$$E = m_r(f_D - 1) - \frac{m_r^2}{2(M+m)}(f_D - 1)^2, \tag{4}$$

where we introduce the dimensionless Dirac energy, which for the $n = 2$ states is

$$f_D(Z\alpha;2s) = f_D(Z\alpha;2p_{1/2}) = \sqrt{\frac{1+\sqrt{1-(Z\alpha)^2}}{2}},$$
$$f_D(Z\alpha;2p_{3/2}) = \sqrt{1-\frac{(Z\alpha)^2}{4}}. \tag{5}$$

Since the energy of the solutions of EDE is expressed in the terms of the Dirac energy only, with the $2s_{1/2}$ and $2p_{1/2}$ levels having the same energy, the leading EDE energy (4) does not contribute to the Lamb shift. That does not cover two important corrections due to the Barker-Glover and Brodsky-Parsons terms:

BG Barker and Glover [14] obtained a complete contribution in order $(Z\alpha)^4m$ exactly in m/M on the base of the Breit-type equation approach. It happens that this contribution splits the $2s_{1/2}$ and $2p_{1/2}$ energy levels in order $(Z\alpha)^4(m/M)^2m$. (The related correction is a small, but observable contribution to the Lamb shift of ordinary and muonic hydrogen.)

BP The Brodsky-Parsons Dirac equation [both ordinary and within EDE, cf. Eq. (4)] does not take into account the HFS interaction. Once we take it into account, we note that the matrix element between the $2p_{1/2}(F=1)$ and $2p_{3/2}(F=1)$ states does not vanish. In the meantime, the difference in their energy is very small. That produces an enhanced second-order perturbation-theory term (with the HFS interaction as the perturbation). The term is denoted as Δ in Fig. 2 and it is of order $(Z\alpha)^4(m/M)^2m$.

There is also a standard nonrelativistic explanation of the term [10,11]. While starting with the Breit-type equation approach, the wave functions are nonrelativistic ones and the perturbation includes in particular the diagonal nonrecoil relativistic contributions (which are dominant for the fine structure) and the diagonal HFS contributions (which are dominant for the HFS interval). There are also off-diagonal contributions due to the nonvanishing matrix element between the $2p_{1/2}(F=1)$ and $2p_{3/2}(F=1)$ states. Starting with the nonrelativistic wave functions, the $2p_{1/2}(F=1)$ and $2p_{3/2}(F=1)$ states are degenerate and in such a case the standard perturbation theory suggests the rediagonalization procedure, which "mixes" the $2p_{1/2}(F=1)$ and $2p_{3/2}(F=1)$ (see Appendix for more explanation). Indeed, the result is the same as within the relativistic approach, based on the Dirac equation.

In both approaches there are higher-order corrections of this type (see, e.g., [6] for detail).

We study all those corrections in Sec. III A.

Once we solve the unperturbed Coulomb problem and, if necessary, add the quantum-mechanical corrections, we turn to a calculation of various QED effects. Quantum electrodynamics in the terms of the Lagrangian and Feynman rules is very similar for ordinary and muonic atoms. However, the solution of the Coulomb bound problem has very different scales and because of that the relative importance of various QED effects in ordinary and muonic atoms is very different.

The specific part of the muonic-atom QED theory includes the contributions with closed electron loops [see, e.g., Fig. 3 for $\alpha(Z\alpha)^2m$ and Fig. 4 for $\alpha^2(Z\alpha)^2m$ contributions]. The characteristic atomic momentum $Z\alpha m_r/2$ (see Table I) in light muonic atoms (for $n = 2$) is comparable with the electron mass m_e, which enhances the contributions. In the ordinary atoms the vacuum polarization is responsible for a small part of the Lamb shift, while in muonic atoms it dominates.

These corrections are considered in detail in Sec. III A. They include the electronic vacuum polarization (eVP) contributions

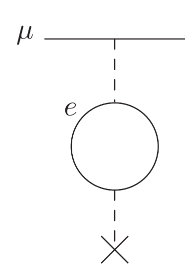

FIG. 3. The leading contribution to the Lamb shift in muonic hydrogen (eVP1). The related potential is known as the Uehling potential.

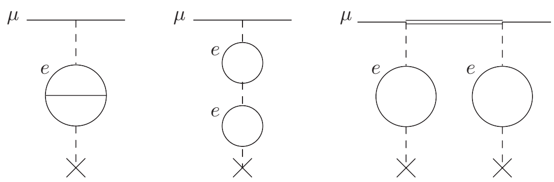

FIG. 4. Characteristic diagrams for the second-order eVP contributions (eVP2). The first diagram denotes the complete irreducible two-loop vacuum polarization contribution. The potential related to the first two diagrams is often referred to as the Källen-Sabry potential. The double line in the third graph is for the reduced Coulomb Green's function of the orbiting muon, which is to appear in the second order of the nonrelativistic perturbation theory. A number of suitable presentations for that nonrelativistic Coulomb Green's function in coordinate space are available (see, e.g., [15–18]).

up to the third order, as well as the contribution of the virtual scattering of light by light.

The muon-specific contributions dominate, but still there are many pieces of the QED theory which are basically the same as in ordinary atoms. Those are the contributions without the closed fermion loops or with the muon vacuum polarization. They can easily be "rescaled" from the conventional theory (see, e.g., [5]).

The remaining part of the theory is due to the nuclear-structure contributions. The paper is basically on the QED theory. However, the purpose of the study of the light muonic atoms is to find the rms nuclear charge radius R_N. In order to do that we present the result for the observable transitions as

$$\text{point like numerical value} + \text{coefficient} \times R_N^2 + \text{some nonpointlike corrections}.$$

To find an accurate value of the radius we have in particular to find QED corrections to the "coefficient" in the expression above. We have also to take care that the "pointlike" and "nonpointlike" physics are defined consistently.

III. QED THEORY OF THE MUONIC ATOMS WITH $A = 3$

A. Effective Dirac equation and the "unperturbed" energy levels

As we already mentioned, using the effective Dirac equation (see the previous section), we express the energy in the terms of the Dirac equation (with the reduced mass). The Dirac equation leaves the $2s_{1/2}$ and $2p_{1/2}$ states degenerate. The contributions to the $2s_{1/2} - 2p_{1/2}$ appear only through the terms beyond (4). Those are the BG and BP terms. Both affect only the $2p$ states. The former is of the form [14]

$$\Delta E_{\rm BG}(nl) = \frac{(Z\alpha)^4 m_r^3}{2n^3 M^2}\left(\frac{1}{j+1/2} - \frac{2}{3}\right)(1-\delta_{l0}). \quad (6)$$

The numerical results are the quantum-mechanical two-body contributions to the energy. They are summarized for the Lamb-shift interval in Table II. The term referred to here as the BP term includes higher-order corrections, which are considered in detail in the Appendix (cf. [6]).

The most important contributions are given in the table with the bold letters. Those in muonic hydrogen are relevant for the so-called proton radius puzzle (see, e.g., [1,2,4,6]). We use the bold italic for the Brodsky-Parsons term to stress that they are relevant for the puzzle and they are defined differently in different papers. We include the BP contributions as a part of the Lamb shift and fine structure. The alternative way of the definition is to completely exclude them from LS and FS and to keep Δ as a separate term.

B. Electronic vacuum polarization and light-by-light scattering contributions

The leading contribution to the Lamb shift in muonic atoms (the so-called Uehling-potential term, see Fig. 3) is very different from the leading contribution to the Lamb shift in ordinary hydrogen. It is of the order $\alpha(Z\alpha)^2 m$, which makes the Lamb shift (i.e., the $2s - 2p$ interval) larger than the fine-structure $2p_{3/2} - 2p_{1/2}$ interval. [See Figs. 1 and 2 for the level structure in muonic hydrogen, tritium, and helium-3 ion. The fine-structure interval is of the order $(Z\alpha)^4 m$.]

The complete account of muon-specific contributions to the Lamb shift in muonic hydrogen, tritium, and helium-3 ion is presented in Table III. Some contributions are rather trivial, since the analytic expressions have been known for arbitrary atoms, such as the leading Uehling term and the related relativistic correction (see Fig. 3).

The relativistic-recoil correction to the Uehling-potential contribution (no. 1.3 in Table III) was calculated for a number of occasions; however, initially a number of the earlier results were either incorrect or incomplete (see [6,19] and the references therein). The correct result was obtained in [20] and [19] for light muonic atoms, including muonic hydrogen and the muonic helium-3 ion. The related result for muonic tritium is found here following [19].

The two-loop electronic vacuum polarization (term no. 2 in Table III, see Fig. 4) has been studied in [11,21] in the

TABLE II. Contributions to the "unperturbed" energy levels for the Lamb-shift interval $\Delta E(2p_{1/2} - 2s_{1/2})$ in muonic tritium and the muonic helium-3 ion. The corrections marked with an asterisk (*) are exact in m/M. The order shown is the leading order in m/M. Such a notation is used for all the tables through the paper. $(Z\alpha)^{4+}$ stands for $(Z\alpha)^4$ and all higher-order terms in $(Z\alpha)$. * Here we present the complete BP term (given with the bold italic font), not only its leading term (see Appendix for details). The most important contributions are given with the bold font.

No.	Designation	Order	Refs.	$\Delta E(\mu$H) [meV]	$\Delta E(\mu$T) [meV]	$\Delta E(\mu^3$He) [meV]
0.1	Rel	$(Z\alpha)^{4+}m$		0	0	0
0.2	Rel-Rec*	$(Z\alpha)^4 m^2/M$		0	0	0
0.3	BG*	$(Z\alpha)^4(m/M)^2 m$	[14]	**0.057 47**	**0.007 91**	**0.1265**
0.4	BP*	$(Z\alpha)^4(m/M)^2 m$	Table XIV	***−0.108 42***	***−0.163 89***	***−0.1298***

TABLE III. Specific muonic-atom contributions to the Lamb-shift interval $\Delta E(2p_{1/2} - 2s_{1/2})$ in muonic tritium and the muonic helium-3 ion due to closed e-loops. ⋆The LbL contribution is a combination of terms with a different Z dependence (cf. Table VI). We follow the notation of [6].

No.	Designation	Order	$\Delta E(\mu\mathrm{H})$ [meV]	$\Delta E(\mu\mathrm{T})$ [meV]	$\Delta E(\mu^3\mathrm{He})$ [meV]
1.1	eVP1 (NR)*	$\alpha(Z\alpha)^2 m$	**205.007 37**	**235.911 10**	**1641.8860**
1.2	eVP1 (Rel)	$\alpha(Z\alpha)^4 m$	0.020 84	0.023 43	0.5257
1.3	eVP1 (Rel-Rec)*	$\alpha(Z\alpha)^4 \frac{m^2}{M}$	−0.002 08	−0.000 885	−0.0163
2	eVP2 (NR)*	$\alpha^2(Z\alpha)^2 m$	**1.658 85**	**1.903 51**	**13.0843**
3	eVP3 (NR)*	$\alpha^3(Z\alpha)^2 m$	0.007 52	0.008 74(9)	0.073(3)
4	LbL*⋆	$\alpha^5 m$	−0.000 89(2)	−0.000 99(1)	−0.013 4(6)
5	eVP+SE	$\alpha^2(Z\alpha)^4 m$	−0.002 54	−0.0031(3)	−0.0627
6	SE[eVP]	$\alpha^2(Z\alpha)^4 m$	−0.001 52	−0.001 87	−0.0299

nonrelativistic approximation for muonic hydrogen (cf. [22]). The relativistic corrections were considered in Refs. [23] and [24]. The nonrelativistic muonic helium result, which is only relevant for $\alpha^5 m$ theory, is taken from [23]. The muonic tritium result is found here following [23,24].

The third-order vacuum polarization contributions (term 3 in Table III, see Fig. 5) were studied for muonic hydrogen in Refs. [25–27]. The adjustment of the μH result to the Lamb shift in light muonic atoms was done in [28]; however, the list of those atoms included neither muonic helium-3, nor muonic tritium. Following [28], the result for the former was previously given by us in [23], the details of which are presented here in Table IV. We have also generalized the results [28] for the muonic tritium atom, which are summarized in Table V.

The contributions, which include the insertion of the virtual scattering of light by light (term 4 in Table III, see Fig. 6), have been studied for muonic hydrogen for a while (see, [6,28,29] and references therein). The complete results for all three types of diagrams in Fig. 6 were obtained in [28,29] for a few light muonic atoms, including muonic hydrogen and excluding the $A = 3$ atoms. Following [28,29], the result for the muonic helium-3 ion was previously given by us in [23] (see Table VI here for individual contributions) and for muonic tritium in this paper (see Table VI for the contributions of the individual diagrams).

The combined contributions of the self-energy and electron vacuum polarization (see Fig. 7) have order $\alpha^6 m$, but they are enhanced by a large logarithm. Term 5 in Table III includes the Uehling correction to the muon self-energy and was calculated for all the muonic atoms in the table in [30]. The electron vacuum polarization insertion into the self-energy was studied in [5,23,31] for some light muonic atoms except for muonic tritium. Here we adjust the result for the latter following [5,31].

TABLE IV. The individual eVP3 contribution to the $\Delta E(2p_{1/2} - 2s_{1/2})$ interval in the muonic helium-3 (see Fig. 5). The notation follows [25].

No.	$E(2s)$ $[\frac{\alpha^3}{\pi^3}(Z\alpha)^2 m_r]$	$E(2p)$ $[\frac{\alpha^3}{\pi^3}(Z\alpha)^2 m_r]$	$E(2p-2s)$ $[\frac{\alpha^3}{\pi^3}(Z\alpha)^2 m_r]$
c3	−0.032 80	−0.039 04	−0.006 24
c111	−0.029 90	0.001 64	0.031 54
c2(1)			0.0195(20)
c12	−0.128 35	0.003 63	0.132 00
c1c2	−0.061 79	−0.011 20	0.050 59
c1c11	−0.038 39	−0.000 06	0.038 33
c1c1c1	−0.002 47	−0.000 35	0.002 12

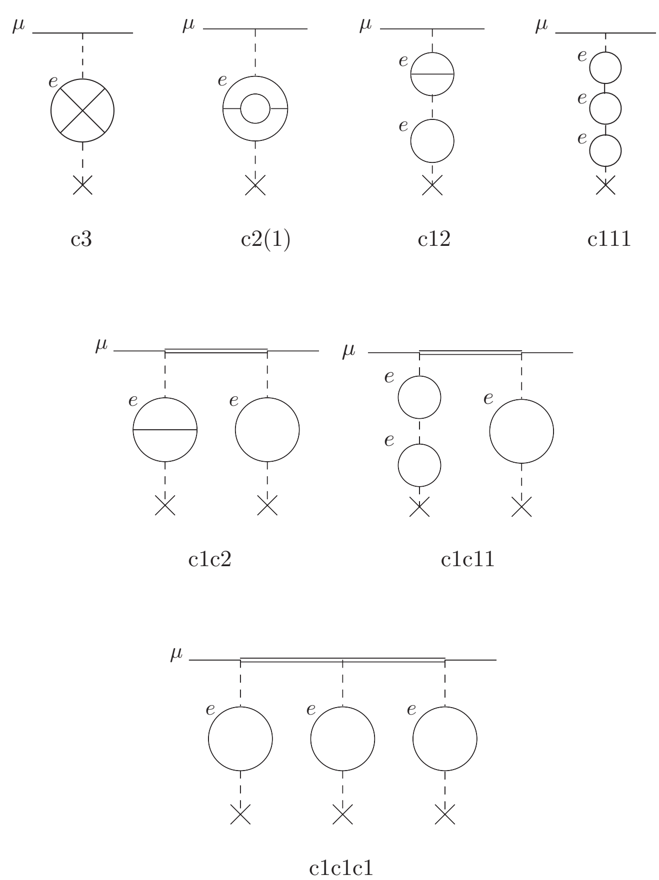

FIG. 5. Characteristic vacuum polarization contributions in order $\alpha^3(Z\alpha)^2 m$ (eVP3). The first graph is for the complete irreducible eVP of the third order without any internal eVP loops, while the second is for the complete irreducible contribution with an internal eVP. The other types of contributions are either due to the reducible part of the three-loop eVP or for iterations of the Uehling and Källen-Sabry potentials.

TABLE V. The individual eVP3 contribution to the $\Delta E(2p_{1/2} - 2s_{1/2})$ interval in muonic tritium (see Fig. 5). The notation follows [25].

No.	$E(2s)$ $[\frac{\alpha^3}{\pi^3}(Z\alpha)^2 m_r]$	$E(2p)$ $[\frac{\alpha^3}{\pi^3}(Z\alpha)^2 m_r]$	$E(2p-2s)$ $[\frac{\alpha^3}{\pi^3}(Z\alpha)^2 m_r]$
c3	−0.027 55	−0.011 22	0.016 33
c111	−0.007 11	0.000 255	0.007 365
c2(1)			0.0144(14)
c12	−0.049 09	0.002 79	0.051 88
c1c2	−0.027 89	−0.000 902	0.026 99
c1c11	−0.010 63	0.000 037	0.010 67
c1c1c1	−0.001 29	−0.000 010	0.001 28

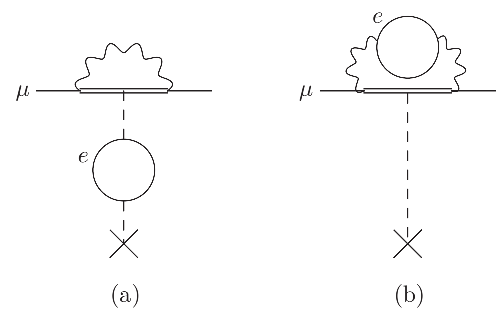

FIG. 7. Logarithmically enhanced $\alpha^6 m$ contributions. Left: a characteristic Feynman diagram for the Uehling-potential correction to the muon self-energy; right: a characteristic graph of the contribution of the two-loop muon self-energy with an eVP insertion.

C. Rescaled QED terms

While the dominant contributions to the Lamb shift of light muonic atoms are from the muon-specific terms, there is a set of contributions which are of the same form for muonic and electronic hydrogenlike atoms. The collection of the related contribution can be found, e.g., in [5]. The summary of relevant contributions is presented in Table VII (cf. [6]). The leading rescaled term is of the order $\alpha^5 m$. The numerical coefficients of the universal contributions used to be larger than those of the specific contributions and we include a part of the $\alpha^6 m$ terms.

IV. NUCLEAR-STRUCTURE AND QED EFFECTS

A. Nuclear-structure contributions: Leading term and higher-order corrections

The Lamb shift in light muonic atoms is affected by various nuclear-structure effects stronger than the Lamb shift in an ordinary atom. There are three types of contributions, which we are to discuss.

(1) There are nuclear-size contributions, i.e., the contributions, which can be parameterized by the size parameters, such as the rms nuclear charge radius R_N, and which vanish once we set $R_N = 0$. The leading contribution is of the form

$$\Delta E_{\rm lead}(nl) = \frac{2\pi}{3}(Z\alpha)\,R_N^2\,|\psi_{nl}(0)|^2 = \frac{2}{3}(Z\alpha)^4 m_r^3\,R_N^2\,\frac{\delta_{l0}}{n^3}$$
$$= \begin{cases} \left(41.580\,r_p^2\,\frac{\delta_{l0}}{n^3}\right) & \text{meV, for } \mu\text{H}, \\ \left(51.263\,r_t^2\,\frac{\delta_{l0}}{n^3}\right) & \text{meV, for } \mu\text{T}, \\ \left(820.18\,r_h^2\,\frac{\delta_{l0}}{n^3}\right) & \text{meV, for } \mu^3\text{He}, \end{cases} \quad (7)$$

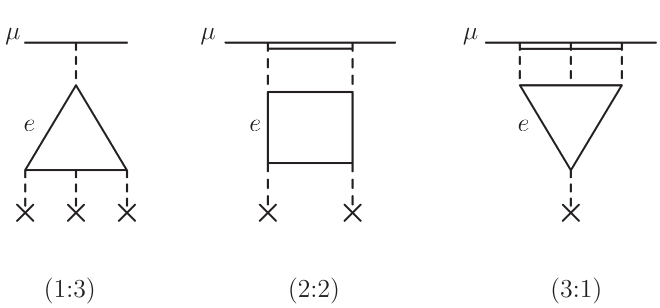

FIG. 6. Characteristic diagrams induced by the light-by-light scattering.

where $\psi_{nl}(\mathbf{r})$ is the wave function of the nonrelativistic Coulomb problem for the light muonic atom of interest (with the reduced mass introduced) and r_N is the numerical value (in the fermis, aka femtometers) of the rms charge radius of the nucleus R_N. This correction is used as the main "signal" to determine the rms charge radius.

There are a number of QED corrections to this leading term which have the same form (i.e., $\propto R_N^2$), and their clarification is important for an accurate determination of R_N. There are also contributions of a more complicated form.

(2) There are the nuclear polarizability contributions, i.e., the two-photon contributions, where the intermediate nuclear state is an excited one. In the case of $A = 3$ there is no discrete excited nuclear states and the "excited states" means the continuous spectrum of disintegrated nuclear constituents.

(3) A pointlike particle cannot possess an anomalous magnetic moment.[1] Its magnetic moment should be equal to $Ze/2M$ [cf. (1) and (2)]. However, the contributions, which vanish with zero anomalous magnetic moment κ_N, do not necessary belong to the nuclear-size contributions, i.e., they do not vanish if we set the nuclear radius to zero. (In a phenomenological consideration, such parameters as the anomalous magnetic moment and the charge and magnetic radii are disentangled, while in a fundamental theory they are expressed in the terms of the same fundamental parameters and therefore can go to zero limit only altogether.) If we consider a compound nucleus within the quantum mechanics, we have to deal with an anomalous magnetic moment in a pointlike limit.

[1]There may be a certain confusion about that. Various effects of the electron structure are measurable, such as the extended size, polarizability, and anomalous magnetic moment. The electron is called "pointlike" and "structureless" because we can in principle calculate all of them. They originate from the same type of diagrams. For example, the vertex diagram produces the anomalous magnetic moment of the electron and its magnetic form factor. Structure effects come together with the anomalous magnetic moment.

The Lagrangian for a theory of a pointlike particle with a nonzero anomalous magnetic moment would lead to additional divergences and a problem with the renormalization. At the level of the Lagrangian we have to choose either to deal with a pointlike particle without any anomalous magnetic moment or to have an effective field theory for a particle with an anomalous magnetic moment and internal structure.

TABLE VI. The individual light-by-light contributions to the $\Delta E(2p_{1/2} - 2s_{1/2})$ interval in muonic tritium at order $\alpha^5 m$. The notation follows [29] (see Fig. 6).

No.	Designation	Order	$\Delta E(\mu\mathrm{H})$ [meV]	$\Delta E(\mu\mathrm{T})$ [meV]	$\Delta E(\mu^3\mathrm{He})$ [meV]
4.1	LbL (1:3)	$\alpha(Z\alpha)^4 m$	–0.001 018(4)	–0.001 12	–0.019 73(6)
4.2	LbL (2:2)	$\alpha^2(Z\alpha)^3 m$	0.001 15(1)	0.001 26(1)	0.011 3(4)
4.3	LbL (3:1)	$\alpha^3(Z\alpha)^2 m$	–0.001 02(1)	–0.001 12(1)	–0.005 0(2)

Such a contribution is of the order $(Z\alpha)^5(m/M)^3 m$. We refer to such a contribution as the κ term.

All these three types of contributions are considered below.

We start with the complete r_N^2 term up to order $\alpha^5 m$. The results are summarized in Table VIII. The leading term [see Eq. (7)] is well known. Term 11 is the relativistic correction to this term [32,33] known in a closed analytic form.

The relativistic higher-order nuclear-finite-size correction, referred to as 11 in Table VIII, is of the order $(Z\alpha)^6 m$. There are two approaches to estimate it. One, following [5,34], deals with the contribution in the logarithmic approximation. The other uses a complete evaluation [33,35]. As it is suggested in [5,34], we expect that the logarithmic approximation is sufficient because the contribution under question is a small one. Besides, any complete evaluation requires a model of the charge distribution and therefore it is model dependent. However, the expression in logarithmic approximation following (7.69) of [5] (and copied later in many reviews and compilations including ours [6]) contains a misprint. The correct one can be found from [33,35]. We use the correction in the form

$$\Delta E_{\mathrm{fns:rel}} = (Z\alpha)^2 \left[1 - \frac{2}{3}(m_r R_N)^2\right] \Delta E_{\mathrm{lead}} \ln \frac{1}{Z\alpha m R_N},$$

where ΔE_{lead} is defined in (7), and assign them the uncertainty of 50%.

Term 12 is the correction to Eq. (7) due to the Uehling potential (see Fig. 8), which can be parameterized as

$$\Delta E_{\mathrm{eVP}}(nl) = C_{\mathrm{eVP}}(nl) \frac{\alpha}{\pi} \frac{(Z\alpha)^4}{n^3} m_r^3 R_N^2. \tag{8}$$

The related coefficients are summarized in Table IX. The result for muonic hydrogen has been known for a while (see [5,6] and references therein). Here, for the muonic tritium and helium-3 we follow [26,36,37] for the $2s$ states and [6] for the $2p$ states. Our result for muonic helium-3 ion is consistent with that in [38].

Term 13 is the one-loop QED correction to the leading nuclear-finite-size contribution in Eq. (7) (see [5]). It is of order $\alpha(Z\alpha)^5 m$ and it is known in a closed analytic form. We do not include it into the final theoretical summary.

There are many contributions which come from the "hard" two-photon exchange (TPE), which accounts for the nuclear structure. Two of the most important nuclear-structure effects are the higher-order finite-nuclear-size contribution (the so-called Friar terms and the recoil corrections to it) and the nuclear polarizability contribution. We briefly overview them in Sec. VI A. However, we consider one specific hard TPE term. It is a recoil contribution with a pointlike nucleus with a nonzero anomalous magnetic moment (see [6]), denoted as term 14.κ in Table VIII. The result has been found in [39] in the closed analytic form

$$\Delta E_\kappa = -\frac{(Z\alpha)^5 m_r^4}{\pi} \frac{\kappa_N}{8 M^2 (M-m)} \left[-3(1+\kappa_N) \ln \frac{M}{m} + (1-\kappa_N)\left(\ln 2 - \frac{1}{4}\right) + B\left(\frac{m}{M}\right)^2\right], \tag{9}$$

where κ_N is the anomalous magnetic moment (2). The value of $B \approx O(1)$, known in detail [39], is negligible and contributes neither to the central value of the Lamb shift nor to its uncertainty.

B. Nuclear-line QED terms

Following [5], we introduce the leading nuclear-line QED correction as

$$\Delta E_{\mathrm{N:QED}}(nl) = \frac{4(Z^2\alpha)(Z\alpha)^4}{\pi n^3} \frac{m_r^3}{M^2} \left\{ \left[\frac{1}{3} \ln \frac{M}{(Z\alpha)^2 m_r} + \frac{11}{72}\right] \delta_{l0} - \frac{1}{3} \ln k_0(nl) \right\}, \tag{10}$$

TABLE VII. The rescaled QED terms, originating from the theory of ordinary hydrogen. The results are presented for the Lamb-shift $\Delta E(2p_{1/2} - 2s_{1/2})$ interval in muonic hydrogen and tritium atoms and the muonic helium-3 ion. Here, the designation is not unique, but together with the order of the terms it is sufficient to distinguish the corrections. We follow the notation of [6].

No.	Designation	Order	$\Delta E(\mu\mathrm{H})$ [meV]	$\Delta E(\mu\mathrm{T})$ [meV]	$\Delta E(\mu^3\mathrm{He})$ [meV]
7.1	QED (Rad)*	$\alpha(Z\alpha)^4 m$	**– 0.663 45**	**–0.809 73**	**–10.6525**
7.2	QED (Rad)	$\alpha(Z\alpha)^5 m$	–0.004 43	–0.005 47	–0.1749
7.3	QED (Rad-Rec)	$\alpha(Z\alpha)^5 \frac{m^2}{M}$	0.000 19	0.000 088	0.002 81
8	QED (Rec)*	$(Z\alpha)^5 m^2/M$	**– 0.044 97**	**–0.018 79**	**–0.558 11**

TABLE VIII. Some nuclear-structure contributions to $\Delta E(2p_{1/2} - 2s_{1/2})$ in muonic tritium and the muonic helium-3 ion. Here we summarize the finite-nuclear-size contribution (FNS) and the "pointlike" TPE κ term. We use two related equations, where r_N is the numerical value of R_N in the fermis. The results given in italic are not used directly and are given for reference purposes only. The numerical values are given for $R_p = 0.84$ fm, $R_t = 1.8$ fm, $R_h = 2.0$ fm to characterize the contributions.

			ΔE for H [meV]		ΔE for T [meV]		ΔE for ^{3}He [meV]	
No.	Designation	Order	Value	Estimation	Value	Estimation	Value	Estimation
10	FNS (NR)	$(Z\alpha)^4(mR_N)^2m$	$\mathbf{-5.1974}\ r_p^2$	***−3.7***	$\mathbf{-6.4078}\ r_t^2$	***−21***	$\mathbf{-102.52}\ r_h^2$	***−410***
11	FNS (Rel)	$(Z\alpha)^6(mR_N)^2m$	$-0.0016\ r_p^2$		$-0.0017\ r_t^2$		$-0.090\ r_h^2$	
			$+0.000\,24(r_p^2)^2$	*−0.0001*	$+0.000\,30(r_t^2)^2$	*−0.0023*	$+0.016(r_h^2)^2$	*−0.11*
12	FNS (eVP)	$\alpha(Z\alpha)^4(mR_N)^2m$	$-0.0282\ r_p^2$	*−0.020*	$-0.0363\ r_t^2$	*−0.12*	$-0.85\ r_h^2$	*−3.4*
13	FNS (SE+μVP)	$\alpha(Z\alpha)^5(mR_N)^2m$	$0.0006\ r_p^2$	*0.0005*	$0.0008\ r_t^2$	*0.002*	$0.02\ r_h^2$	*0.10*
14.κ	TPE (κ)	$(Z\alpha)^5m^4/M^3$	$-0.003\,05$		$-0.002\,98$		-0.0187	

where $\ln k_0(nl)$ is the Bethe logarithm. The related numerical results for the muonic atoms of interest are

$$\Delta E_{\mathrm{N:QED}}(2p_{1/2} - 2s_{1/2}) = \begin{cases} -0.010\,41\ \mathrm{meV}, & \text{for } \mu\mathrm{H}, \\ -0.001\,58\ \mathrm{meV}, & \text{for } \mu\mathrm{T}, \\ -0.088\,30\ \mathrm{meV}, & \text{for } \mu^3\mathrm{He}. \end{cases} \tag{11}$$

C. Hadronic vacuum polarization

The calculation of the hadronic vacuum polarization produces rather a marginal contribution (cf. [23]),

$$\Delta E_{\mathrm{hVP}}(2p_{1/2} - 2s_{1/2}) = \begin{cases} 0.0106(10)\ \mathrm{meV}, & \text{for } \mu\mathrm{H}, \\ 0.0132(10)\ \mathrm{meV}, & \text{for } \mu\mathrm{T}, \\ 0.21(2)\ \mathrm{meV}, & \text{for } \mu^3\mathrm{He}, \end{cases} \tag{12}$$

with accuracy sufficient for the comparison of theory and experiment for the Lamb shift. The result is consistent with those in [35,38]. While applying this correction, one has to remember that it is not that important what exact model has been used for the calculations. It is more important that the hadronic vacuum polarization should be taken into account consistently in all the competitive evaluations (for spectroscopy of ordinary atoms, the Lamb shift in muonic atoms, elastic electron-proton scattering, etc.).

V. QED SUMMARY

We consider above all the $\alpha^5 m$ contributions. Some of them have small numerical values, however, it is important to consider all of them. The smallness of the result for a particular contribution does not mean that it is negligible without any calculations. It may be not clear *a priori* that such a contribution is small. We have also taken into account the logarithmically enhanced $\alpha^6 m$ terms. The results for muonic hydrogen, tritium, and helium-3 ions are summarized in Table X. We include in the QED table pure QED contributions as well as the QED contributions to the r_N^2 term.

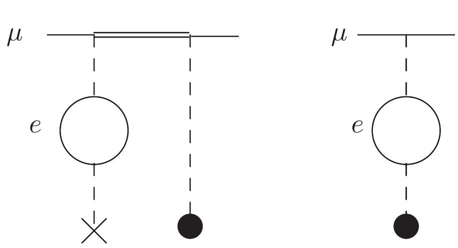

FIG. 8. The characteristic diagrams for the eVP correction to the finite-nuclear-size term. The first diagram represents contributions similar to (7) but with the wave function corrected due to eVP. The second diagram is for the finite-nuclear-size correction to the eVP potential.

The muonic hydrogen result,

$$\Delta E^{\mathrm{QED}}(2p_{1/2} - 2s_{1/2}) = \left[205.9211(10) - 5.2271(8)\,r_p^2 + 0.0002(1)\left(r_p^2\right)^2\right]\ \mathrm{meV}, \tag{13}$$

is slightly different from our result in [6], because we have corrected a minor error for the BP contribution in Table II. We also excluded the $\alpha(Z\alpha)^5(mR_N)^2m$ term and included a $(Z\alpha)^6(mR_N)^2m$ contribution, since the former is not logarithmically enhanced and the latter is.

The result for the muonic helium-3 ion is found to be

$$\Delta E^{\mathrm{QED}}(2p_{1/2} - 2s_{1/2}) = \left[1644.17(2) - 103.47(5)\,r_h^2 + 0.02(1)\left(r_h^2\right)^2\right]\ \mathrm{meV}. \tag{14}$$

The contributions have been previously reviewed in [35,38]. Those reviews present a theory of the $\alpha^5 m$ contributions and a partial consideration of the $\alpha^6 m$ terms. In the case of [38] the consideration of some particular higher-order terms is incomplete (cf., e.g., the three-loop electronic vacuum polarization [38], where the third-order diagram is missing, with our complete calculation in [23]). The difference in the $\alpha^5 m$ theory is the minor one. In the journal version of [35] the result on the relativistic-recoil correction to the Uehling term is incorrect, which has been fixed in the subsequent eprint versions. The term no. 14.κ is not taken into consideration in [35,38], however, it plays rather a marginal role. Therefore we

TABLE IX. The Uehling correction (8) to the leading nuclear-finite-size term for the $n = 2$ states in muonic hydrogen, tritium, and helium-3 ion.

	μH	μT	μ^3He
$C_{\mathrm{eVP}}(2s)$	1.543 01	1.611 63	2.359 36
$C_{\mathrm{eVP}}(2p)$	− 0.011 64	− 0.012 49	− 0.021 73

TABLE X. The QED summary table on the Lamb-shift interval $\Delta E(2p_{1/2} - 2s_{1/2})$ in muonic tritium and the muonic helium-3 ion. We follow the notation in [6]. The uncertainty in the *total* values is due to the estimation of the higher-order contributions.

No.	$\Delta E(\mu\mathrm{H})$ [meV]	$\Delta E(\mu\mathrm{T})$ [meV]	$\Delta E(\mu^3\mathrm{He})$ [meV]	Refs.
		Unperturbed quantum mechanics		
0	**−0.050 95**	**−0.155 98**	**−0.0032**	Table II
		Specific QED		
1	**205.026 13**	**235.933 65**	**1642.3956**	Table III
2	**1.658 85**	**1.903 51**	**13.084 27**	Table III
3	0.007 52	0.008 74(9)	0.073(3)	Table III
4	−0.000 89(2)	−0.000 99(2)	−0.0134(6)	Table III
5	−0.002 54	−0.004 12	−0.062 69	Table III
6	−0.001 52	−0.001 87	−0.0299	Table III
		Rescaled QED		
7	**−0.667 69**	**−0.815 11**	**−10.8246**	Table VII
8	**−0.044 97**	**−0.018 79**	**−0.5581**	Table VII
		Nuclear-line QED		
9	−0.010 41	−0.001 58	−0.0883	Eq. (11)
		Nuclear finite size		
10	**−5.1974** $\boldsymbol{r_p^2}$	**−6.407 8** $\boldsymbol{r_t^2}$	**−102.52** $\boldsymbol{r_h^2}$	Table VIII
11	$-0.0016\, r_p^2$	$-0.0017\, r_t^2$	$-0.091\, r_h^2$	
	$+0.000\,24\, (r_p^2)^2$	$+0.0003\, (r_t^2)^2$	$+0.016\, (r_h^2)^2$	Table VIII
12	$-0.0282\, r_p^2$	$-0.0363\, r_t^2$	$-0.85\, r_h^2$	Table VIII
14.κ	−0.003 05	−0.002 98	−0.0187	Table VIII
		Hadronic VP		
16	0.010 6(10)	0.013 2(10)	0.21(2)	Eq. (12)
Total	$205.9211(10) - 5.2271(8)\, r_p^2$	$236.8577(11) - 6.446(8)\, r_t^2$	$1644.16(2) - 103.47(5)\, r_h^2$	
	$+0.000\,24(12)\, (r_p^2)^2$	$+0.0003(2)\, (r_t^2)^2$	$+0.016(8)\, (r_h^2)^2$	

consider the QED theory of the muonic helium-3 ion as well established.

The result for muonic tritium,

$$\Delta E^{\mathrm{QED}}(2p_{1/2} - 2s_{1/2}) = \left[236.8577(11) - 6.446(8)\, r_t^2 + 0.0003(2)\left(r_t^2\right)^2\right] \mathrm{meV}, \quad (15)$$

has been obtained here. The first term differs from that for muonic hydrogen [see Eq. (13)] by approximately 15%, which is basically an effect of the different values of the reduced mass in the leading term (cf. term no. 1 in Table X).

VI. THE HARD TWO-PHOTON EXCHANGE

A. Nuclear polarizability, Friar term, etc.

Two-photon-exchange contributions play a certain role in the central value of the theoretical prediction for the Lamb shift in light muonic atoms and a crucial role in the determination of its uncertainty. There are three kinds of the TPE contributions.

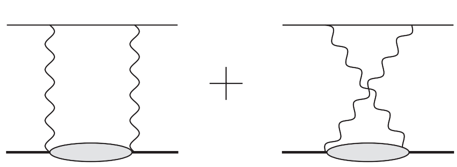

FIG. 9. Two-photon exchange: the inelastic contribution (the proton polarizability). The intermediate state is not equal to the reference nuclear state.

(1) There are indeed pure QED TPE contributions. They are well known (see term no. 8 in Table VII in Sec. III C and in the QED summary table in the previous section).

(2) The other contributions involve the nuclear-structure effects. The elastic part of TPE deals with the form factors of the nucleus. If the nucleus is literally described as a certain extended object, then there is a small part that does not vanish once we put the nuclear size to zero. It is related to the anomalous magnetic moment and has been already taken into account (see term 14.κ in Table VIII in Sec. IV A and in the QED summary table in the previous section). The rest of the elastic contribution, where the so-called Friar term dominates, we discuss in this section.

(3) The calculation of the TPE contributions involves an intermediate state of the nucleus. The elastic contributions are those where such an intermediate state relates to the unchanged nuclear state. If the intermediate state changes, then that is a part of the inelastic term, aka the nuclear polarizability contribution (see Fig. 9).[2] In the case of the helion and triton there is only one bound state, the ground state of the nucleus. Any excited states actually belong to the continuum of the states of the disintegrated nucleus, either a nucleon +

[2]While calculating the TPE contribution to the Lamb shift in muonic hydrogen, the dispersion approach has been used. In this case the term *inelastic* is often used, which is presented as a dispersion integral, while the subtraction term is considered separately (see, e.g., [6] and references therein).

deuteron, or three separate nucleons. A more hard excitation may produce additional particles, such as mesons, or turn a nucleon into the Δ particle or another baryon.

The basic theoretical expressions for the muonic atoms with a nucleus with spin $1/2$ are the same as for the muonic hydrogen (cf. [34,40–47]). However, their relative importance is different, as well as the methods by which they are calculated.

The elastic part of TPE can be presented in the form [6,39]

$$\begin{aligned}
\Delta E_{\rm eTPE}(nl) &= \Delta E_{\rm Fr}(nl) + \Delta E_{\rm rec}(nl),\\
\Delta E_{\rm rec}(nl) &= -\frac{16(Z\alpha)^5\, m_r^4}{\pi}\, I_{\rm rec}\frac{\delta_{l0}}{n^3},\\
I_{\rm rec} &= I_\kappa + I_{\rm EF} + I_{\rm M1} + I_{\rm M2},\\
I_\kappa &= \kappa\int_0^\infty \frac{dq}{q^4}\{(2+\kappa)f_{M1} + f_{M2}\},\\
I_{\rm EF} &= \int_0^\infty \frac{dq}{q^4} f_{EF}(m,M;q^2)([G_E(q^2)]^2 - 1),\\
I_{\rm M1} &= \int_0^\infty \frac{dq}{q^4} f_{M1}([G_M(q^2)]^2 - (1+\kappa)^2),\\
I_{\rm M2} &= \int_0^\infty \frac{dq}{q^4} f_{M2}[G_M(q^2)G_E(q^2) - (1+\kappa)],
\end{aligned} \tag{16}$$

where the Friar term is defined as (see, e.g., [5,11])

$$\begin{aligned}
\Delta E_{\rm Fr}(2s_{1/2}) &= -\frac{2(Z\alpha)^5\, m_r^4}{\pi}\, I_{\rm Fr},\\
I_{\rm Fr} &= \int_0^\infty \frac{dq}{q^4}([G_E(q^2)]^2 - 1 - 2G'_E(0)\,q^2),
\end{aligned} \tag{17}$$

where G_E and G_M are electric and magnetic form factors and the f functions are given in [6], following [41].

Let us consider the Friar term in more details. All the contributions to the $\Delta E_{\rm rec}$ are the recoil contributions and they are suppressed by m/M. Therefore, the Friar term in Eq. (17) is the leading elastic TPE contribution. We can present it in the form [32,33]

$$I_{\rm Fr} = \frac{\pi}{48}\int d^3r\, d^3r'\,\hat\rho_E({\bf r})\hat\rho_E({\bf r}')|{\bf r}-{\bf r}'|^3 \equiv \frac{\pi}{48}\langle r^3\rangle_2, \tag{18}$$

where the function $\hat\rho_E({\bf r})$ is the Fourier transform of the electric charge form factor

$$\hat\rho_E({\bf r}) = \frac{1}{(2\pi)^3}\int d^3{\bf q}\; e^{i{\bf rq}}\; G_E({\bf q}^2),$$

which somewhat differs from the charge density $\rho_E({\bf r})$ in the sense of classical physics and nonrelativistic quantum mechanics. The difference comes from the fact that the nucleus at rest after absorbing a photon with a nonzero momentum is the subject of a kick. It is not at rest afterwards. If the momentum is essentially smaller than Mc, then the velocity of the motion after the kick is much smaller than c and therefore the recoil effect is a relativistic correction which can be neglected. (The nuclear-recoil consideration is the same as the consideration of the nonrelativistic nucleus neglecting its velocity.) In the TPE calculations for the light nuclei (which are somewhat larger than the proton and some heavier) the nonrelativistic consideration of the nucleus is more accurate than for the proton. Neglecting the recoil, we can substitute $\hat\rho_E({\bf r})$ for the "real" charge density $\rho_E({\bf r})$, which can be described by the nuclear wave function in a straightforward way.

The same can be done for the inelastic contributions. Neglecting the recoil effects, i.e., expanding in the ratio of a characteristic momentum to Mc, one can describe the TPE contributions, both elastic and inelastic, with the help of the quantum mechanics. The consideration of the recoil correction is one of the questions for the validation of the accuracy of the polarizability calculations.

Another approximation used in the calculations is in limiting the consideration with the pointlike nucleons. That is one more problem for the estimation of the accuracy. Once these two approximations have been done, we should discuss two kinds of problems. The first type is due to the validity of the pointlike nonrecoil picture in general with the given model. That can be checked by comparing actual pointlike results with the results of the calculations. Indeed, we do not have any measurements for the pointlike nuclei, however, assuming that the internal nucleon structure is not much affected by the binding effects, one can easily express, e.g., the result for the rms charge radius for the pointlike nuclei in the terms of the actual experimental values of R_N^2 and the rms charge radii of the proton and neutron.

The other source of the problems is due to approximations accepted for a calculation of a particular variable, such as TPE contributions. In comparison with the rms charge radius, it resulted from additional approximations.

First we present results recently found by different authors for the polarizability contribution to the Lamb shift in muonic atoms with $A = 3$ and afterwards we examine their accuracy. We start with the results of [48] for the nuclear-structure TPE contribution,

$$\Delta E_{\rm ns:TPE}(2p_{1/2} - 2s_{1/2}) = \begin{cases} 0.768(25)\ \text{meV}, & \text{for } \mu\text{T},\\ 15.40(39)\ \text{meV}, & \text{for } \mu^3\text{He}, \end{cases} \tag{19}$$

which are the only recent results on the issue. Previously the same authors obtained the results on the nuclear polarizability contribution in the muonic deuterium [49] and muonic helium-4 [50], and later the same method was applied to the $A = 3$ muonic atoms. (It is not that important that the results [48–50] are obtained by the *same* authors with the *same* methods but that they are based on the *same* experimental data and the *same* approximations.) They did two independent evaluations [48–50], based on an effective two-nucleon potential plus three-nucleon forces, taken for one of two calculations from the phenomenological description [51,52] and the other from the effective chiral perturbation theory [53,54]. Both potentials deal with nonrelativistic pointlike nucleons.

The nucleons inside a nucleus have their own structure contributions. The chiral perturbative theory is not a good way to describe the nucleon polarizability. The latter can be described either by using another effective field theory (cf. [43,47]) or by using a combination of theory and experimental data (cf. [34,40–42,46]).

B. Accuracy of the nuclear-structure TPE contribution

Let us outline in detail the "road map" of the evaluation of the nuclear polarizability and elastic TPE contributions related to the result [48].

(1) To start one has to choose the appropriate level of approximation (which is N^{3}LO $\chi^{\rm PT}$ for the calculation in muonic deuterium [49], tritium and helium-3 [48], and helium-4 [50])[3] and to write the $\chi^{\rm PT}$ Lagrangian. In a renormalizable theory the Lagrangian does not depend on the level of approximation. In an effective nonrelativistic theory, a number of new contact terms of different shapes should be generated and this very number depends on the level of approximation. See Refs. [55,56] for details on N^{3}LO $\chi^{\rm PT}$.

The use of an effective field theory, which exploits the chiral perturbation theory, suggested in [57], provides us with a possibility to consider a broad range of phenomena and, in contrast to the phenomenological fits, use the information from many other channels. Neither the LO approximation nor the NLO and NNLO ones were good for the description of the nucleon-nucleon scattering. The N^{3}LO approximation appears to be the first approximation consistent with the NN data (see, e.g., [53,55]).

(2) The Lagrangian contains a number of parameters which should be determined from experiment. Therefore, one has to calculate a certain number of observable variables and to compare theory to experiment to find the numerical values of the parameters. That has been done in [53,55]; however, the accuracy of the determination of the parameters is unclear as well as the correlation of their uncertainties. Therefore, its outcome for the accuracy of the subsequent stages is also uncertain.

(3) Once the Lagrangian and its parameters are defined, one could in principle calculate any low-energy property of the nucleons and light nuclei. The first stage is to calculate an effective nucleon-nucleon (NN) potential. The later is found in [53], where a comparison with $p-n$ and $p-p$ scattering is also given (see also [56]). It is also necessary to find three-nucleon forces, which was done in [54].

(4) The NN potential (together with three-nucleon forces) allows us to describe a few-nucleon nucleus by solving the related nonrelativistic quantum-mechanics problem. The problem was solved for the NN potential induced by N^{3}LO $\chi^{\rm PT}$ for the deuteron [49], triton and helion [48], and α particle [50].

(5) With the wave functions (and other details of the spectrum) in hand, one can calculate various nuclear and atomic variables. The variables of the interest are the nuclear polarizability contribution and the elastic term, determined through the nuclear charge distribution. There are also a number of benchmark quantities, such as the rms nuclear charge radius, which are good to calculate as a test. The results for the light muonic atoms were found in [48–50]. Note that the calculation of various nuclear and atomic variables may involve certain approximations which are specific for a variable. In the case of the polarizability, there is a kind of partial wave expansion suggested in [58] and widely used (see, e.g., [49,59–61]). Particular variables also may involve specific corrections. Their accuracy and their comparison with the experimental values or theoretical results, obtained by different methods, may be in such a case rather specific than generic. Indeed, a better agreement for a test variable means a somewhat better description; however, it is not clear quantitatively, how to interpret the [dis]agreement with the test variables into the uncertainty of the calculation of the variable of interest.

(6) A part of the evaluation has been done with a certain regulator function, which was introduced additionally to the Lagrangian. While the Lagrangian is theoretically motivated, the regulator function is not. To consider the approach as a model-independent one, one has to check that the result does not really depend on the details of the regularization, which in practice can be done only with certain reservations (see, e.g., [49]).

(7) All this development has been done for nonrelativistic pointlike nucleons. The related corrections have to be found. However, they are required on all the stages, not only on the final one as it is done in [48–50].

(8) To verify the accuracy one may check the results on the test quantities (see, e.g., [49,59–61] for deuteron, [48] for triton and helion, and [50] for an α particle). However, some of them are known not very accurately, and they may be affected by the nucleon-finite-size and relativistic corrections in a different way than the TPE contributions.

(9) One may also check for the scatter of the results. The most accurate competitive results for muonic deuterium [49,61] follow from a phenomenological AV18 potential [51,52] (see also a result from the zero-range-potential method [59]). As for the other muonic atoms, the AV18 results are presented in [48,50] together with the N^{3}LO $\chi^{\rm PT}$ ones.

This phenomenological potential is an alternative to N^{3}LO $\chi^{\rm PT}$ calculations. The AV18 potential is a result of fitting the NN scattering and the deuteron data [51]. Notably, the N^{3}LO $\chi^{\rm PT}$ results for the NN scattering have a certain systematic deviation from the AV18 fit (see [53]). Actually, one has to consider this departure area by area and to determine the uncertainty. After that, using such an area-by-area uncertainty, one should look for their impact on a particular calculation of the TPE effects. The overall integral departure in the comparison of two approaches may be not sufficient to estimate the uncertainty.

The use of a theoretically motivated description is a breakthrough. It enables us to use the data from different channels and it sets some constraints, which would be unclear from phenomenological fits. While the NLO and NNLO results were not adequate to the description of the NN scattering, the N^{3}LO approximation has managed to produce results which are consistent with the data rather well. Indeed, one may wonder, what should be the outcome of N^{4}LO $\chi^{\rm PT}$, which is rather too difficult to achieve. A further progress in the evaluation of nucleon-nucleon potential and few-nucleon forces is possible and actually it has been achieved (see [62–64] and references therein), but it has not yet been applied for the subsequent study of the polarizability contribution. The use of higher-order terms of the perturbation theory should allow reduction of errors in many variables. Working with functions and many variables, one should be able to take advantage of

[3]LO is for the "leading order," N for the "next," e.g., NNLO is the next to the next to the leading order approximation, etc., and $\chi^{\rm PT}$ stands for the chiral perturbation theory.

such a general progress. In the meantime, in the case of light muonic atoms we are rather interested in one figure per atom, i.e., in the related TPE contribution to the Lamb shift, which is the sum of the polarizability and elastic term. A general improvement of the accuracy is not sufficient to be sure that this is the only value of interest that also improved. Besides, increasing the number of the fitting parameters should make their correlations more complicated.

The comparison of the experimental result [2][4] and the theoretical prediction for muonic deuterium [66], where the polarizability is taken as a certain average of the results from [49,59,61], demonstrates an approximately 2.5-σ deviation. To understand whether that is a real contradiction we have to have a *controlled* uncertainty. The control of the uncertainty is a crucial issue.

The uncertainty presented in (19) is that of the original paper [48] of the Hebrew-TRIUMF group (as well as the results on other muonic atoms [49,50]). As we explain above there are a number of stages which could produce the uncertainty. That may be done only by a big collaboration, which would include all the contributors at the intermediate stage. The Hebrew-TRIUMF group can control only a part of the stages, namely, the solution of the nonrelativistic problem for the nucleus consisting of the pointlike nucleons and the calculation of the nuclear polarizability within the nonrelativistic pointlike approximation, as well as partially investigating the contributions to the polarizability beyond this approximation. Such [partial] estimations have been performed [48].

As we mentioned, the same group has performed also a calculation for muonic deuterium [49]. Their part of the uncertainty in the cases of $A = 2$ and $A = 3$ has somewhat different but comparable values. The former one is 1.2% for muonic deuterium, while the latter is 3% for muonic tritium and 2.5% for muonic helium-3. The muonic-deuterium theoretical prediction, based on [49], disagrees with the experiment [2] and the discrepancy is as large as approximately 4.5% of the polarizability contribution.

The estimations in [48–50] do not control the uncertainty due to the uncertainty of the parameters, due to corrections to the nucleon-nucleon potential, etc. However, we can expect that they are comparable for muonic deuterium and muonic atoms with $A = 3$. We estimate this part of the uncertainty as 5%, which expands the total uncertainty in (19).

VII. CONCLUSIONS

Finally, combining the QED theory from Table X, the original results [48] for the nuclear-structure TPE contribution in (19), and the additional uncertainty of 5% to them as explained above, for the Lamb shift for the muonic atoms of interest we arrive at

$$\Delta E(2p_{1/2} - 2s_{1/2}) = \big[237.626(46) - 6.446(8)\,r_t^2 + 0.000\,3\,\big(r_t^2\big)^2\big]\ \mathrm{meV} \tag{20}$$

for muonic tritium and

$$\Delta E(2p_{1/2} - 2s_{1/2}) = \big[1659.58(86) - 103.47(8)\,r_h^2 + 0.02\,\big(r_h^2\big)^2\big]\ \mathrm{meV} \tag{21}$$

for the muonic helium-3 ion. The accuracy of the theoretical prediction is determined by the accuracy of the calculation of the nuclear-structure effects. It is not completely under control, and here it is extended in comparison to the original estimation [48]. After our paper was finished we became aware of two more calculations of the nuclear-structure effects. One [67] is an update of the result [48] used in this paper. The other [68] presents alternative calculations. They are consistent with the result [48].

The theoretical prediction above is not for the transitions which are measured directly (cf. Fig. 1). Let us suggest that in the muonic tritium and helium-3 the same transitions will be measured, namely,

$$\Delta E_t \equiv \Delta E(2p_{3/2}(F=2) - 2s_{1/2}(F=1)) \tag{22}$$

and

$$\Delta E_s \equiv \Delta E(2p_{3/2}(F=1) - 2s_{1/2}(F=0)). \tag{23}$$

Their values may be rearranged as

$$\begin{aligned}\tfrac{1}{4}\Delta E_s + \tfrac{3}{4}\Delta E_t &= \Delta E(2p_{1/2} - 2s_{1/2}) + \Delta E(2p_{3/2} - 2p_{1/2}) \\ &\quad + \tfrac{1}{8}\Delta E_{\mathrm{HFS}}(2p_{3/2}),\\ \Delta E_s - \Delta E_t &= \Delta E(2s_{1/2}(F=1) - 2s_{1/2}(F=0)) \\ &\quad - \Delta E_{\mathrm{HFS}}(2p_{3/2}).\end{aligned} \tag{24}$$

To compare them with the theoretical predictions (20) and (21), we find

$$\Delta E(2p_{1/2} - 2s_{1/2}) = \big[\tfrac{1}{4}\Delta E_s + \tfrac{3}{4}\Delta E_t\big] + \Delta_1, \tag{25}$$

$$\begin{aligned}\Delta E_{\mathrm{HFS}}(2s) &\equiv \Delta E(2s_{1/2}(F=1) - 2s_{1/2}(F=0)) \\ &= [\Delta E_s - \Delta E_t] + \Delta_2,\end{aligned} \tag{26}$$

where for muonic tritium

$$\Delta_1 = -\big[9.778\,95(6) + 0.000\,06\,r_t^2\big]\ \mathrm{meV},$$
$$\Delta_2 = 3.958\,26(1)\ \mathrm{meV},$$

and for the muonic helium-3 ion

$$\Delta_1 = -\big[141.958(5) + 0.004\,r_h^2\big]\ \mathrm{meV},$$
$$\Delta_2 = -24.292\,5(7)\ \mathrm{meV}.$$

The value of $\Delta E(2p_{1/2} - 2s_{1/2})$, the Lamb shift, is considered in this paper, while the $2s$ hyperfine interval $\Delta E_{\mathrm{HFS}}(2s)$ needs a separate consideration (see, e.g., [35,69]).

ACKNOWLEDGMENTS

The authors are grateful to John Arrington, Sonia Bacca, Nir Barnea, Misha Gorchtein, Julian J. Krauth, Krzysztof Pachucki, Randolf Pohl, and Ingo Sick for stimulating and useful discussions. The work was supported in part by DFG (under Grant No. HA 1457/9-1) and by RFBR (under Grant No. 14-02-91332).

[4]The full comparison includes the results on the $R_p^2 - R_d^2$ from the isotopic shift in hydrogen and deuterium [4,65], on R_p^2 from the Lamb shift in muonic hydrogen [1], and on R_d^2 from the Lamb shift in muonic deuterium [2].

TABLE XI. Theory of the $2p$ fine-structure interval $\Delta E(2p_{3/2} - 2p_{1/2})$ in muonic tritium and the muonic helium-3 ion. The radiative correction (item 7) is for the fine structure completely determined by the muon anomalous magnetic moment. The uncertainty in the *total* values is due to the estimation of the higher-order contributions.

No.	Designation	Order	Refs.	$\Delta E(\mu\mathrm{H})$ [meV]	$\Delta E(\mu\mathrm{T})$ [meV]	$\Delta E(\mu^3\mathrm{He})$ [meV]
			Unperturbed quantum mechanics			
0.1	Rel	$(Z\alpha)^{4+}m$		**8.415 64**	**9.023 881**	**144.3955**
0.2	Rel-Rec*	$(Z\alpha)^6 m^2/M$		5.1×10^{-6}	2.1×10^{-6}	0.000 13
0.3	BG*	$(Z\alpha)^4(m/M)^2m$	[14]	**−0.086 21**	**−0.011 859**	**−0.1898**
0.4	BP (tot)*	$(Z\alpha)^4(m/M)^2m$	Table XIV	***0.162 63(2)***	***0.245 83(2)***	***0.1947(2)***
			Specific QED			
1	eVP1 Rel*	$\alpha(Z\alpha)^4m$		0.005 02	0.006 03	0.2697
			Rescaled QED			
7	$(g-2)_\mu$	$\alpha(Z\alpha)^4m$		0.017 64	0.020 28	0.3245
			Nuclear finite size			
11	FNS (Rel)	$(Z\alpha)^6m$		$-0.000\,05\,r_p^2$	$-0.000\,06\,r_t^2$	$-0.004\,r_h^2$
Total				8.514 72(6) $-0.000\,05\,r_p^2$	9.284 17(6) $-0.000\,06\,r_t^2$	144.995(5) $-0.004\,r_h^2$

APPENDIX: THE 2p FINE AND HYPERFINE STRUCTURE

Once we use the effective Dirac equation (see Secs. II and III A), the leading contribution to the energy is expressed in the terms of the solution of the Dirac equation with the reduced mass. In such a case, there are no additional $(Z\alpha)^4m^2/M$ corrections to the Dirac equation contribution for the fine structure. The correction in order $(Z\alpha)^6m^2/M$ (see Table XI) comes from the second term in (4), while the BG and BP terms produce corrections in the order $(Z\alpha)^4m^3/M^2$ (see Sec. II).

The theory in order α^5m is summarized in Tables XI (for the $2p$ fine structure), XII (for the $2p_{3/2}$ HFS interval), and XIII (for the $2p_{1/2}$ HFS interval), where we follow the notation of [6].

All the contributions there except of the Brodsky-Parsons term [10,11] have been already discussed in detail for the Lamb shift and they are very similar for the fine and hyperfine structure. The leading BP contribution is of the form [10] (cf. [11])

$$\Delta E_{\mathrm{BP:lead}}(2p_{1/2}(F=1)) = -\frac{\langle 2p_{1/2}(F=1)|H_{\mathrm{HFS}}|2p_{3/2}(F=1)\rangle\langle 2p_{3/2}(F=1)|H_{\mathrm{HFS}}|2p_{1/2}(F=1)\rangle}{E(2p_{3/2})-E(2p_{1/2})} \tag{A1}$$

and

$$\Delta E_{\mathrm{BP:lead}}(2p_{3/2}(F=1)) = -\Delta E_{\mathrm{BP:lead}}(2p_{1/2}(F=1)). \tag{A2}$$

One can consider it as an enhanced second-order contribution with the Dirac equation as the unperturbed theory. In the Dirac theory the $2p_{1/2}$ and $2p_{3/2}$ states are not degenerate, but their energy splitting is much smaller than $(Z\alpha)^2m$, which is the characteristic value for the gross-structure intervals. In the meantime the hyperfine interaction, considered as the perturbation, has a nonvanishing off-diagonal matrix element between the closely lying $2p_{1/2}(F=1)$ and $2p_{3/2}(F=1)$. Under such conditions, the second-order contribution of the perturbation theory with the hyperfine interaction has an enhanced term (see Fig. 10). [It may be also obtained from the theory based on the Coulomb-Schrödinger equation as a result of the rediagonalization of the

TABLE XII. Theory of the $2p_{3/2}$ hyperfine interval $\Delta E(2p_{3/2}(F=2) - 2p_{3/2}(F=1))$ in muonic tritium and the muonic helium-3 ion. Here, the *leading* term includes the relativistic corrections. The uncertainty in the *total* values is due to the estimation of the higher-order contributions.

Designation	Order	Refs.	$\Delta E(\mu\mathrm{H})$ [meV]	$\Delta E(\mu\mathrm{T})$ [meV]	$\Delta E(\mu^3\mathrm{He})$ [meV]
		Unperturbed quantum mechanics			
Leading	$(Z\alpha)^{4+}m^2/M$		**3.040 79**	**3.998 72**	**−22.8484**
Rel-Rec*	$(Z\alpha)^4(m/M)^2m$		0.351 39	0.177 47	−1.2432
BP (tot)*	$(Z\alpha)^4(m/M)^2m$	Table XIV	***−0.144 56(2)***	***−0.218 52(2)***	***−0.1730***
		Specific QED			
eVP1*	$\alpha(Z\alpha)^4m^2/M$		0.001 28	0.001 74	−0.0346
		Rescaled QED			
$(g-2)^*_\mu$	$\alpha(Z\alpha)^4m^2/M$		− 0.000 89	− 0.001 17	0.0067
Total			3.248 01(1)	3.958 26(1)	−24.2925(7)

TABLE XIII. Theory of the $2p_{1/2}$ hyperfine interval $\Delta E(2p_{1/2}(F=1) - 2p_{1/2}(F=0))$ in muonic tritium and the muonic helium-3 ion. Here, the *leading* term includes the relativistic corrections. The uncertainty in the *total* values is due to the estimation of the higher-order contributions.

Designation	Order	Refs.	$\Delta E(\mu\mathrm{H})$ [meV]	$\Delta E(\mu\mathrm{T})$ [meV]	$\Delta E(\mu^3\mathrm{He})$ [meV]
		Unperturbed quantum mechanics			
Leading	$(Z\alpha)^{4+}m^2/M$		**7.602 90**	**9.997 80**	**−57.1436**
Rel-Rec*	$(Z\alpha)^4(m/M)^2m$		**0.351 39**	**0.177 47**	**−1.243 15**
BP (tot)*	$(Z\alpha)^4(m/M)^2m$	Table XIV	**−0.144 56(2)**	**−0.218 52(2)**	**−0.1730**
		Specific QED			
eVP1*	$\alpha(Z\alpha)^4m^2/M$		0.005 67	0.008 11	−0.1220
		Re-scaled QED			
$(g-2)^*_\mu$	$\alpha(Z\alpha)^4m^2/M$		0.004 43	0.005 83	−0.0333
Total			7.819 83(1)	9.970 69(1)	−58.7150(7)

$2p_{1/2}(F=1)$ and $2p_{3/2}(F=1)$ states, which are degenerate in that theory (cf. [10,11]).] One can resummate all the perturbation theory (in the Dirac approach) with the hyperfine interaction as the perturbation and the $2p_{1/2}(F=1)$ and $2p_{3/2}(F=1)$ as the intermediate states. The result is

$$\Delta E(2p_{1/2}(F=1)) = -\Delta_{\rm BP}$$
$$= -\sqrt{\left(\frac{E(2p_{3/2}(F=1)) - E(2p_{1/2}(F=1))}{2}\right)^2 + (\langle 2p_{1/2}(F=1)|H_{\rm HFS}|2p_{3/2}(F=1)\rangle)^2}$$
$$+\frac{E(2p_{3/2}(F=1)) - E(2p_{1/2}(F=1))}{2}, \tag{A3}$$

$$\Delta E_{\rm BP}(2p_{3/2}(F=1)) = -\Delta E_{\rm BP}([2p_{1/2}(F=1)). \tag{A4}$$

(In the nonrelativistic theory this identity appears immediately through the rediagonalization.)

The related equations for the main terms [for both Eqs. (A1) and (A3)] are given in [10] and [11] for hydrogen and muonic hydrogen, respectively. To adjust the results for the other nuclei with spin 1/2, we have to find their g factors in the terms of particle physics [see Eq. (1) in Sec. I].

While the expression for the leading term of the hyperfine intervals requires a value of the magnetic moment of the nucleus as a whole, various corrections may involve the Dirac part and the anomalous part of the magnetic moment and, therefore, of the g factor separately. The calculation of ΔE_κ in (9) is one such example. The other example is the calculation

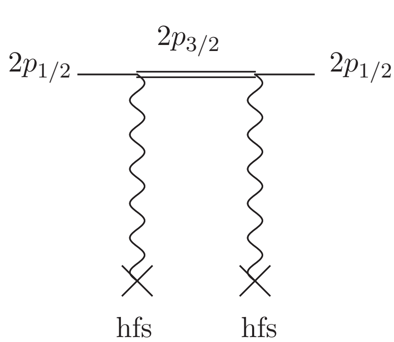

FIG. 10. The diagram for the leading Brodsky-Parsons term (A1) for the $E(2p_{1/2}(F=1)$ in the theory based on the Dirac equation, with an enhanced term in the second-order perturbation series for the $2p_{1/2}(F=1)$ with the HFS interaction as the perturbation.

of the off-diagonal matrix element of the HFS interaction over $2p_{1/2}(F=1)$ and $2p_{3/2}(F=1)$, which also requires such a separation. The calculation of various corrections to the leading BP term is very similar to that in muonic hydrogen [6]. The results are summarized in Table XIV.

Our results for the $\alpha^5 m$ contributions for muonic helium-3 ion are consistent with [70] and with the subsequent review [35]. We exclude here the results on $\alpha^2(Z\alpha)^4m$ from [70] for the fine and hyperfine structure, since they present only a part of the corrections of this type, ignoring terms of the third order in perturbation theory (cf. [23]).

In the literature, two different definitions of the composition of the energy levels may be used. We here consider a three-term composition, namely, the Lamb shift (the $2p-2s$ splitting), and the fine and hyperfine structure, the definitions of which are summarized in Eq. (3). The contributions of the Brodsky-Parsons term (aka the $2p$ mixing term) are included into those three types of terms. Such a composition is consistent with most "by default" phenomenological definitions.

The other possible composition, introduced in [10,11], is often used specifically in hydrogen and muonic hydrogen. It requires four terms. The BP term, denoted as Δ, is treated as a kind of separate effect.

To convert our three-term scheme to a four-term one, one has to simply exclude the BP terms from the Lamb-shift, and fine and hyperfine intervals presented here. The related results are given for the fine and hyperfine intervals directly in the summary tables (see term 0.4 in Tables XI, XII, and XIII). For

TABLE XIV. Contributions to the Brodsky-Parsons term ($\Delta_{\rm BP}$) in muonic tritium and the muonic helium-3 ion. Here we use the name of the BP term for the complete account of the enhanced contributions (as explained in the text). The main term, marked with *, contains the fine structure as follows from Eq. (4) with the BG correction (6) added and the appropriate HFS interval which includes the "NR" and "Rel-Rec" terms from Tables XIII and XII. The uncertainty in the *total* values is due to the estimation of the higher-order contributions.

No.	Designation	Order	$\Delta E(\mu\mathrm{H})$ [meV]	$\Delta E(\mu\mathrm{T})$ [meV]	$\Delta E(\mu^3\mathrm{He})$ [meV]
0.4.1	BP**	$(Z\alpha)^4 m^3/M^2$	**0.145 19**	**0.219 63**	**0.173 10**
0.4.2	BP$[(g-2)_\mu]^\star$	$\alpha(Z\alpha)^4 m^3/M^2$	−0.000 82	−0.001 45	−0.000 70
0.4.3	BP[eVP1]	$\alpha(Z\alpha)^4 m^3/M^2$	0.000 19	0.000 33	0.000 61
0.4	BP (tot)		0.144 56(2)	0.218 52(2)	0.173 02(2)

the Lamb shift it is a part of term 0 in the summary table (see Table X), which is given explicitly in a subordinate table (see term 0.4 in Table II). The value Δ, needed for the four-term presentation, is given by us in Table XIV. The BP contributions are given in the tables mentioned with bold italic to facilitate their exclusion if desired.

[1] A. Antognini, F. Nez, K. Schuhmann, F. D. Amaro, F. Biraben, J. M. R. Cardoso, D. S. Covita, A. Dax, S. Dhawan, M. Diepold, L. M. P. Fernandes, A. Giesen, A. L. Gouvea, T. Graf, T. W. Hänsch, P. Indelicato, L. Julien, Cheng-Yang Kao, P. Knowles, F. Kottmann, E.-O. Le Bigot, Y.-W. Liu, J. A. M. Lopes, L. Ludhova, C. M. B. Monteiro, F. Mulhauser, T. Nebel, P. Rabinowitz, J. M. F. dos Santos, L. A. Schaller, C. Schwob, D. Taqqu, J. F. C. A. Veloso, J. Vogelsang, and R. Pohl, Science **339**, 417 (2013).

[2] R. Pohl, F. Nez, L. M. P. Fernandes, F. D. Amaro, F. Biraben, J. M. R. Cardoso, D. S. Covita, A. Dax, S. Dhawan, M. Diepold, A. Giesen, A. L. Gouvea, T. Graf, T. W. Hänsch, P. Indelicato, L. Julien, P. Knowles, F. Kottmann, E.-O. Le Bigot, Y.-W. Liu, J. A. M. Lopes, L. Ludhova, C. M. B. Monteiro, F. Mulhauser, T. Nebel, P. Rabinowitz, J. M. F. dos Santos, L. A. Schaller, K. Schuhmann, C. Schwob, D. Taqqu, J. F. C. A. Veloso, and A. Antognini, Science **353**, 669 (2016).

[3] T. Nebel, F. D. Amaro, A. Antognini, F. Biraben, J. M. R. Cardoso, D. S. Covita, A. Dax, L. M. P. Fernandes, A. L. Gouvea, T. Graf, T. W. Hänsch, M. Hildebrandt, P. Indelicato, L. Julien, K. Kirch, F. Kottmann, Y.-W. Liu, C. M. B. Monteiro, F. Nez, J. M. F. dos Santos, K. Schuhmann, D. Taqqu, J. F. C. A. Veloso, A. Voss, and R. Pohl, Hyperfine Interact. **212**, 195 (2012).

[4] P. J. Mohr, D. B. Newell, and B. N. Taylor, Rev. Mod. Phys. **88**, 035009 (2016).

[5] M. I. Eides, H. Grotch, and V. A. Shelyuto, *Theory of Light Hydrogenic Bound States* (Springer, Berlin, 2007).

[6] S. G. Karshenboim, E. Yu. Korzinin, V. A. Shelyuto, and V. G. Ivanov, J. Phys. Chem. Ref. Data **44**, 031202 (2015).

[7] I. Sick, in *Precision Physics of Simple Atoms and Molecules*, edited by S. G. Karshenboim, Lecture Notes in Physics Vol. 745 (Springer, Berlin, 2007), pp. 57–77.

[8] K. A. Olive *et al.* (Particle Data Group), Chin. Phys. C **38**, 090001 (2014).

[9] R. B. Firestone, V. S. Shirley, C. M. Baglin, S. Y. F. Chu, and J. Zipkin, *Table of Isotopes* (John Wiley & Sons, Inc., New York, 1996).

[10] S. J. Brodsky and R. G. Parsons, Phys. Rev. **163**, 134 (1967).

[11] K. Pachucki, Phys. Rev. A **53**, 2092 (1996).

[12] H. Grotch and D. R. Yennie, Eur. Phys. J. A **202**, 425 (1967).

[13] J. Sapirstein and D. R. Yennie, in *Quantum Electrodynamics*, edited by T. Kinoshita (World Scientific, Singapore, 1990), pp. 560–672.

[14] W. A. Barker and F. N. Glover, Phys. Rev. **99**, 317 (1955).

[15] L. Hostler, J. Math. Phys. **5**, 591 (1964).

[16] H. F. Hameka, J. Chem. Phys. **47**, 2728 (1967); **48**, 4810(E) (1968).

[17] L. C. Hostler, Phys. Rev. **178**, 126 (1969).

[18] P. P. Pavinsky and A. I. Sherstyuk, Vestnik Len. Gos. Univ. **22**, 11 (1968) [in Russian]; L. Hostler, J. Math. Phys. **11**, 2966 (1970); S. V. Khristenko and S. I. Vetchinkin, Opt. Spektrosk. **31**, 503 (1971) [Opt. Spectrosc. **31**, 269 (1971)]; S. A. Zapryagaev, N. L. Manakov, and L. P. Rapoport, Yad. Fiz. **15**, 508 (1972) [Sov. J. Nucl. Phys. **15**, 282 (1972)]; S. V. Khristenko, Teor. Mat. Fiz. **22**, 35 (1975) [Theor. Math. Phys. **22**, 21 (1975)].

[19] S. G. Karshenboim, V. G. Ivanov, and E. Yu. Korzinin, Phys. Rev. A **85**, 032509 (2012).

[20] U. D. Jentschura, Phys. Rev. A **84**, 012505 (2011).

[21] A. Di Giacomo, Nucl. Phys. B **11**, 411 (1969).

[22] E. Borie and G. A. Rinker, Phys. Rev. A **18**, 324 (1978); E. Borie, Z. Phys. A **297**, 17 (1980); E. Borie and G. A. Rinker, Rev. Mod. Phys. **54**, 67 (1982).

[23] E. Yu. Korzinin, V. G. Ivanov, and S. G. Karshenboim, Phys. Rev. D **88**, 125019 (2013).

[24] S. G. Karshenboim, E. Y. Korzinin, and V. G. Ivanov, Phys. Rev. A **89**, 032129 (2014).

[25] T. Kinoshita and M. Nio, Phys. Rev. Lett. **82**, 3240 (1999); **103**, 079901(E) (2009).

[26] V. G. Ivanov, E. Yu. Korzinin, and S. G. Karshenboim, Phys. Rev. D **80**, 027702 (2009).

[27] V. G. Ivanov, E. Yu. Korzinin, and S. G. Karshenboim, arXiv:0905.4471.

[28] S. G. Karshenboim, V. G. Ivanov, E. Yu. Korzinin, and V. A. Shelyuto, Phys. Rev. A **81**, 060501 (2010).

[29] S. G. Karshenboim, E. Yu. Korzinin, V. G. Ivanov, and V. A. Shelyuto, JETP Lett. **92**, 8 (2010).

[30] U. D. Jentschura and B. J. Wundt, Eur. Phys. J. D **65**, 357 (2011).

[31] R. Barbieri, M. Caffo, and E. Remiddi, Lett. Nuovo Cimento **7**, 60 (1973); H. Suura, and E. Wichmann, Phys. Rev. **105**, 1930 (1957); A. Peterman, *ibid.* **105**, 1931 (1957).
[32] L. A. Borisoglebsky and E. E. Trofimenko, Phys. Lett. B **81**, 175 (1979).
[33] J. L. Friar, Ann. Phys. (NY) **122**, 151 (1979).
[34] K. Pachucki, Phys. Rev. A **60**, 3593 (1999).
[35] E. Borie, Ann. Phys. (NY) **327**, 733 (2012).
[36] U. D. Jentschura, G. Soff, V. G. Ivanov, and S. G. Karshenboim, Phys. Rev. A **56**, 4483 (1997); S. G. Karshenboim, U. Jentschura, V. Ivanov, and G. Soff, Eur. Phys. J. D **2**, 209 (1998); D. Eiras and J. Soto, Phys. Lett. B **491**, 101 (2000); K. Jungmann, V. G. Ivanov, and S. G. Karshenboim, in *Hydrogen Atom: Precision Physics of Simple Atomic Systems*, edited by S. G. Karshenboim *et al.* (Springer, Berlin, 2001), p. 446.
[37] S. G. Karshenboim, E. Y. Korzinin, and V. G. Ivanov, JETP Lett. **88**, 641 (2008); **89**, 216(E) (2009).
[38] A. A. Krutov, A. P. Martynenko, G. A. Martynenko, and R. N. Faustov, J. Exp. Theor. Phys. **120**, 73 (2015).
[39] S. G. Karshenboim, E. Y. Korzinin, V. A. Shelyuto, and V. G. Ivanov, Phys. Rev. D **91**, 073003 (2015).
[40] A. P. Martynenko, Yad. Phys. **69**, 1344 (2006); Phys. At. Nucl. **69**, 1309 (2006).
[41] C. E. Carlson and M. Vanderhaeghen, Phys. Rev. A **84**, 020102 (2011).
[42] M. C. Birse and J. A. McGovern, Eur. Phys. J. A **48**, 120 (2012).
[43] C. Peset and A. Pineda, Nucl. Phys. B **887**, 69 (2014).
[44] A. Pineda, Phys. Rev. C **71**, 065205 (2005).
[45] D. Nevado and A. Pineda, Phys. Rev. C **77**, 035202 (2008).
[46] M. Gorchtein, F. J. Llanes-Estrada, and A. P. Szczepaniak, Phys. Rev. A **87**, 052501 (2013).
[47] J. M. Alarcón, V. Lensky, and V. Pascalutsa, Eur. Phys. J. C **74**, 2852 (2014).
[48] N. Nevo Dinur, C. Ji, S. Bacca, and N. Barnea, Phys. Lett. B **755**, 380 (2016).
[49] O. J. Hernandez, C. Ji, S. Bacca, N. Nevo Dinur, and N. Barnea, Phys. Lett. B **736**, 344 (2014).
[50] C. Ji, N. Nevo Dinur, S. Bacca, and N. Barnea, Phys. Rev. Lett. **111**, 143402 (2013).
[51] R. B. Wiringa, V. G. J. Stoks, and R. Schiavilla, Phys. Rev. C **51**, 38 (1995).
[52] B. S. Pudliner, V. R. Pandharipande, J. Carlson, and R. B. Wiringa, Phys. Rev. Lett. **74**, 4396 (1995).
[53] D. R. Entem and R. Machleidt, Phys. Rev. C **68**, 041001 (2003).
[54] P. Navrátil, Few-Body Syst. **41**, 117 (2007).
[55] D. R. Entem and R. Machleidt, Phys. Rev. C **66**, 014002 (2002).
[56] E. Epelbaum, W. Glockleb, and U.-G. Meissner, Nucl. Phys. A **747**, 362 (2005).
[57] S. Weinberg, Phys. Lett. B **251**, 288 (1990); Nucl. Phys. B **363**, 3 (1991).
[58] J. L. Friar and G. L. Payne, Phys. Rev. A **56**, 5173 (1997).
[59] J. L. Friar, Phys. Rev. C **88**, 034003 (2013).
[60] K. Pachucki, Phys. Rev. Lett. **106**, 193007 (2011).
[61] K. Pachucki and A. Wienczek, Phys. Rev. A **91**, 040503(R) (2015).
[62] E. Epelbaum, H. Krebs, and U.-G. Meissner, Phys. Rev. Lett. **115**, 122301 (2015).
[63] H. Krebs, A. Gasparyan, and E. Epelbaum, Phys. Rev. C **85**, 054006 (2012); **87**, 054007 (2013).
[64] S. Binder, A. Calci, E. Epelbaum, R. J. Furnstahl, J. Golak, K. Hebeler, H. Kamada, H. Krebs, J. Langhammer, S. Liebig, P. Maris, Ulf-G. Meißssner, D. Minossi, A. Nogga, H. Potter, R. Roth, R. Skibiński, K. Topolnicki, J. P. Vary, and H. Witała (LENPIC Collaboration), Phys. Rev. C **93**, 044002 (2016).
[65] A. Huber, T. Udem, B. Gross, J. Reichert, M. Kourogi, K. Pachucki, M. Weitz, and T. W. Hänsch, Phys. Rev. Lett. **80**, 468 (1998); C. G. Parthey, A. Matveev, J. Alnis, R. Pohl, T. Udem, U. D. Jentschura, N. Kolachevsky, and T. W. Hänsch, *ibid.* **104**, 233001 (2010).
[66] J. J. Krauth, M. Diepold, B. Franke, A. Antognini, F. Kottmann, and R. Pohl, Ann. Phys. (NY) **366**, 168 (2016).
[67] O. J. Hernandez, N. Nevo Dinur, C. Ji, S. Bacca, and N. Barnea, Hyperfine Interact. **237**, 158 (2016).
[68] C. E. Carlson, M. Gorchtein, and M. Vanderhaeghen, Phys. Rev. A **95**, 012506 (2017).
[69] A. P. Martynenko and E. N. Elekina, Phys. At. Nucl. **73**, 2074 (2010).
[70] E. N. Elekina and A. P. Martynenko, Phys. At. Nucl. **73**, 1828 (2010).